Relative Stability of Black Hole Threshold Solutions and the Dynamical Fate of the $n = 1$ Bartnik McKinnon Solution

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Outline

- Motivation
- Spherically symmetric Einstein / SU(2) Yang-Mills (EYM)
- Critical phenomena (black hole threshold) review
- Relative stability of critical solutions
- Relative stability of scalar / YM Type II solutions
- (One) dynamical fate of n=1 Bartnik-McKinnon solution

Motivation

Why study Einstein-SU(2) Yang Mills?

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- Rich phenomenology in context of BH critical phenomena
- Provides good model in which to study relative stability of BH critical solutions

Spherically Symmetric SU(2) EYM
(with Eric Hirschmann)

- Consider SU(2) Yang-Mills (gauge) field, minimally coupled to Einstein gravity in spherical symmetry

- General form for spherically symmetric metric ($G = c = 1$)

$$
\begin{aligned}
ds^2 &= \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2a^2\beta\, dtdr + a^2\, dr^2 + r^2 b^2\, d\Omega^2 \\
&= \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2a^2\beta\, dtdr + a^2\, dr^2 + R^2\, d\Omega^2
\end{aligned}
$$

  where α, β, a, b and R are functions of r and t; R measures proper surface area ("areal radius")

- Gravitating mass well defined in spherically symmetry (at least in vacuum regions)

$$
m(R,t) = \frac{1}{2} R \left(1 - R^{;\mu} R_{;\mu}\right)
$$

  m, dm/dR are useful diagnostic quantities

- Action for general Einstein / Yang-Mills theory

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{1}{g^2} F^a_{\mu\nu} F^{a\mu\nu} \right]$$

where a is the group index and g is the YM coupling constant that will be set to unity after this slide

- Einstein field equations

$$\begin{aligned} \frac{1}{16\pi} G_{\mu\nu} &= T_{\mu\nu} \\ &= \frac{1}{g^2} \left(2F^a_{\mu\lambda} F^{a}{}_{\nu}{}^{\lambda} - \frac{1}{2} g_{\mu\nu} F^a_{\alpha\beta} F^{a\alpha\beta} \right) \end{aligned}$$

- Yang-Mills field equations:

$$D_\mu F^{a\mu\nu} = 0$$

where D_μ is the gauge-covariant/spacetime covariant derivative

- Now specialize to SU(2)—most general spherically-symmetric parameterization of the gauge connection is (Witten, PRL **38**, 121 (1977))

$$
\begin{aligned}
A = u\tau^r dt \quad &+ \quad v\tau^r dr + (w\tau^\theta + \tilde{w}\tau^\phi)d\theta \\
&+ \quad \left(\cot\theta\tau^r + w\tau^\phi - \tilde{w}\tau^\theta\right)\sin\theta d\phi
\end{aligned}
$$

  where u, v, w and $\tilde{w}$ are all functions of r and t and the τ^a are the spherical projection of the Pauli spin matrices and form an anti-Hermitian basis for SU(2), satisfying

$$[\tau^a, \tau^b] = \epsilon^{abc}\tau^c \quad a, b, c \in \{r, \theta, \phi\}$$

- Field strength is then

$$
\begin{aligned}
F \quad &= \quad \tau^r(\dot{v} - u')dt \wedge dr \\
&+ \quad [(\dot{w} - u\tilde{w})dt + (w' - v\tilde{w})dr] \wedge (\tau^\theta d\theta + \tau^\phi \sin\theta d\phi) \\
&+ \quad \left[(\dot{\tilde{w}} + uw)dt + (\tilde{w}' + vw)dr\right] \wedge (\tau^\phi d\theta - \tau^\theta \sin\theta d\phi) \\
&- \quad (1 - w^2 - \tilde{w}^2)\tau^r d\theta \wedge \sin\theta d\phi
\end{aligned}
$$

  where $^{\cdot} \equiv \partial/\partial t, ' \equiv \partial/\partial r$

- Convenient to write EOM in first-order-in-time form; to this end define auxiliary variables

$$
\begin{aligned}
\Pi &= \frac{a}{\alpha}\left[\dot{w} - u\tilde{w} - \beta(w' - v\tilde{w})\right] \\
\Phi &= w' - v\tilde{w} \\
P &= \frac{a}{\alpha}\left[\dot{\tilde{w}} + uw - \beta(\tilde{w}' + vw)\right] \\
Q &= \tilde{w}' + vw \\
Y &= \frac{b^2 r^2}{2\alpha a}(\dot{v} - u')
\end{aligned}
$$

- Then have the following EOM for the YM field:

$$
\begin{aligned}
\dot{\Phi} &= \left(\frac{\alpha}{a}\Pi + \beta\Phi\right)' + uQ - v\left(\frac{\alpha}{a}P + \beta Q\right) - \tilde{w}\,\frac{2\alpha a}{b^2 r^2}Y \\
\dot{Q} &= \left(\frac{\alpha}{a}P + \beta Q\right)' - u\Phi + v\left(\frac{\alpha}{a}\Pi + \beta\Phi\right) + w\,\frac{2\alpha a}{b^2 r^2}Y \\
\dot{\Pi} &= \left(\frac{\alpha}{a}\Phi + \beta\Pi\right)' + uP - v\left(\frac{\alpha}{a}Q + \beta P\right) + \frac{\alpha a}{b^2 r^2}w(1 - w^2 - \tilde{w}^2) \\
\dot{P} &= \left(\frac{\alpha}{a}Q + \beta P\right)' - u\Pi + v\left(\frac{\alpha}{a}\Phi + \beta\Pi\right) + \frac{\alpha a}{b^2 r^2}\tilde{w}(1 - w^2 - \tilde{w}^2) \\
\dot{Y} &= \frac{\alpha}{a}(\tilde{w}\Phi - wQ) + \beta(\tilde{w}\Pi - wP) \\
Y' &= \tilde{w}\Pi - wP \\
u' &= -\frac{2\alpha a}{r^2}Y
\end{aligned}
$$

