INITIAL DATA FOR BLACK HOLE COLLISIONS

- Motivation

- The Arnowitt-Deser-Misner (ADM, 3+1) formalism (brief review)

- The Initial Value Problem for General Relativity
  - The York Conformal Approach
  - Constructing *Black Hole* Initial Data—the "Imaging" Technique
  - Solutions of the Momentum Constraint

- Three Codes for solving the Hamiltonian Constraint.
- Comparison of the Codes (Results).
  - Setting up the comparison.
  - Typical results.

- Conclusions.

(G. Cook, M. Dubal, S. Klasky, R. Matzner, S. Olivera, M. Choptuik)

MOTIVATION

- **Why study black hole collisions?**

  - Astrophysical interest: Black hole collisions (presumably from binaries) should be relatively efficient/strong generators of gravitational radiation. Anticipate terrestrial detection by planned GWO's such as LIGO.

  - Physical interest: Black holes are the "point masses" of general relativity. Study of interactions seems no less important/relevant than the Kepler problem in Newtonian gravity. High probability of unearthing new information concerning gravitational interaction.

  - Computational interest: Difficult/challenging problem for numerical relativists—solution will take us a long way toward the long-anticipated goal of computing "the general solution of Einstein's equations".

- **How do we study black hole collisions?**

  - Nonlinearity, complexity of Einstein's equations necessitate *numerical* treatment, particularly in strong-field, highly time dependent regimes.

  - Field equations are set of non-linear, coupled, hyperbolic & elliptic PDE's—space-plus-time approach involves specification of *initial data* followed by integration of equations of motion to generate *future* (or past) data.

  - **Determining appropriate initial data for is a non-trivial task.** Involves solution of elliptic PDE's since general relativity is a *constrained* theory.

The Arnowitt-Deser-Misner (ADM) or "3+1" Formalism for GR

(`Arnowitt, Deser and Misner`, in *The Dynamics of General Relativity*, `L. Witten ed.`)

- Pose general relativity as a *dynamical* theory (geometrodynamics).

  View geometry of *spacetime* as "time history" of the geometry of a *spacelike hypersurface* ("instant of time"), $\Sigma(t)$

- Geometry of spacetime is described by a *4-metric*, ${}^{(4)}g_{\mu\nu}$
  Geometry of Σ is described by a *3-metric*, ${}^{(3)}g_{ij}$

- Spacetime displacement in 3+1 language ($x^\mu \equiv \left(t, x^1, x^2, x^3\right) \equiv \left(t, x^i\right)$):

$$
\begin{aligned}
{}^{(4)}ds^2 &= {}^{(4)}g_{\mu\nu} dx^\mu \, dx^\nu \\
&= -\alpha^2 \, dt^2 + {}^{(3)}g_{ij} \left(dx^i + \beta^i dt\right) \left(dx^j + \beta^j dt\right)
\end{aligned}
$$

- Two types of variables:
  1) Kinematical: *lapse function*, α and *shift vector*, β^i (4 functions)
  2) Dynamical: three metric, ${}^{(3)}g_{ij}$ (6 functions)

- Kinematical variables are essentially *freely specifiable* and embody the 4-fold coordinate freedom (general covariance) of general relativity

- The ADM formalism is a *Hamiltonian* formulation of the dynamics of the gravitational field—i.e. the formulation casts Einstein's equations into a set of first-order-in-time equations.

- In addition to the metric components ${}^{(3)}g_{ij}$ ("generalized coordinates", q^i in "ordinary" Hamiltonian mechanics) we have the dynamically conjugate variables ("generalized momenta", p_i), K_{ij}

- The components of this *extrinsic curvature tensor*, K_{ij}, are defined by

$$K_{ij} = K_{ji} = -\nabla_j \, n_i$$

  where $n_\mu = {}^{(4)}g_{\mu\nu} n^\nu$, and n^μ is the unit timelike (future directed) normal vector field to the spacelike hypersurfaces. ∇ is the 4-covariant derivative operator (defined by parallel transport in the 4-geometry).

- The extrinsic curvature components can be expressed as

$$K_{ij} = \frac{1}{2\alpha} \left(-\frac{\partial g_{ij}}{\partial t} + D_i \, \beta_j + D_j \, \beta_i \right)$$

  where D is the 3-covariant derivative operator (defined by parallel transport in the 3-geometry, and also by *projection* of ∇.)

- The extrinsic curvature of a given $\Sigma\,(t)$ describes the manner in which the hypersurface is embedded in the enveloping 4-geometry/spacetime.

- The set $\{g_{ij}, K_{ij}\}$ constitute a complete set of variables for the vacuum IVP for GR.

The Initial Value Problem for General Relativity

(`York and Piran`, in *Spacetime & Geometry*, `Matzner & Shepley eds.`)

- Of the 10 vacuum Einstein equations:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0$$

  the 4 equations

$$G^{0\nu} = 0$$

  do *not* involve second time derivatives of the metric functions ${}^{(4)}g_{\mu\nu}$ but rather can be written entirely in terms of the basic dynamical variables $\{g_{ij}, K_{ij}\}$ and their spatial derivatives.

- These equations therefore represent *constraints* on the $\{g_{ij}, K_{ij}\}$ which must be satisfied on all hypersurfaces, $\Sigma(t)$ and on the *initial hypersurface* in particular.

