THE (UNSTABLE) THRESHOLD OF BLACK HOLE FORMATION

- Review of Critical Scalar Collapse and Related Early Results

- Self-similarity and Perturbation Analysis

- Other Models & Recent Developments

- Open Issues & Future Work

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MASSLESS SCALAR FIELD COLLAPSE

(**Choptuik**, *Phys. Rev. Lett.*, **70**, 9, 1993)

FLAT SPACETIME (WEAK-FIELD) FORMALISM

- **Spherical symmetry:** coordinates (t, r, θ, φ), no dependence on θ or φ

- **Metric:** ("geometric units": $G = c = 1$)

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

- **Scalar field equation of motion:**

$$\Box\phi \;=\; 0 \quad \Longrightarrow \quad \frac{\partial^2}{\partial t^2}\,(r\phi) \;=\; \frac{\partial^2}{\partial r^2}\,(r\phi)$$

- **General solution:** ingoing & outgoing waves:

$$r\phi\,(r, t) \;\sim\; u\,(r + t) \;+\; v\,(r - t)$$

- **Initial data:** specify ingoing profile, $f\,(r)$, outgoing profile, $g\,(r)$

$$\begin{aligned} r\phi\,(r, 0) \;&=\; f\,(r) \;+\; g\,(r) \\ \frac{\partial}{\partial t}\, r\phi\,(r, 0) \;&=\; f'\,(r) \;-\; g'\,(r) \end{aligned}$$

Typically set $g(r) = 0$

CURVED SPACETIME (STRONG-FIELD) FORMALISM

- **Metric:** Generalization of Schwarzschild coordinates:

$$ds^2 = \; -\alpha^2(r,t) \;\; dt^2 + a^2(r,t) \;\; dr^2 + r^2 \; \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

- **Geometric radial coordinate:** r; *measures proper surface area*
- **Geometric time coordinate:** T_0; *measures proper time of a central observer*:

$$T_0 \equiv \int_0^t \alpha\left(0, \tilde{t}\right) \, d\tilde{t}$$

- **Auxiliary scalar field variables:**

$$\Phi\left(r,t\right) \equiv \frac{\partial \phi}{\partial r}\left(r,t\right) \qquad\qquad \Pi\left(r,t\right) \equiv \frac{a}{\alpha}\frac{\partial \phi}{\partial t}\left(r,t\right)$$

- **Equations of motion:**

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial r}\left(\frac{\alpha}{a}\,\Pi\right) \qquad\qquad \frac{\partial \Pi}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\alpha}{a}\,\Phi\right)$$

$$\frac{1}{\alpha}\frac{d\alpha}{dr} - \frac{1}{a}\frac{da}{dr} + \frac{1-a^2}{r} = 0$$

$$\frac{1}{a}\frac{da}{dr} + \frac{a^2-1}{2r} - 2\pi r\left(\Pi^2 + \Phi^2\right) = 0$$

CURVED SPACETIME FORMALISM (cont.)

- Equations of motion are invariant under rescalings (homogeneity)

$$r \to k\,r \qquad t \to k\,t \qquad k > 0$$

- Useful to discuss dynamics in terms of vbls. which are *form invariant* under these transformations:

$$X\,(r,t) \equiv \sqrt{2\pi}\,\frac{r}{a}\,\Phi = \sqrt{2\pi}\,\frac{r}{a}\,\frac{\partial\phi}{\partial r}$$

$$Y\,(r,t) \equiv \sqrt{2\pi}\,\frac{r}{a}\,\Pi = \sqrt{2\pi}\,\frac{r}{\alpha}\,\frac{\partial\phi}{\partial t}$$

- **Total mass,** M_{ADM}, of spacetime is

$$M_{ADM} = \int_0^\infty \frac{dm}{dr}\,dr = \int_0^\infty X^2 + Y^2\,dr$$

$$a\,(r,t)^2 = \left(1 - \frac{2m\,(r,t)}{r}\right)^{-1}$$

- Coordinate system cannot penetrate interior of black holes. However, black hole formation clearly signaled in calculation by:

$$\frac{2m}{r} \to 1 \qquad \text{for some} \qquad r = R_{BH} = 2M_{BH}$$

COMPETITION

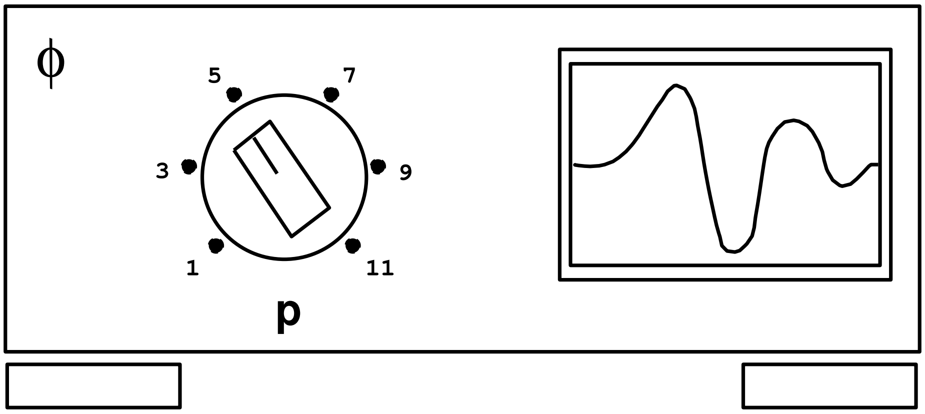
Where can energy in system end up?

K. E. & REPULSIVE SELF-INTERACTION: $\Longrightarrow \quad r = \infty$

GRAVITATIONAL P. E.: $\Longrightarrow \quad r < r_{BH}$

The Threshold of Black Hole Formation: Critical Behaviour

- Consider parametrized families of collapse solutions
- Parameter, p, controls degree of self-gravitation in evolution



- Demand that family "interpolates" between flat spacetimes and spacetimes containing black holes:
  - Low setting: no black hole forms
  - High setting: black hole forms

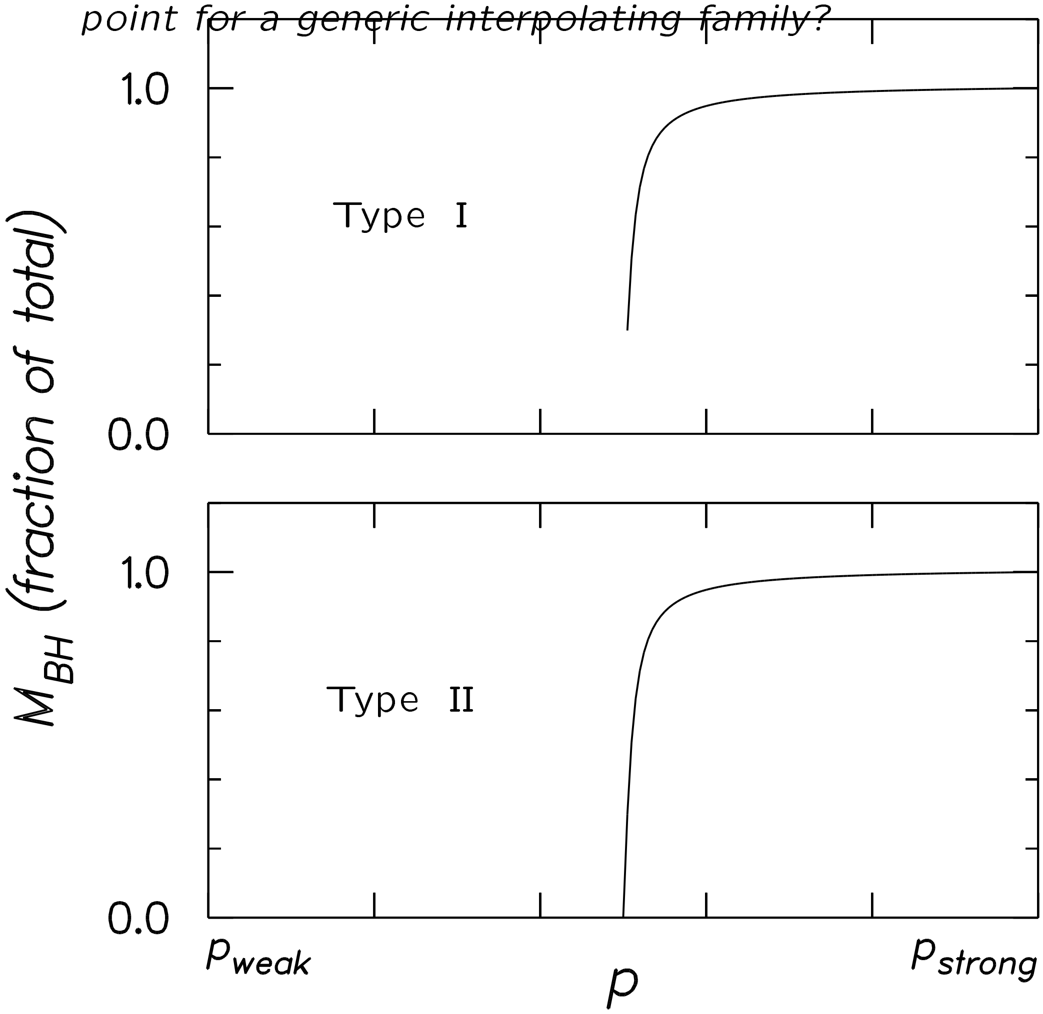
Black hole formation "turns on" at some threshold value $p^\star$

Phenomena in near-threshold regime $\equiv$ *Critical Phenomena*

KEY QUESTION

(D. Christodoulou, 1987)

What is the behaviour of the black hole mass at a critical point for a generic interpolating family?