SU(2) EYM—Purely Magnetic Ansatz

- Assume electric charge density is identically 0; $\Longrightarrow Y(r,t) \equiv 0$
- Can set $v = 0$ by gauge transformation; $Y = 0$ then implies $u = \mathrm{const.}$ Further gauge transformation makes $u = 0$; EOM then imply that we can set $\tilde{w} = 0$ without loss of generality (i.e. that $\tilde{w}$ is pure gauge in this case)
- Thus, in the context of the (dynamically self-consistent) "purely magnetic" ansatz, the dynamics of the YM field is described by the single "field", $w(r,t)$
- Regularity at the origin, and finite-energy require that $w(r,t)$ be in one of two vacuum states at $r = 0$ and $r = \infty$:

$$w(0,t) = \pm 1 \qquad w(\infty,t) = \pm 1$$

- Hereafter, will also work in polar/areal (Schwarzschild-like) coordinates

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$$

- Equations of motion simplify considerably:

$$
\begin{aligned}
\dot{\Phi} &= \left(\frac{a}{\alpha}\Pi\right)' \\
\dot{\Pi} &= \left(\frac{a}{\alpha}\Phi\right)' + \frac{\alpha a}{r^2} w \left(1 - w^2\right) \\
\frac{a'}{a} &+ \frac{1 - a^2}{2r} + \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}\left(1 - w^2\right)^2\right) \\
\frac{\alpha'}{\alpha} &+ \frac{a^2 - 1}{2r} + \frac{1}{r}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}\left(1 - w^2\right)^2\right) \\
w' &= \Phi
\end{aligned}
$$

- Initial conditions

$$
\begin{aligned}
w(0, r) &= f(r) \\
\dot{w}(0, r) &= g(r)
\end{aligned}
$$

where in practice typically choose $g(r)$ so that data is time-symmetric ($g \equiv 0$), or (almost) purely ingoing (imploding).

SU(2) EYM—General t-dependent Spherical Ansatz

- Now allow for *both* electric/magnetic charge densities

- Can still set $v(t,r) \equiv 0$ via gauge transformation, but now must apparently retain both $u(t,r)$ *and* $\tilde{w}(t,r)$ in addition to $w(t,r)$, although there is clearly gauge freedom left in u, w, $\tilde{w}$ (e.g. no evolution equation for u, and will see "gauge" effects in animations to come)

- Regularity (YM field must again be in vacuum state at origin)

$$\lim_{r\to 0}\left(w(t,r)^2 + \tilde{w}(t,r)^2\right) = 1 + O(r^2)$$

- Via gauge freedom can take

$$\begin{aligned} w(t,0) &= 1 + O(r^2) \\ \tilde{w}(t,0) &= O(r^2) \\ u(t,0) &= O(r^2) \end{aligned}$$

- Equations of motion

$$
\begin{aligned}
\dot{w} &= \frac{\alpha}{a}\Pi + u\tilde{w} \\
\dot{\tilde{w}} &= \frac{\alpha}{a}P - uw \\
\dot{\Pi} &= \left(\frac{\alpha}{a}w'\right)' + uP + \frac{\alpha a}{r^2}w\left(1 - w^2 - \tilde{w}^2\right) \\
\dot{P} &= \left(\frac{\alpha}{a}\tilde{w}'\right)' - u\Pi + \frac{\alpha a}{r^2}\tilde{w}\left(1 - w^2 - \tilde{w}^2\right) \\
u' &= -\frac{2\alpha a}{r^2}Y \\
Y' &= \tilde{w}\Pi - wP \\
\frac{\alpha'}{\alpha} &= \frac{a^2 - 1}{2r} + 4\pi r a^2 {S^r}_r = \cdots \\
\frac{a'}{a} &= \frac{1 - a^2}{2r} + 4\pi r a^2 \rho = \cdots
\end{aligned}
$$

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$$
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w(r,0) &= f(r) \\
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- Note: In all of calculations described below, outgoing radiation conditions (Sommerfeld conditions), possibly corrected by relevant non-differentiated terms, work well

Review of Black Hole Critical Phenomena

- Consider parameterized families of solutions to Einstein equations, typically coupled to one or more matter fields (but vacuum case can also be considered); focus on collapse of matter/energy and black hole formation

- Family parameter, p, viewed as "control parameter" for initial data, and hence for subsequent dynamical evolution

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- Demand that family "interpolates" through the black hole threshold, i.e. that there exists a critical value, $p = p^\star$, such that

  1. $p < p^\star$: No black hole forms
  2. $p > p^\star$: Black hole forms

- Empirically (and for some models, analytically) scenarios 1. and 2. characterized by long-time, stable "end-states" of evolution, may be *only* such states

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- Solution in near-critical regime $p \sim p^\star \equiv$ black hole critical phenomena

- Use "competition" (loosely, kinetic energy vs potential energy) inherent in collapse models, and fine-tuning to dynamically evolve to *unstable* critical solution

Critical Phenomenology

- Critical solutions $Z^\star$, *do* exist (for all models considered thus far) and are locally unique (in solution space sense and up to certain symmetry transformations)—details of initial data, parameterization irrelevant

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- Critical solutions belong to two broad classes, that can conveniently be labelled by behaviour of black hole mass at threshold (which can be viewed as an order parameter)

  1. Type I: Black hole formation turns on at *finite* mass
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- Near-critical solutions characterized by scaling of dimensionful quantities (defines additional critical exponents)

Critical Phenomenology

- Although *unstable*, critical solutions tend to be *minimally* so, in the sense of having *one* unstable mode in the context of perturbation theory

- Growth factor (Lyapunov exponent), $\mathrm{Re}\lambda_1$, of unstable mode can be immediately related to exponents in scaling relations

Type I Critical Solutions

- Smallest BH has finite mass

- Model will generally have one (or more) intrinsic length scales that will set the minimum mass

- Critical solution exhibits time translational invariance

  1. Continuous: static
  2. Discrete: periodic, defines "exponent", ω

- Scaling law for, e.g., "lifetime" of near-critical configuration during dynamical evolution

$$\tau \sim \sigma \ln |p - p^\star| \qquad \sigma = [\mathrm{Re}\lambda_1]^{-1}$$

Type I Critical Solutions

- Examples (all spherically symmetric)
  - magnetic EYM ($n = 1$ Bartnik-McKinnon solution)
  - real scalar field (unstable oscillons, Brady et al)
  - complex scalar field (unstable mini-boson stars, Hawley, Lai)
  - perfect fluid (neutron star models on unstable branch, Noble)

Type II Critical Solutions

- No minimum BH mass, arbitrarily small BHs possible
- Critical solution exhibits scale invariance
  1. Continuous: continuous self-similarity (CSS)
  2. Discrete: discrete self-similarity (DSS), defines "echoing exponent", Δ
- Scaling law for, e.g., BH masses from super-critical evolutions:

$$\ln M_{\rm BH} \sim \gamma \ln |p - p^\star| \qquad \gamma = [{\rm Re}\lambda_1]^{-1}$$

Type II Critical Solutions

- Examples (spherically symmetric)
  - massless scalar field: $\Delta \approx 3.44$, $\gamma \approx 0.37$
  - magnetic EYM: $\Delta \approx 0.74$, $\gamma \approx 0.20$
  - non-linear sigma models (Choptuik *et al*, Husa *et al*)
  - perfect fluid (Evans & Coleman, Neilsen, Noble)
- Examples (axisymmetric)
  - vacuum gravitational waves (Abraham & Evans)
  - massless scalar field with angular momentum (Pretorius *et al*)

Critical Collapse in Purely Magnetic EYM
(Choptuik, Chmaj, Bizon, PRL 77, 424, (1996))

- See both Type I and Type II transitions, depending on initial data

- Roughly, get Type II transition if, during collapse, configuration becomes sufficiently relativistic (kinetic-energy dominated), i.e. so that self-interaction "potential" term in effective Lagrangian

$$\frac{\left(1 - w^2\right)^2}{r^2}$$

  becomes negligible in comparison to kinetic terms w'^2, $\dot{w}^2$

- Within context of this ansatz, Bartnik and McKinnon demonstrated numerically existence of countable infinity of regular, *static* solutions, $w_n(r)$, $n = 1, 2, \cdots$, to EYM equations, where n counts number of zero crossings of $w(r)$

- Solutions have been extensively studied, generalized since

Critical Collapse in Purely Magnetic EYM

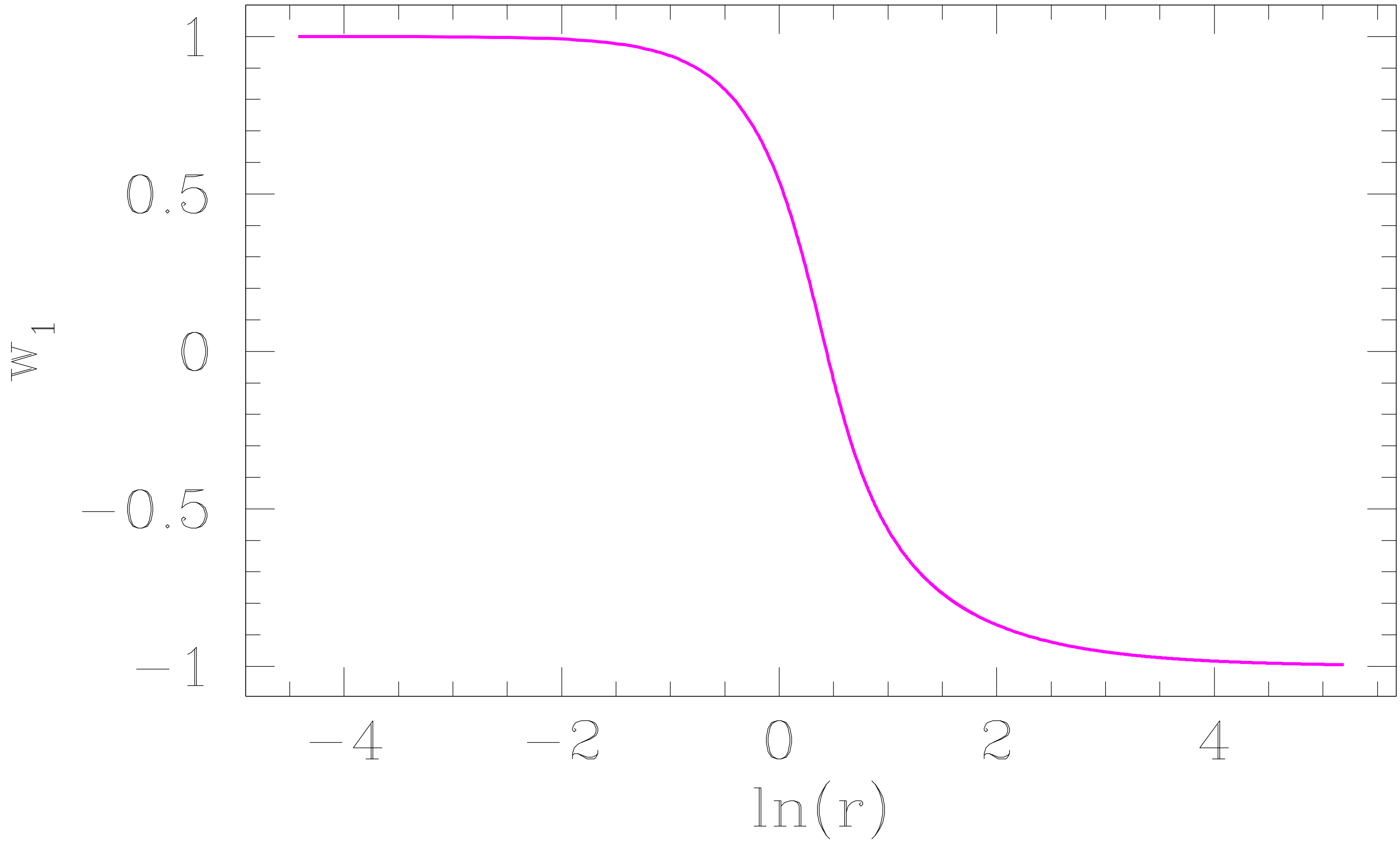
- Key facts

  1. w_n has n unstable perturbative modes in magnetic ansatz
  2. w_n has $2n$ unstable perturbative modes in general ansatz

- In particular, $n = 1$ solution can, and does, act as Type I critical solution for appropriate initial data families

- As mentioned above, Type II solution characterized by $\Delta \approx 0.74$, $\gamma = [\mathrm{Re}\lambda_1]^{-1} \approx 0.20$