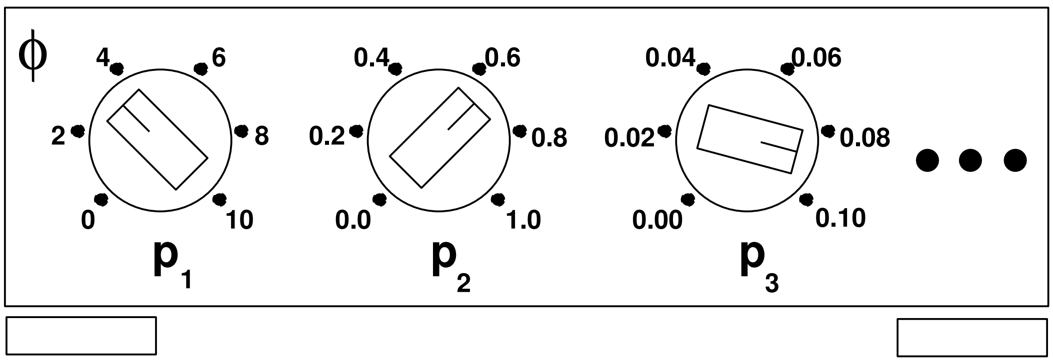


The Threshold of Black Hole Formation

- **Critical Phenomena:**
  - Interpolating families have *critical points* where black hole formation "just occurs"
  - *Phenomenology in critical regime analogous to statistical mechanical critical phenomena:*

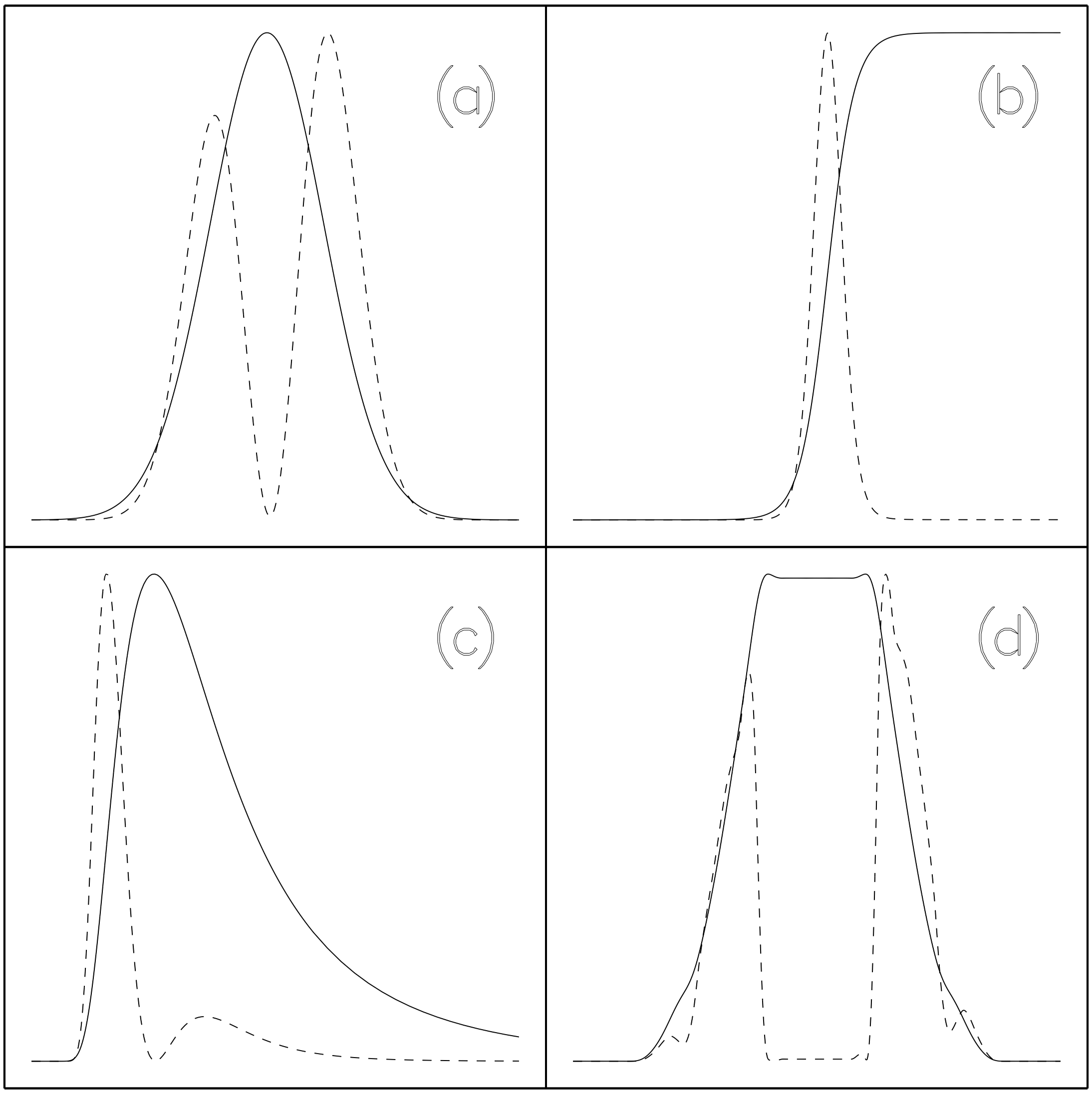
    M_{BH} *plays the role of order parameter*

- No matter what the original parametrization, will always want the following type of controls in order to explore near-critical phenomenology:



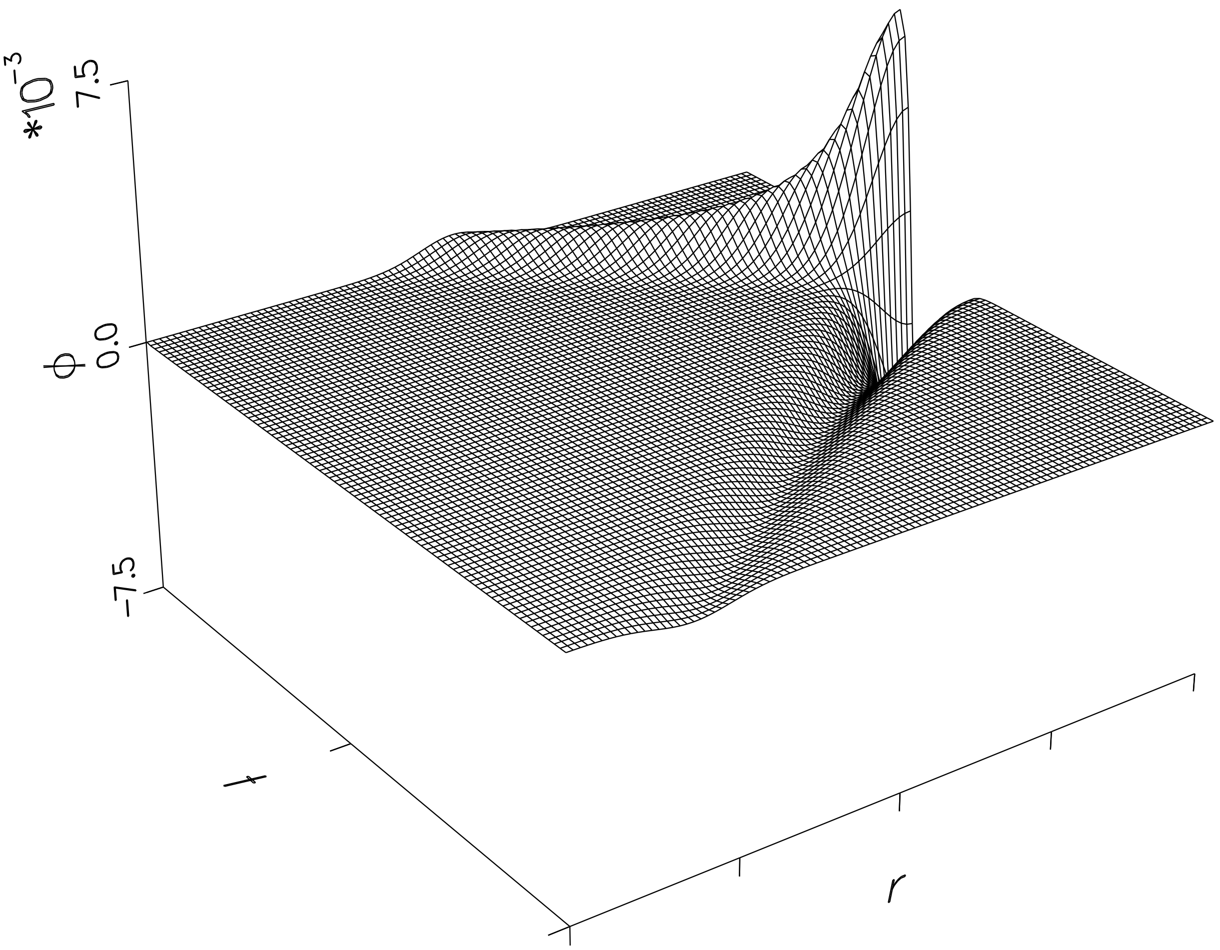
$$\mathbf{p}^* = \mathbf{3.68} \cdots$$

INITIAL DATA FAMILIES

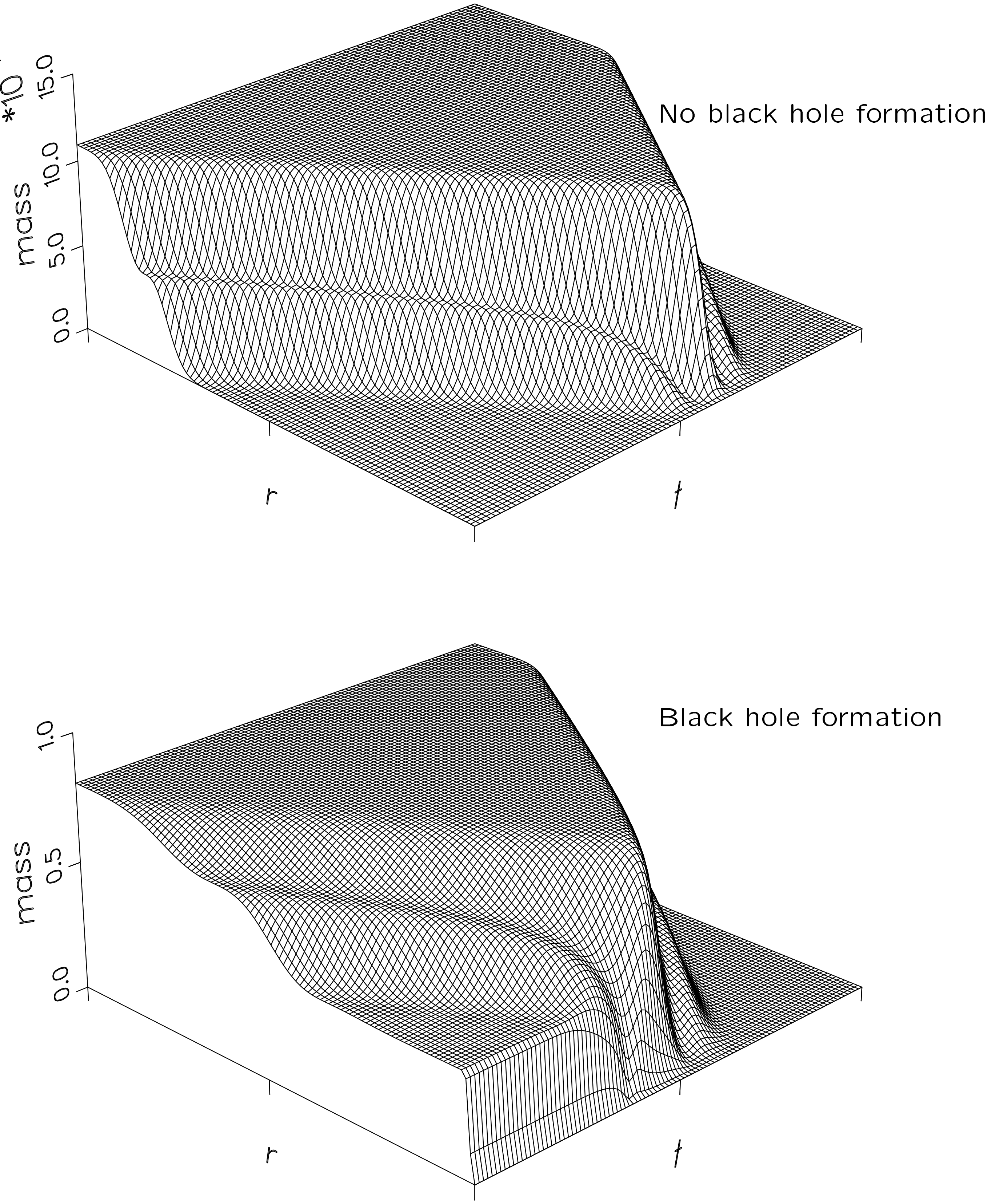


Massless Scalar Collapse: Weak-Field Evolution

$$r\phi(r,t) = \begin{cases} f(t+r) & r \geq t \\ f(t+r) - f(t-r) & r < t \end{cases}$$



Massless Scalar Collapse: "Interpolating" Solutions



CRITICAL PHENOMENA—TYPE II GENERIC FEATURES

- Arbitrarily small black holes possible (definition)
  - Sufficient fine-tuning of initial data can result in regions of spacetime with arbitrarily high curvature
  - Precisely critical solutions contain naked singularities

- Power-law scaling of black hole mass

$$M_{BH} \sim c_f \, |p - p^\star|^{\gamma_m}$$

- Scaling behaviour of critical solution
  - Discrete self-similarity (scale periodicity / "echoing")
    scalar, gravitational, Yang-Mills waves
  - Continuous self-similarity
    perfect fluid, mutiple-scalar systems

SCALAR FIELD CRITICAL PHENOMENA—SUMMARY

- **Discrete self-similarity:** (scale-periodicity, "echoing")

  - Precisely-critical evolution: any "scaling" variable $Z^\star$ (such as ϕ, m/r, dm/dr $\cdots$), satisfies:

  $$Z^\star(\rho \pm n\Delta, \tau \pm n\Delta) \sim Z^\star(\rho, \tau)$$

  where

  $$\rho \equiv \log r \qquad \tau \equiv \log\left(T_0^\star - T_0\right)$$

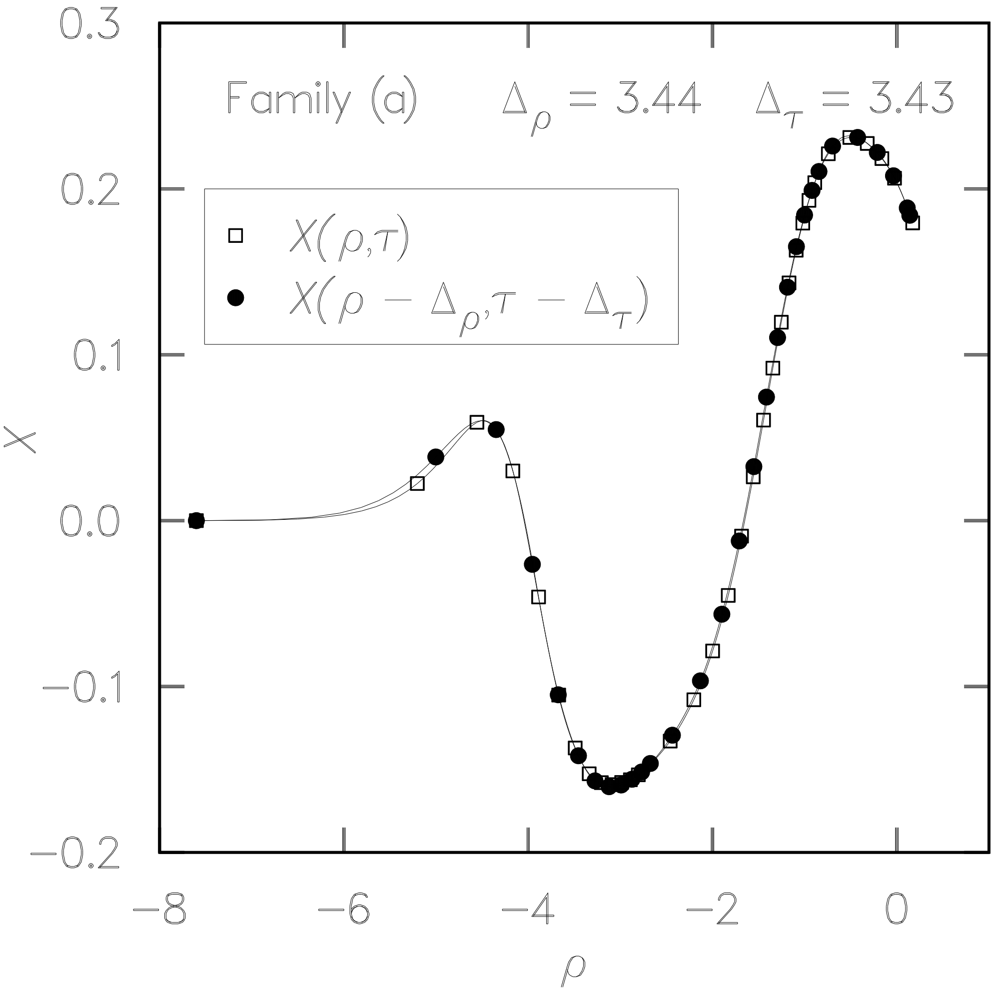
  $T_0^\star$: central proper time at which the infinite series of echos of the precisely-critical solution accumulates

  $$\Delta \approx 3.44 \qquad e^\Delta \approx 31$$

- **Mass scaling:**

$$M_{BH} \sim c_f\,|p - p^\star|^\gamma \qquad \gamma \approx 0.37$$

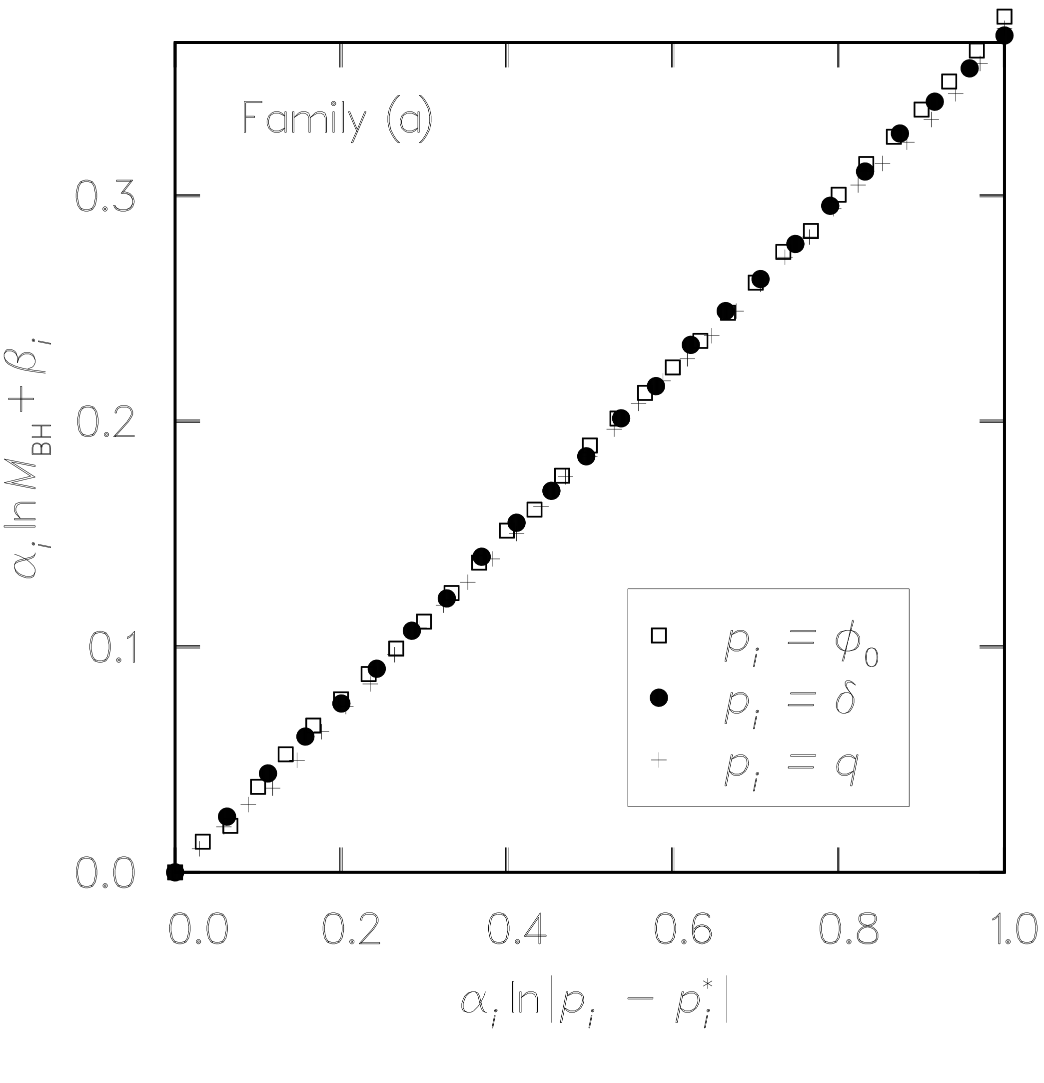
SCALAR COLLAPSE: DISCRETE SELF-SIMILARITY



- **Gundlach (PRD 55, 695)**: working from discretely self-similar ansatz finds: $\Delta = 3.445\ldots$

SCALAR COLLAPSE: BLACK HOLE MASS SCALING

$$\phi(r,0) = \phi_0\, r^3 \exp\left(-\left(\left(r - r_0\right)/\delta\right)^q\right)$$



- Scaling provides remarkably good fit even when $> 90\%$ of pulse energy ends up in black hole

SCALE-PERIODICITY IN GRAVITATIONAL WAVE COLLAPSE

(from **Abrahams & Evans**, PRL, **70**, 2980)

See cited reference for Figure: "$n(sin\,\theta = 1)$ vs $ln(r) + n\Delta$"

$$\Delta \sim 0.6$$

MASS SCALING IN GRAVITATIONAL WAVE COLLAPSE

(from **Abrahams & Evans**, PRL, **70**, 2980)

See cited reference for Figure: "Mass vs a"

$$\gamma \sim 0.36$$

SELF-SIMILARITY IN PERFECT FLUID COLLAPSE

(from **Evans & Coleman**, PRL, **72**, 1782)

See cited reference for Figure: "$a(r)$ vs $log\, r$"