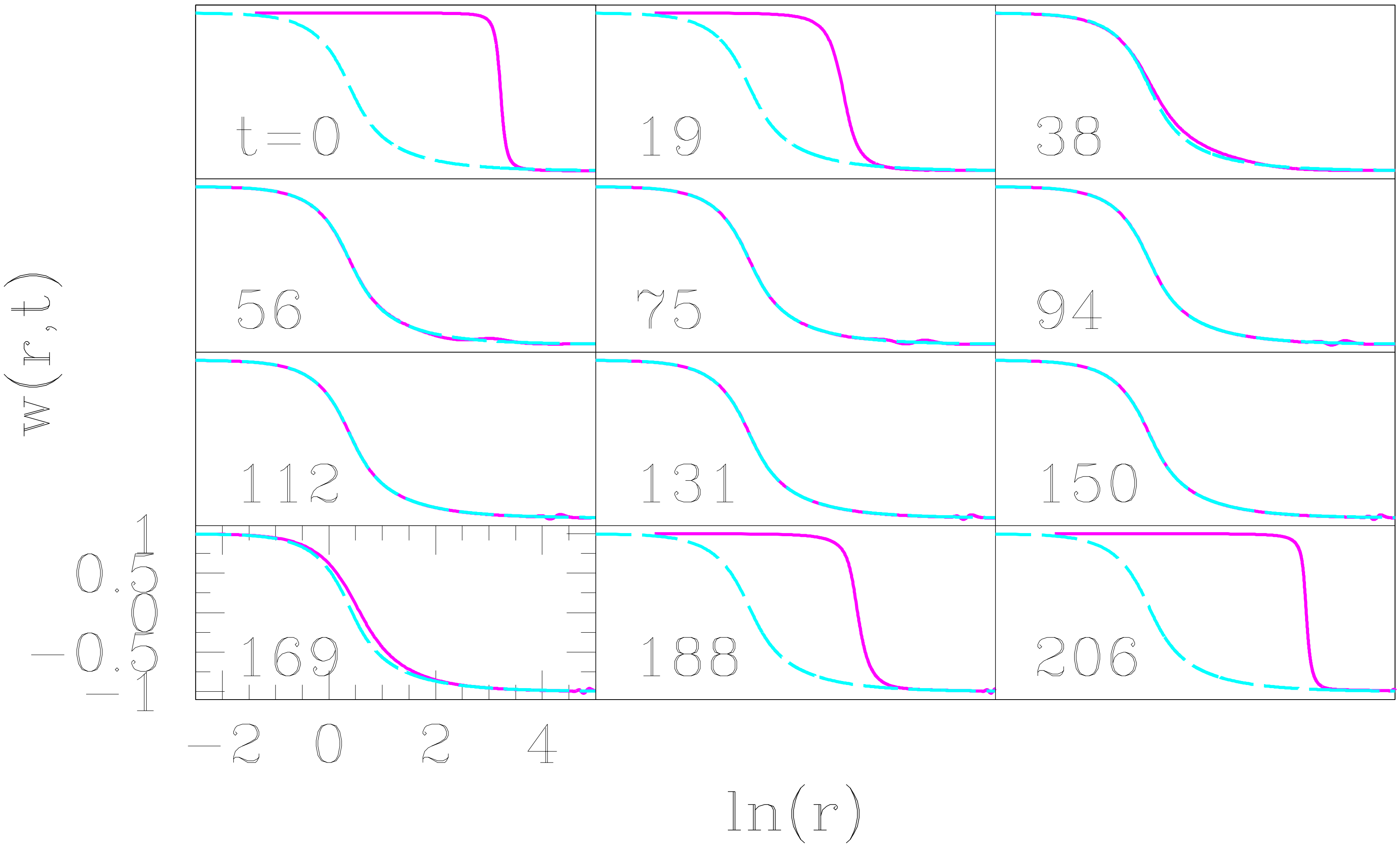
$n = 1$ Bartnik-McKinnon Solution



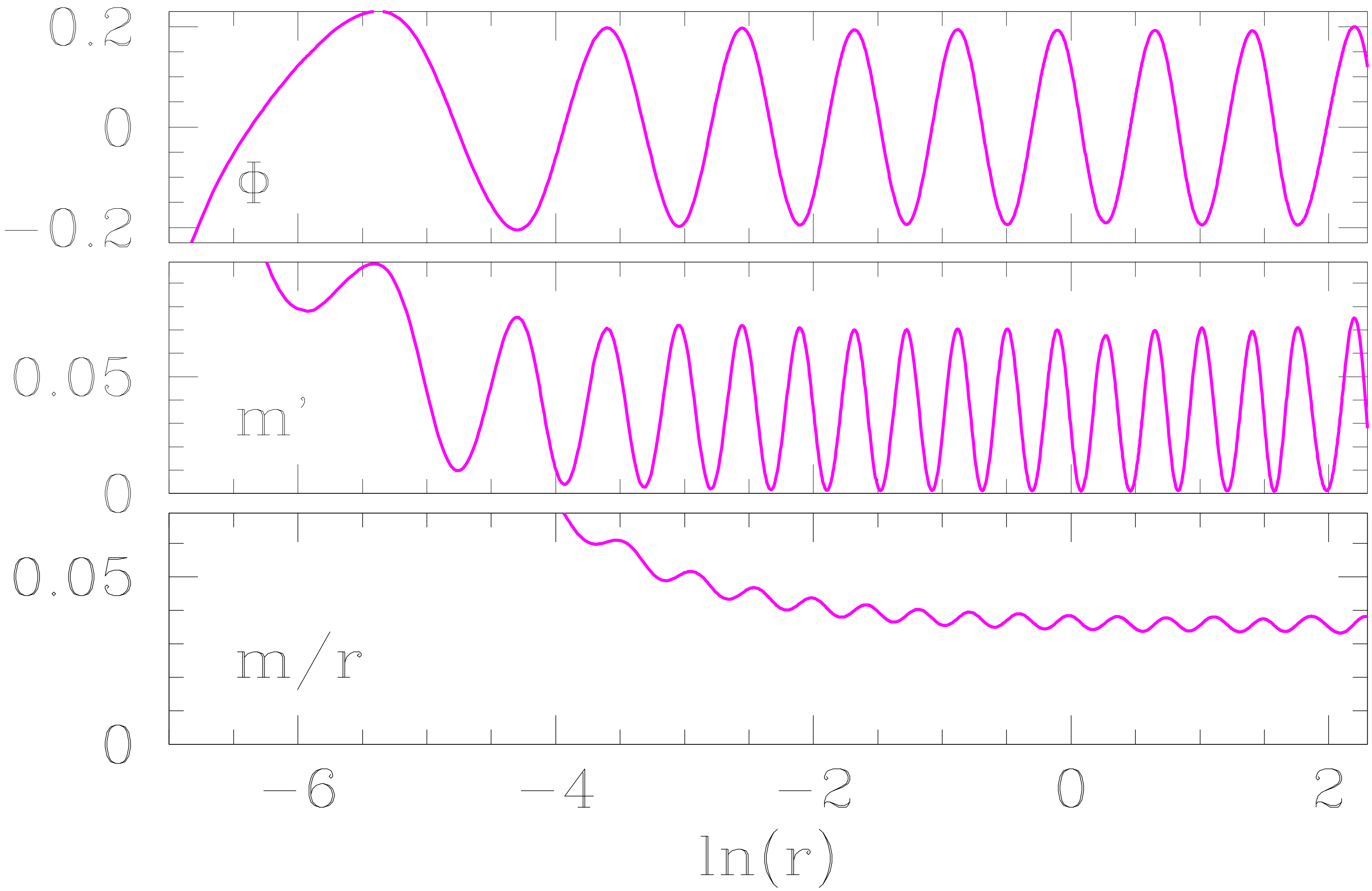
EYM Collapse Animations

- ANIMATION of Type I collapse $(w(r, t))$
- ANIMATION of Type II collapse $((1 - w)/r)$

Type I EYM Collapse



Type II EYM Collapse



Relative Stability of Critical Solutions

QUESTION: Given that critical solutions are *unstable*—i.e. in perturbation theory, always have at least one unstable mode—how does matter of one type behave in presence of critical solution of another type of matter?

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- Can at least partially address this issue by considering *relative stability* of critical solutions as (loosely) defined below
- Will proceed via (approximate) solution of full field equations
- Presumably could also do perturbation theory (perhaps using results from full PDEs as input), but some evidence that pert. theory will not be as effective in the relative stability case

Relative Stability: Basic Setup of Numerical Expts.

- Consider two fields

$$\Psi_1(r,t), \qquad \Psi_2(r,t)$$

  where we are investigating the stability of Ψ_2 w.r.t. critical soln of pure-Ψ_1 model, $\Psi_1^\star$

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3. (Minimally) couple $\Psi_2(t,r)$, with parameterized initial data $\Psi_2(0,r;\,q)$ such that support of $\Psi_2(t,r;\,q)$ during evolution overlaps support of $\Psi_1(t,r;\,p)$. Generically, for pure Ψ_2 evolution $\Psi_2(0,r;\,q^\star)$ will generate critical solution $\Psi_2^\star$

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4. *Fix* initial data $\Psi_2(0,r;\,q)$ (i.e. fix q), then retune $\Psi_1(0,r;\,p)$, determining $p_q^\star$ such that $[\Psi_1(0,r;\,p_q^\star), \Psi_2(0,r;\,q)]$ generates a black hole threshold solution

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5. Study solution phenomenology as function of q, including limits $q \to 0$, $q \to p^\star$

Relative Stability of Scalar & YM Type II Solns.

- YM field w: Adopt dynamical purely magnetic ansatz described above, pure EYM model admits Type II solution, $w^{\star}_{\rm II}$, with

$$\Delta_{\rm YM} \approx 0.74 \qquad \gamma_{\rm YM} \approx 0.20$$

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- Massless scalar field, ϕ: Has Type II solution with EYM model admits Type II solution with

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- Initial data

$$\begin{aligned} w(0,r;\,p) &= p\,e^{-(r-c)^2/s^2} \\ \phi(0,r;\,q) &= q\,e^{-(r-C)^2/S^2} \end{aligned}$$

with constants c, s, C and S chosen to ensure dynamical overlap of the supports of the two fields; $\dot{w}(0,r)$ and $\dot{\phi}(0,r$ chosen to produce ingoing initial data.

Relative Stability of Scalar & YM Type II Solns.

- Setting $\phi \equiv 0$, tune p to $p^\star$ such that $w(0, r; p^\star) \Longrightarrow w^\star_{\mathrm{II}}$

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- Fix q, $\phi(r, 0;\, q)$, retune p, $w(r, 0,\, p)$ to determine $p^\star_q$ that generates critical solution

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- Investigate phenomenology as function of q