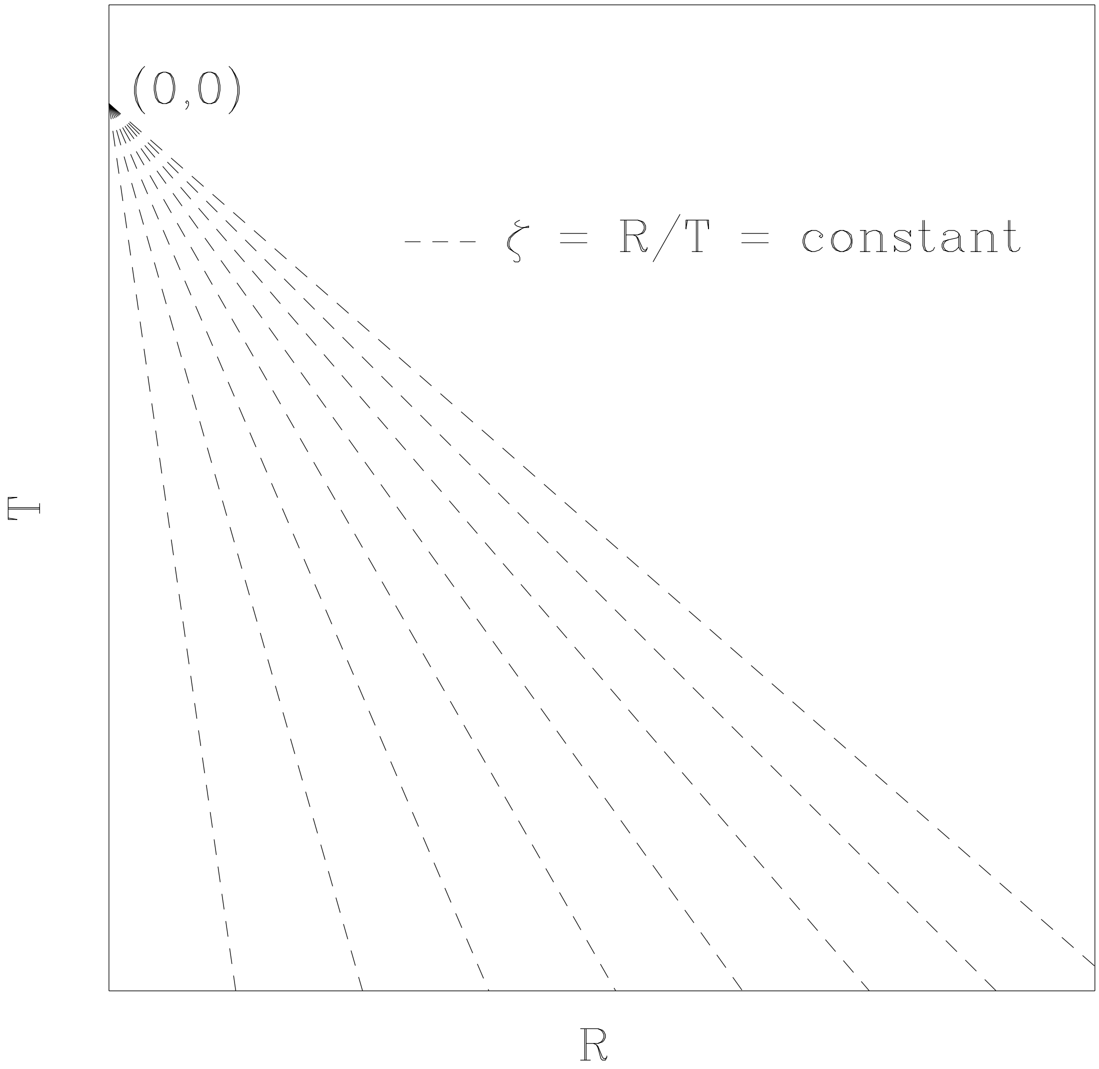
MASS SCALING IN PERFECT FLUID COLLAPSE

(from **Evans & Coleman**, PRL, **72**, 1782)

See cited reference for Figure: “$log(M_{BH})$ vs $log(\eta - \eta_c)$

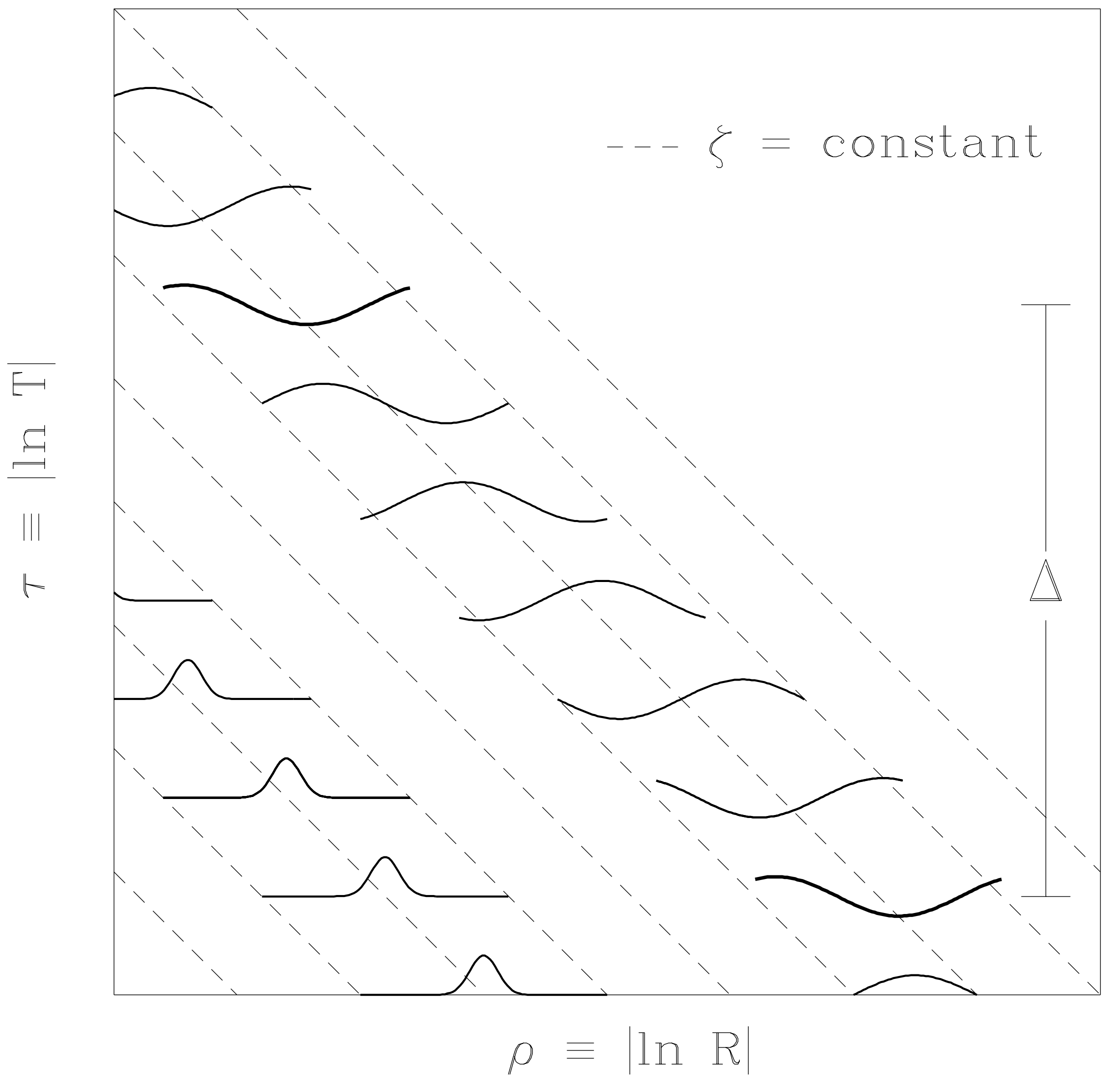
$$\gamma \sim 0.36$$

SELF-SIMILARITY: CONTINUOUS AND DISCRETE



- $T \equiv$ central proper time (measured from "accumulation point")
- $R \equiv$ areal radius
- Similarity variable: $\zeta \equiv R/T$
- Singularity at $(0,0)$

SELF-SIMILARITY: CONTINUOUS AND DISCRETE



- Continuously self-similar (CSS):

$$g(\zeta,\tau) = g(\zeta,\tau') \qquad \tau,\ \tau' \quad \text{arbitrary}$$

- Discretely self-similar (DSS):

$$g(\zeta,\tau) = g(\zeta,\tau + n\Delta) \qquad n = 0, \pm 1, \pm 2, \cdots$$

Δ—*model-dependent "echoing exponent"*

PERTURBATION THEORY

(Evans & Coleman, Koike et al, Maison Eardley & Hirschmann, Gundlach)

- For given model (matter content, couplings, symmetry), find one or more isolated critical (threshold) solutions, $g^\star$ via dynamic evolution of 1-parameter families

- $g^\star$ can generally be constructed "directly" from ansatz which reflects particular self-similarity exhibited by solution

- Consider perturbation theory using $g^\star$ as background—look for eigenmodes, $f_i(\zeta)$, eigenvalues, λ_i:

$$\delta g(\zeta,\rho) \;\equiv\; g(\zeta,\rho) - g^\star(\zeta,\rho) \;\sim\; \exp\left(\lambda_i \rho\right)\, f_i(\zeta)$$

- Observed critical behaviour strongly suggests that "unstable manifold" of $g^\star$ has dimension 1, i.e. *single* eigenmode with

$$\mathrm{Re}\ \lambda_\star > 0$$

PETURBATION THEORY (cont.)