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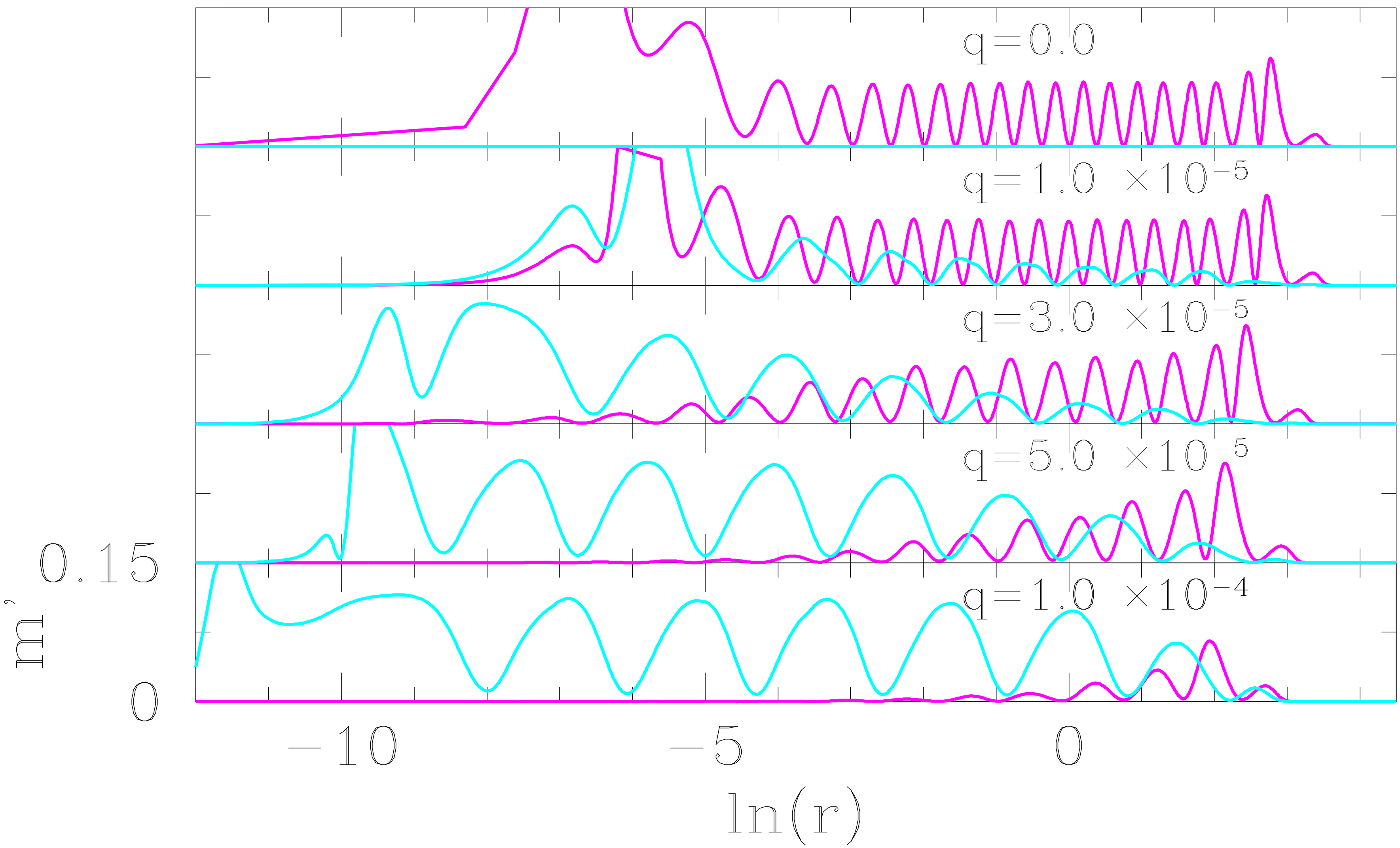
- Fix q, $\phi(r, 0;\, q)$, retune p, $w(r, 0,\, p)$ to determine $p^\star_q$ that generates critical solution
- Investigate phenomenology as function of q
- Note that the YM critical solution has the larger Lyapunov exponent, so naively, one might expect it to be unstable in the presence of the Type II scalar solution

Relative Stability of Scalar & YM Type II Solns.

- ANIMATION of critical solution for $q = 0$
  - $[dm/dr]_{\rm YM}$ and $[dm/dr]_{\rm S} \equiv 0$
- ANIMATION of critical solution for $q = 1.0 \times 10^{-5}$
  - $[dm/dr]_{\rm YM}$ and $10 \times [dm/dr]_{\rm S}$
- ANIMATION of critical solution for $q = 3.0 \times 10^{-5}$
  - $[dm/dr]_{\rm YM}$ and $[dm/dr]_{\rm S}$
- ANIMATION of critical solution for $q = 5.0 \times 10^{-5}$
  - $[dm/dr]_{\rm YM}$ and$[dm/dr]_{\rm S}$
- ANIMATION of critical solution for $1 = 5.0 \times 10^{-4}$
  - $[dm/dr]_{\rm YM}$ and $[dm/dr]_{\rm S}$

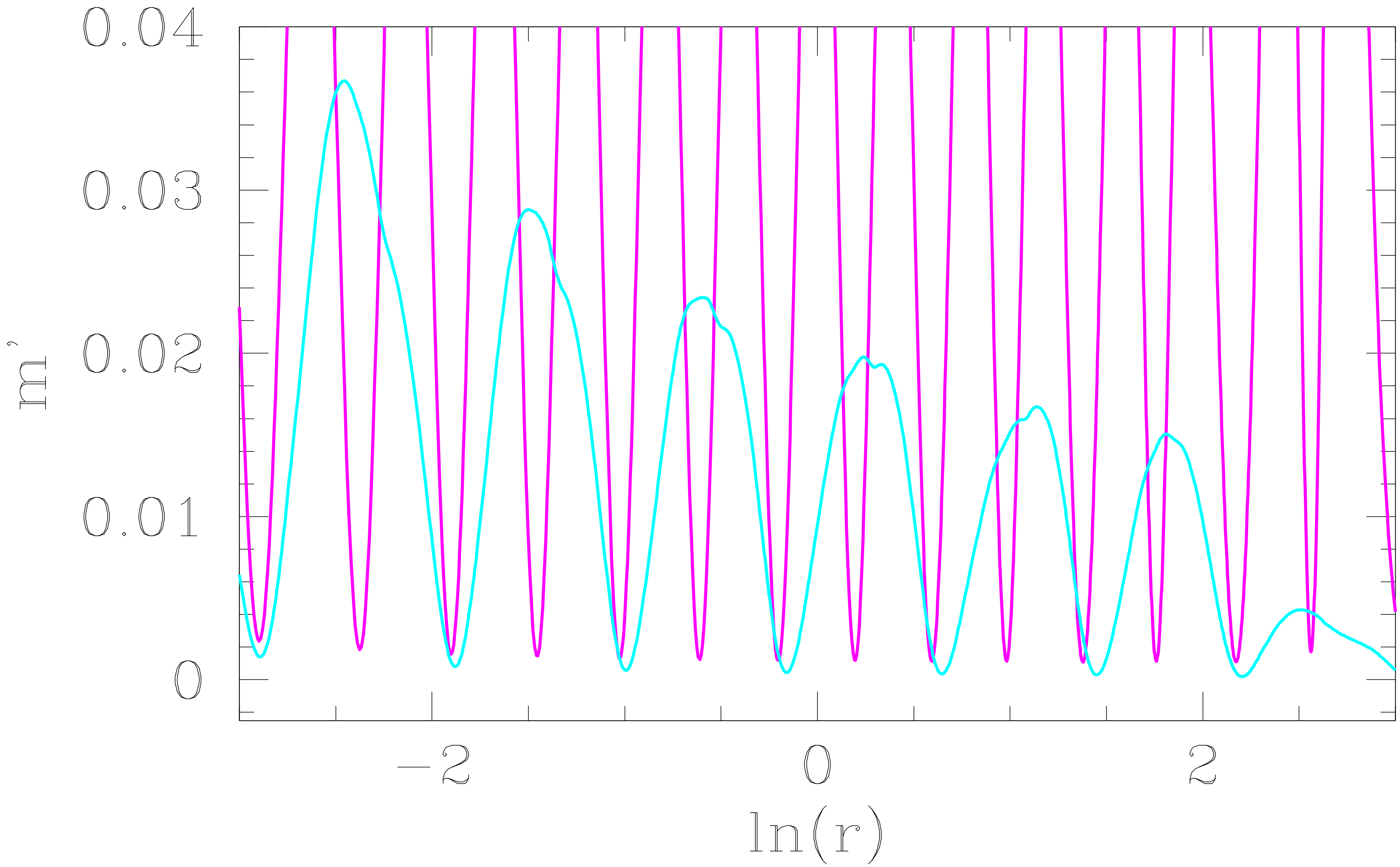
Relative Stability of Scalar & YM Type II Solns.

$[dm/dr]_{\mathrm{YM}}$ **and** $[dm/dr]_{\mathrm{S}}$



Relative Stability of Scalar & YM Type II Solns.

$[dm/dr]_{\mathrm{YM}}$ **and** $[dm/dr]_{\mathrm{S}}$ **(detail)**



Relative Stability of Scalar & YM Type II Solns.

Remarks

- No sign of instability of scalar field in presence of Type I YM solution
- No sign of instability of YM field in presence of Type II scalar solution, as above results indicate
- Interesting dynamical evolution of DSS frequency; scalar field apparently starts oscillating (in scale) with YM frequency, but eventually evolves to its own frequency

Dynamical Fate of $n = 1$ Bartnik-McKinnon Solution

- Title of talk somewhat misleading since actually looking at *highly tuned* dynamical fate

- Adopt most general spherically symmetric ansatz for YM fields; $n = 1$ Bartnik-McKinnon solution, $w_1^\star$ known to have *two* unstable modes in this context

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- Dynamical fields: w and $\tilde{w}$

- Initial data

$$w(0, r;\, p) \equiv \frac{1 + \frac{c^2 - r^2}{p^2}}{\sqrt{\left(1 + \frac{c^2 - r^2}{p^2}\right)^2 + 4r^2}}$$

$$\tilde{w}(0, r;\, q) \equiv q\, e^{-(r-C)^2/S^2}$$

with constants c, C and S chosen to ensure dynamical overlap of two fields and $\dot{w}(0, r)$, $\dot{\tilde{w}}(0, r)$ specified to produce ingoing initial data

Dynamical Fate of $n = 1$ Bartnik-McKinnon Solution

- Tune $p^\star$ so that $w(r, 0; p^\star) \Longrightarrow w_1^\star$
- Fix q, $\tilde{w}(r, 0; q)$, retune p, $w(r, 0, p)$ to determine $p_q^\star$ that generates critical solution
- Investigate phenomenology as function of q
- Useful diagnostic quantity, $z(t, r)$