- Straightforward argument (scaling analysis, RG) then predicts

$$M_{BH} \propto |p - p^\star|^{1/\lambda_\star}$$

i.e.

$$\gamma = \frac{1}{\lambda_\star}$$

- This picture of critical solutions as "intermediate attractors" with 1-dimensional unstable manifolds also explains observed universality (family-independence) of mass-scaling exponent

- Validated for many spherically symmetric models, including
  - Perfect fluid ($p \propto \rho$): **`Koike et al, Maison, Evans`**
  - Axion-dilaton: **`Eardley & Hirschmann, Hamade et al`**
  - Non-linear σ: **`Eardley & Hirschmann`**
  - Massless scalar (DSS): **`Gundlach`**
  - SU(2) Einstein-YM: **`Gundlach`**
  - Scalar electrodynamics: **`Gundlach, Martin-Garcia`** (additional scaling law for charge)

PETURBATION THEORY (cont.)

- Several other threshold (?) solutions have been found starting from a CSS ansatz.
  - Generically have *two or more* growing modes
  - Will not be intermediate attractors for *generic* 1-parameter family of solutions, but possibly for "special" families (**Liebling, work in progress**)

- No equivalent "theory" yet for echoing exponents (Δ) which arise in DSS solutions

- Consideration of general admissible scaling solution in DSS case (**Gundlach, PRD, 55, 695**) predicts "universal wiggle" in mass-scaling law; observed by **Hod & Piran, (PRD, 55, 440)**

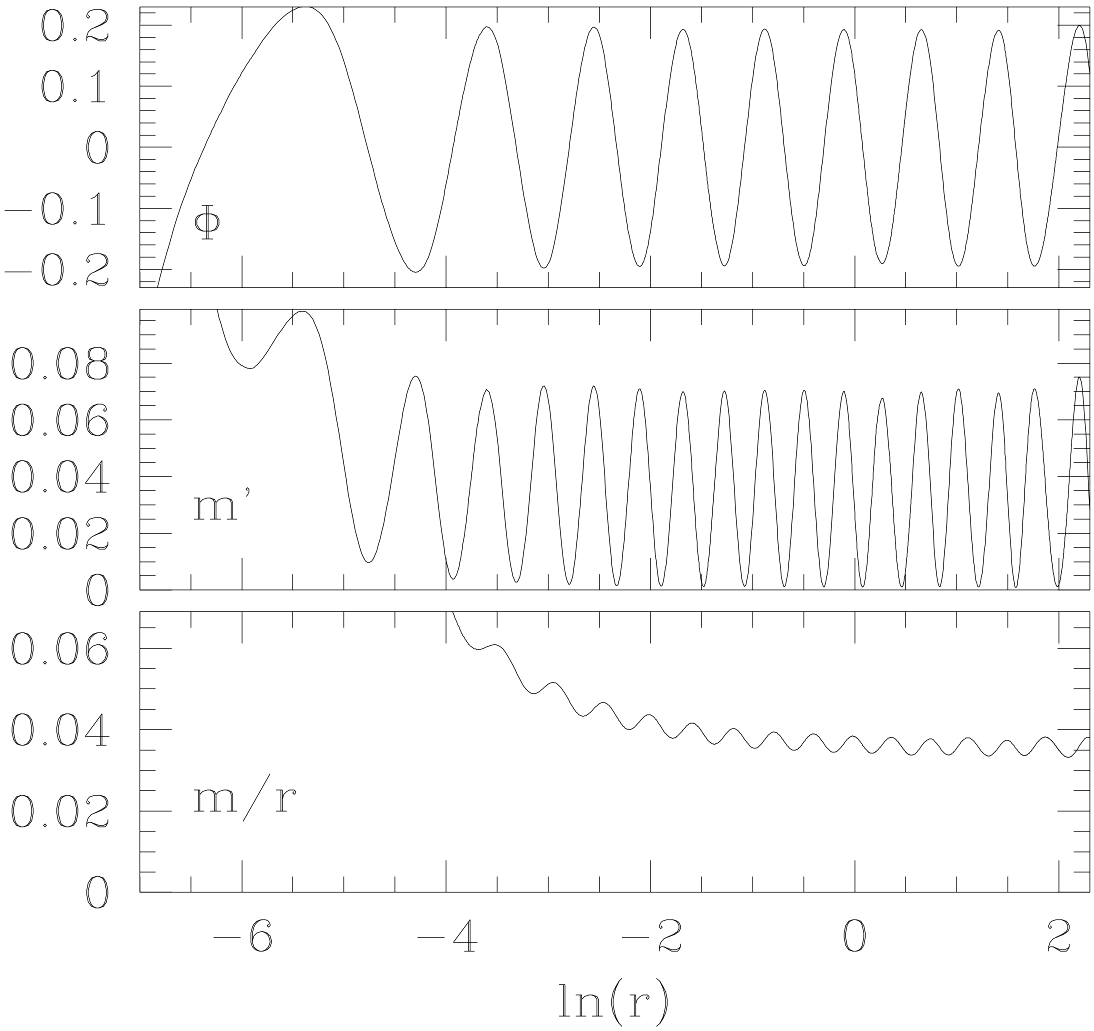
- Interesting recent work by **Garfinkle, (PRD, 56, 3169)** considers separation of Einstein equations into pieces governing
  - Dynamics of overall scale
  - Dynamics of "scale invariant" part of the metric

  Also studies model non-linear dynamical system $(a(t), b(t), c(t))$ which exhibits many of features seen in critical collapse.

EINSTEIN/YANG-MILLS COLLAPSE: TYPE II BEHAVIOUR

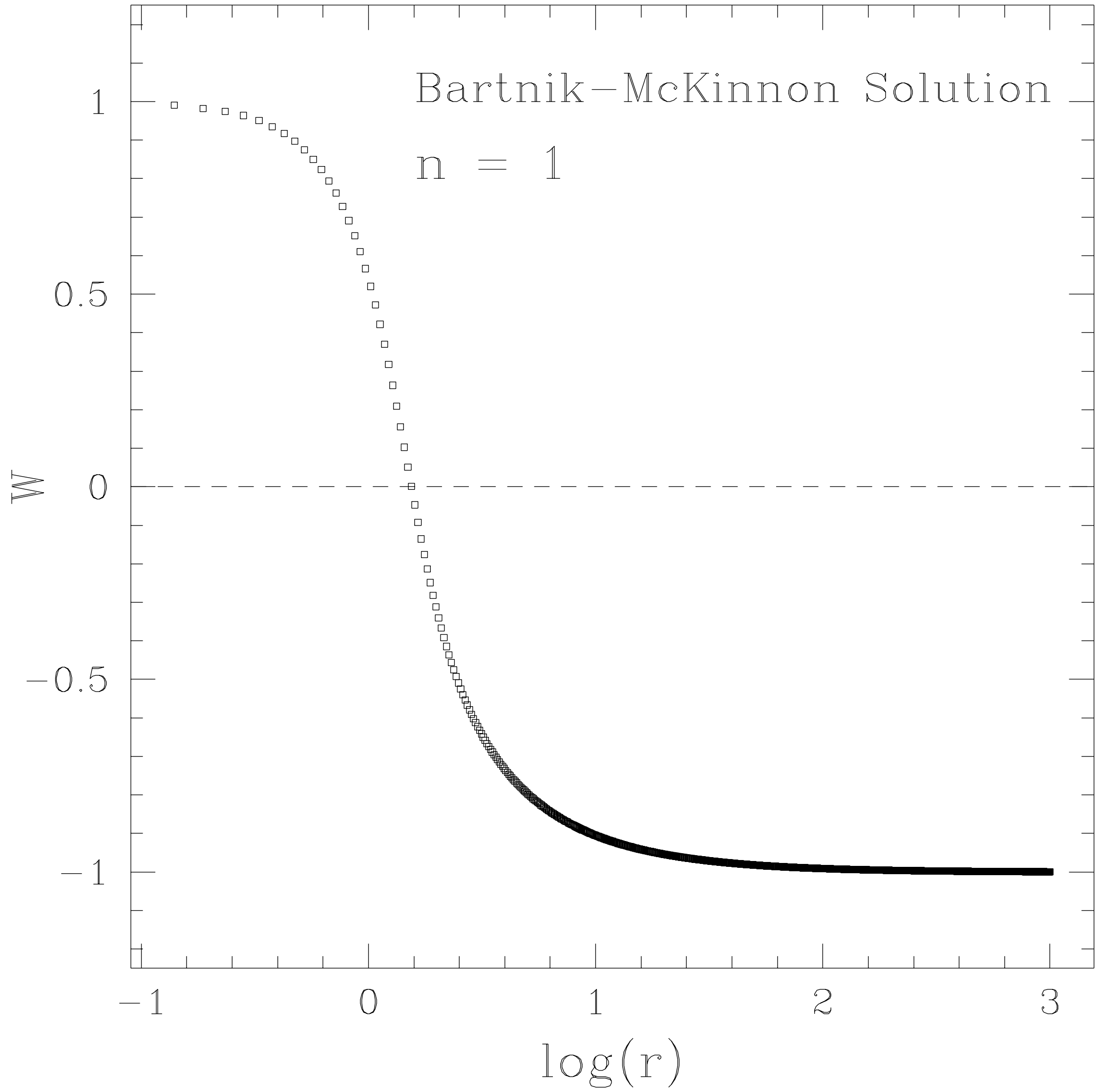
(Choptuik, Chmaj & Bizon, PRL 77, 424)