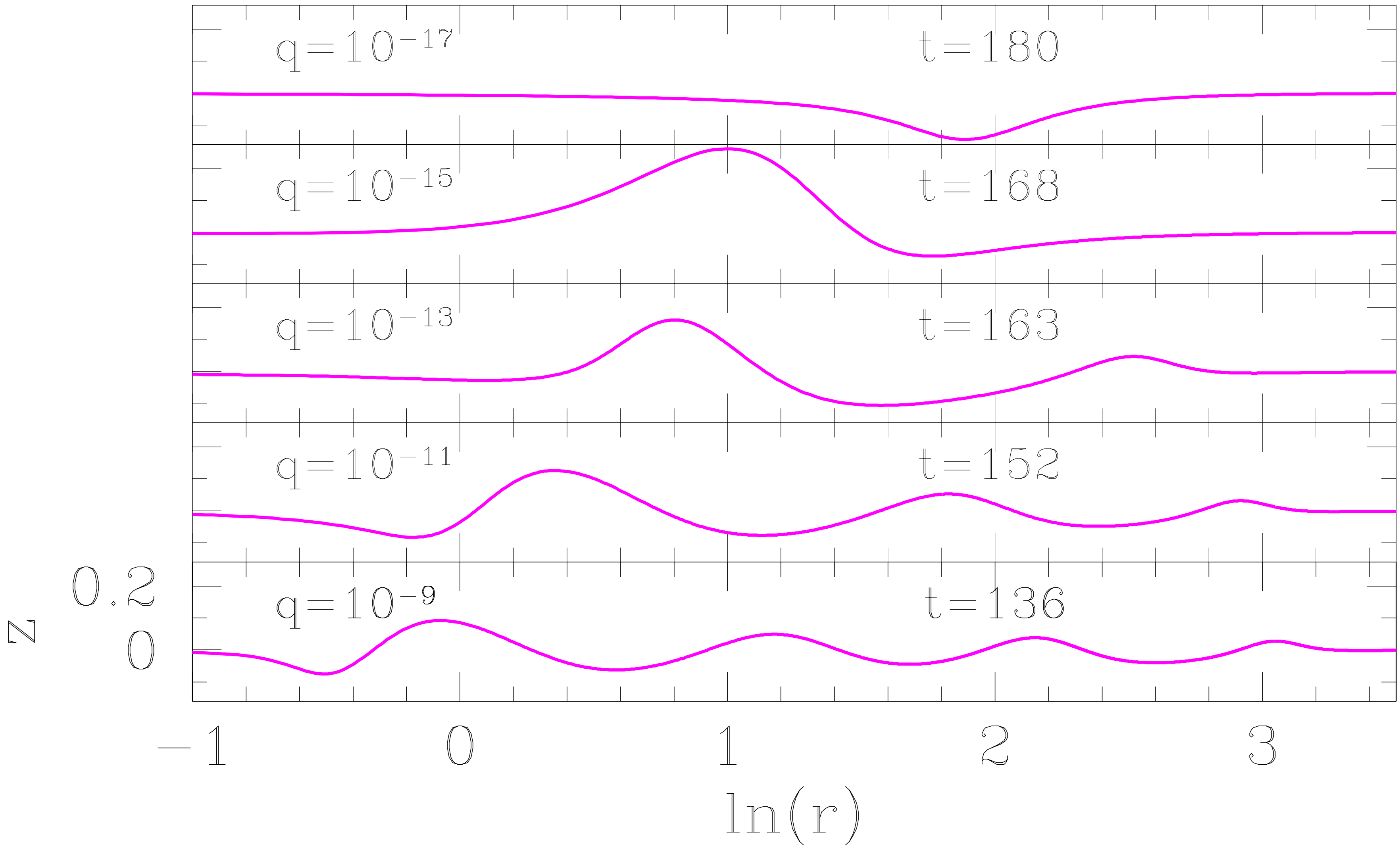
$$z = \frac{w^2 + \tilde{w}^2 - 1}{r}$$

Instability of $n = 1$ Bartnik McKinnon Solution

- ANIMATION of critical solution for $q = 0$
  - w and $\tilde{w} \equiv 0$
- ANIMATION of critical solution for $q = 1 \times 10^{-15}$
  - w and $\tilde{w}$
- ANIMATION of critical solution for $q = 1 \times 10^{-13}$
  - w and $\tilde{w}$
- ANIMATION of critical solution for $q = 1 \times 10^{-11}$
  - w and $\tilde{w}$
- ANIMATION of critical solution for $q = 1 \times 10^{-9}$
  - w and $\tilde{w}$

Instability of $n = 1$ Bartnik McKinnon Solution

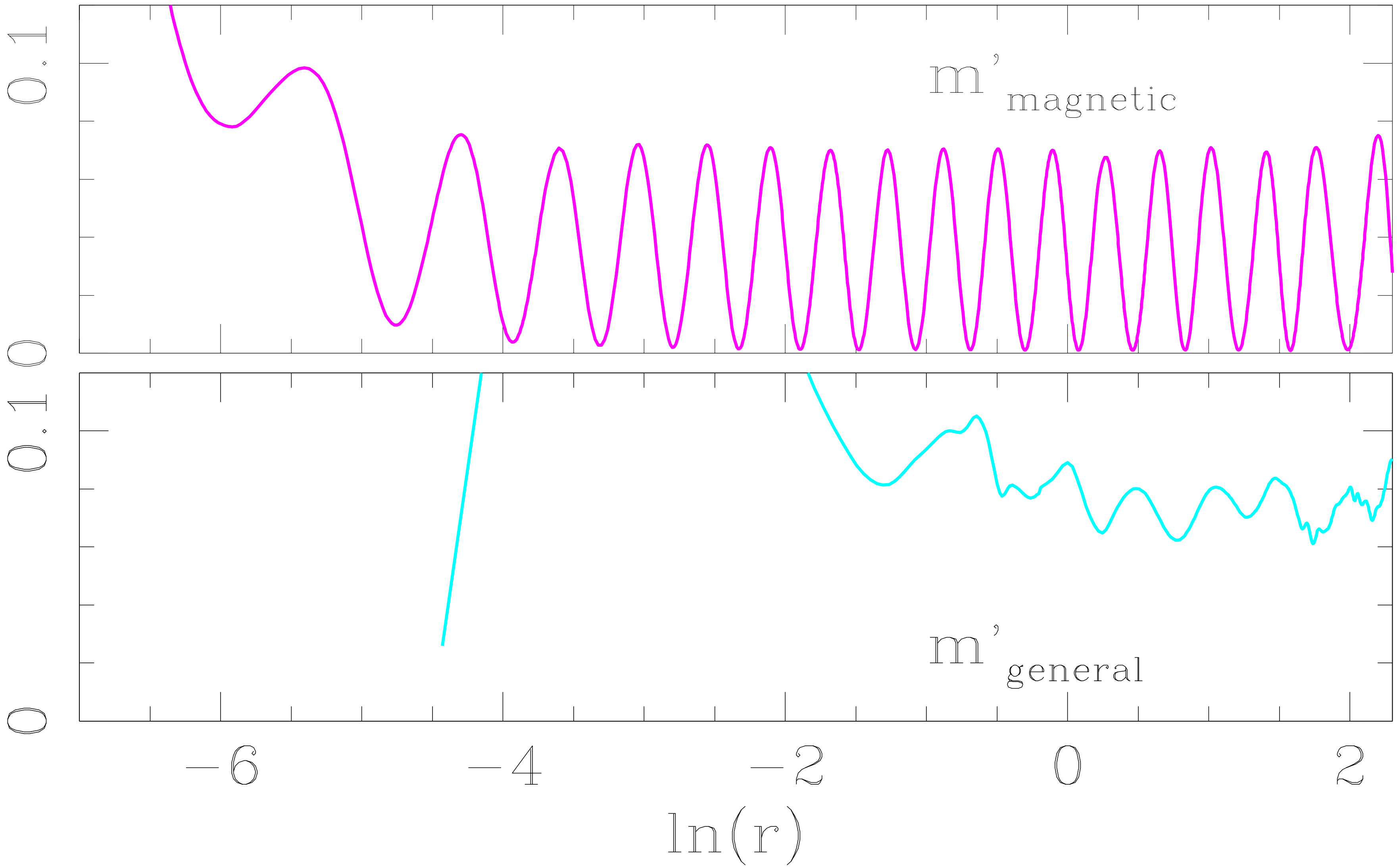
Critical $z \equiv (w^2 + \tilde{w}^2 - 1)/r$ as a function of q



Instability of $n = 1$ Bartnik McKinnon Solution

Comparison of magnetic and general critical solutions

Magnetic solution tuned to 10^{-15}, general to 10^{-12}



Instability of $n = 1$ Bartnik McKinnon Solution
Remarks

- Very interesting that *static* solution, $w_1^\star$, is in some sense *unstable* to collapse to self-similar solution

- Precise nature of self-similar solution(s) at threshold not completely determined yet, but apparently distinct from Type II solution seen in purely magnetic collapse

Concluding Remarks

- Calculations suggestive of an entire *hierarchy* of critical solutions when one considers the case of coupling to all conceivable forms of matter

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  - Quite possible that in completely generic case (i.e. w.r.t matter content), there is only *one* threshold solution
- Relaxation of symmetry assumptions quite likely to lead to additional phenomenology
- One can only contemplate the astrophysical consequences!