- Matter model: Non-abelian gauge field; non-trivial vacuum structure



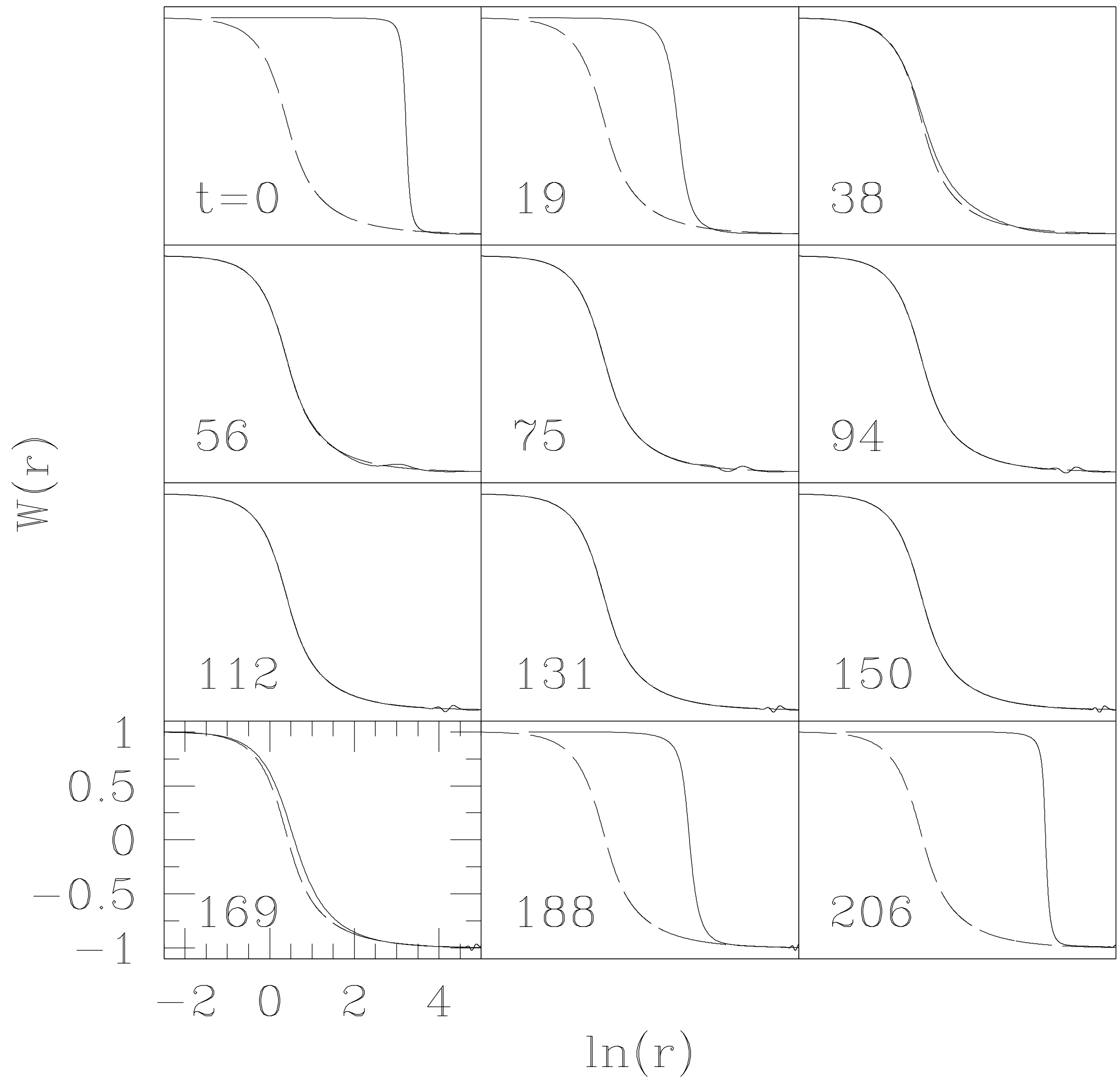
- Scaling exponents: $\Delta \sim 0.74 \quad \gamma \sim 0.20$

EINSTEIN/YANG-MILLS COLLAPSE: TYPE I BEHAVIOUR

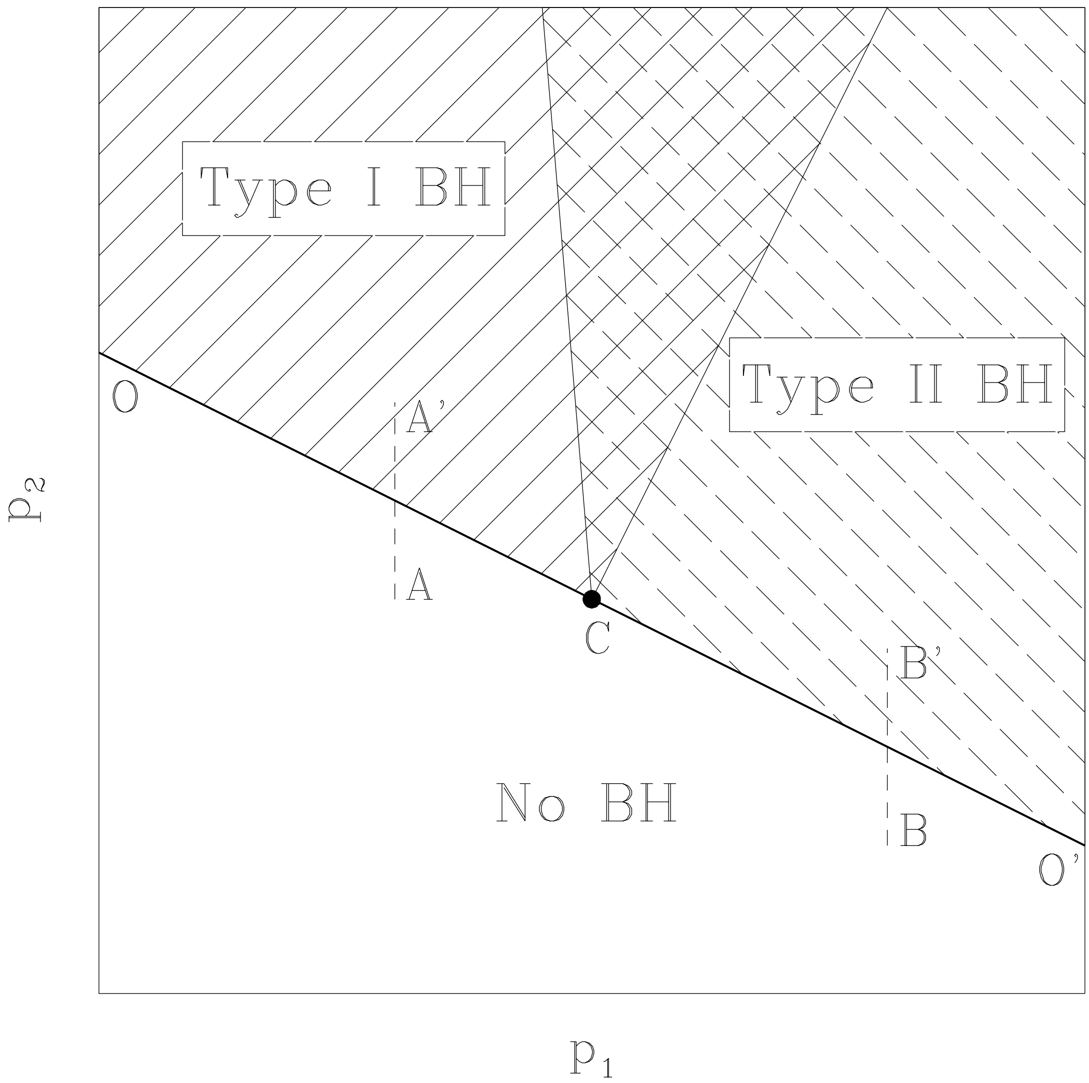


EINSTEIN/YANG-MILLS COLLAPSE: TYPE I BEHAVIOUR

- Critical solution in this case is *static* configuration found by `Bartnik and Mckinnon, PRL 61, 141`



PHASE-SPACE SCHEMATIC FOR EYM COLLAPSE



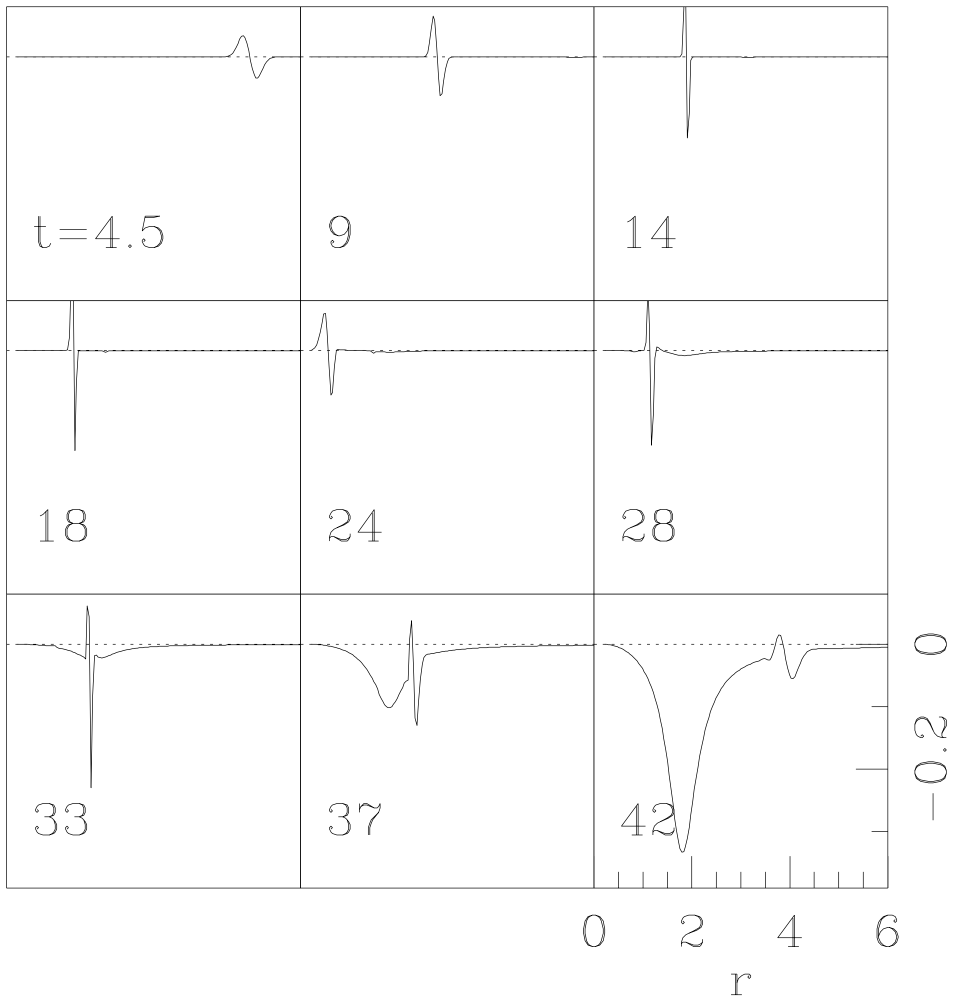
- The two types of threshold behaviour co-exist at C.

- Similar picture now established for *massive* scalar collapse (**Brady, Chambers & Goncalves, PRD, 56, 6057**)

IRREGULAR CRITICAL SOLUTIONS

(**Choptuik, Hirschmann & Liebling, PRD 55, 6014**)

- Following **Van Putten, PRD 54, 5931**, investigate stability of *static* massless scalar solution—irregular at origin. (**Janis, Newman & Winicour, PRL 20, 878** $\cdots$)



- Solution unstable and apparently at black hole threshold, although interpretation of "evaporation" of irregular solution is unclear

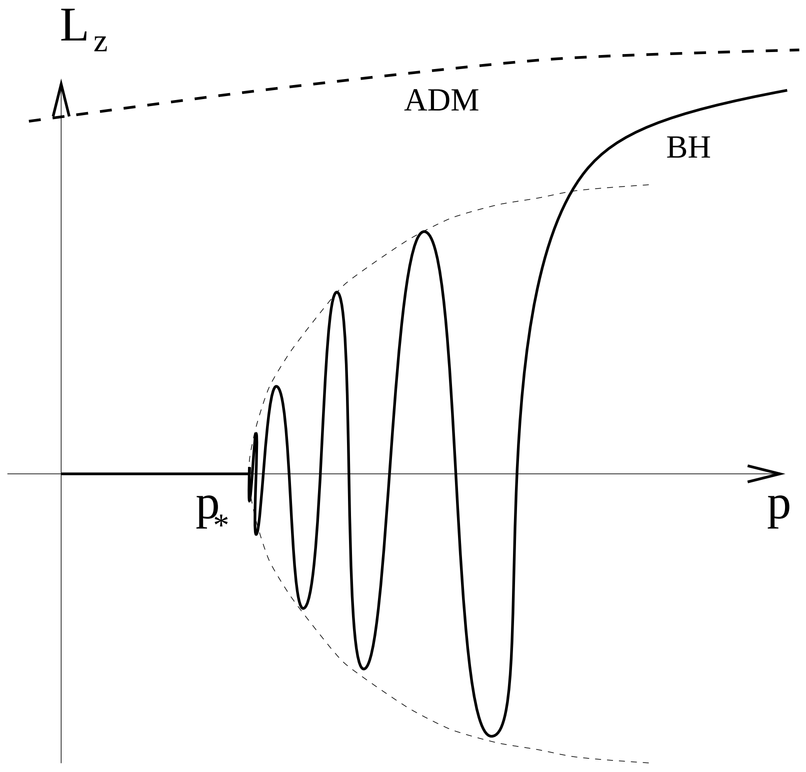
ANGULAR MOMENTUM AT THE BLACK HOLE THRESHOLD

(Carsten Gundlach, gr-qc/9710066, gr-qc/9711079)

- Considers non-spherical perturbations about Evans-Coleman radiation fluid ($p = \rho/3$) critical solution.

- Finds *no* growing non-spherical modes, strongly suggesting that spherical critical solution will persist, at least for small departures from spherical symmetry

- Examines dominant mode, ($\ell = 1$), contributing to angular momentum, L—predicts angular momentum scaling in critical regime with exponent $\mu \approx 0.80$; also predicts that ang. momentum vector will rotate as function of $p - p^\star$ with universal frequency $\omega \approx 0.23$



QUANTUM EFFECTS

What impact does quantization of matter fields have on criticality?

- Picture murky at the current time (semi-classical only):
  - Quantization introduces length scale, mass-gap appears at threshold (**Chiba & Siino**, **Peleg et al, PRD, 55, 4525**)
  - Hawking effect "renormalizes" competition, scaling persists where SC approximation is valid (**Thorlacius & Strominger, PRL, 72, 1584; Ayal & Piran, gr-qc/9704027**)

Every Type II critical solution is strong-field at arbitrarily small scales

Critical solutions demand to be quantized

SOME OPEN ISSUES & PLANNED WORK

- **Axisymmetry; no symmetry (eventually)**
  - New matter fields (electromagnetic), redo gravitational wave collapse studied by `Abrahams & Evans`
  - Angular momentum
- **Effect of multiple fields:**
  - Investigate relative stability of critical solutions
- **Quantum Effects:** ???
- **Further understanding of mode structure, origin of DSS**
- **Global structure of critical spacetimes**
- **Relationship to log-periodicity in CM/RG literature**

RICH PHENOMENOLOGY WAITING TO BE EXPLORED!