- In the ADM space-plus-time approach, the 4 constraints naturally decompose into a scalar equation—the *Hamiltonian constraint*:

$${}^{(3)}R - K^i{}_j \, K^j{}_i + K^2 = 0$$

  and a 3-vector equation—the *momentum constraint(s)*:

$$D_j \, K_i{}^j - D_i \, K = 0$$

  where K is the trace of the extrinsic curvature tensor, $K \equiv {}^{(3)}g^{ij} K_{ij}$, and ${}^{(3)}R$ is the Ricci curvature scalar of the spacelike hypersurface.

- The *initial value problem* for general relativity, then, involves a specification of the topology of the initial spacelike hypersurface, $\Sigma(0)$, together with a specification of the dynamical variables $\{g_{ij}, K_{ij}\}$ such that the constraint equations are satisfied.

- As a consequence of the constraints, we are allowed to choose freely only 8 of the 12 $\{g_{ij}, K_{ij}\}$, the other 4 variables will then be fixed by the constraints. **Key question:** *which* variables should we specify *a priori* and which should be determined from the constraints?

The Lichnerowicz/Ō Murchadha/York Conformal Approach

(Lichnerowicz, *J. Math. Pures Appl.*, **23**, 37 (1944)
Ō Murchadha & York, *Phys. Rev. D*, **10**, 428 (1974))

- Key feature of this approach is that it casts the constraint equations into a set of 4 quasi-linear, elliptic partial differential equations for 4 gravitational "potentials"—this facilitates both theoretical analysis and actual (numerical) solution.

- **Two Key Ideas/Techniques**

    - 1) Specify any physical variable $\{g_{ij}, K_{ij}\}$ only up to a (position-dependent) *conformal scaling*:

    $$f_{\rm physical} = \psi^p \, \hat{f}_{\rm base}$$

    where ψ—the *conformal factor* and one of our "potentials"—will be determined from the Hamiltonian constraint. The scaling exponent, p may vary depending on the physical variable (3-tensor) being scaled. In particular, for the 3-metric g_{ij}, write

    $$g_{ij} = \psi^4 \, \hat{g}_{ij}$$

    then, among other things, scalar curvature ${}^{(3)}R$ transforms as

    $$R = \psi^{-4} \, \hat{R} - 8\psi^{-5} \hat{\Delta}\psi$$

    - 2) Perform a transverse-traceless/longitudinal/trace decomposition of the extrinsic curvature tensor—this introduces three additional "potentials", X^i, which will be fixed by the momentum constraints. As a first step in this decomposition, introduce the *trace-free* part, A^{ij}, of the extrinsic curvature:

    $$A^{ij} \equiv K^{ij} - \frac{1}{3} g^{ij} \, K$$

    where K is now to be regarded as a freely specifiable quantity (trade-off for ψ).

Details of the transverse-traceless/longitudinal decomposition

- Rather than decomposing A^{ij} directly, we again introduce a base/bare tensor $\hat{A}^{ij}$ related to the physical tensor via a conformal transformation:

$$A^{ij} = \psi^{-10}\,\hat{A}^{ij}$$

where the power "-10" was chosen principally because it results in the following property:

$$D_j A^{ij} = \psi^{-10}\,\hat{D}_j \hat{A}^{ij}$$

- As with *any* traceless, symmetric tensor, $\hat{A}^{ij}$ can be decomposed into a piece, $\hat{A}^{ij}_{\mathrm{TT}}$ with vanishing divergence and trace, and another trace-free part which can be obtained from differentiating a vector potential W^i:

$$\begin{aligned}\hat{A}^{ij} &= \hat{A}^{ij}_{\mathrm{TT}} + (\hat{\ell}W)^{ij} \\ &\equiv \hat{A}^{ij}_{\mathrm{TT}} + \hat{D}^i W^j + \hat{D}^j W^i - \frac{2}{3}\,\hat{g}^{ij}\,\hat{D}_k W^k\end{aligned}$$

where the "transverse-traceless" (TT) part, $\hat{A}^{ij}_{\mathrm{TT}}$ satisfies

$$\hat{D}_i \hat{A}^{ij}_{\mathrm{TT}} = 0$$

- In practice, will generally be inconvenient to give the freely specifiable part of the (conformally scaled) extrinsic curvature in terms of a transverse-traceless tensor. Thus, we "reverse decompose" $\hat{A}^{ij}_{\mathrm{TT}}$ as

$$\hat{A}^{ij}_{\mathrm{TT}} = \hat{T}^{ij} - (\hat{\ell}V)^{ij}$$

where the trace-less, symmetric tensor $\hat{T}^{ij}$ is freely specifiable and V^i is another vector field. Then

$$\hat{A}^{ij} = \hat{T}^{ij} + (\hat{\ell}W)^{ij} - (\hat{\ell}V)^{ij} \equiv \hat{T}^{ij} + (\hat{\ell}X)^{ij}$$

- The Hamiltonian and momentum constraint equations then become

$$\hat{\Delta}\psi = \frac{1}{8}\hat{R}\,\psi + \frac{1}{12}K^2\psi^5 - \frac{1}{8}\left(\hat{T}^{ij} + (\hat{\ell}X)^{ij}\right)^2\,\psi^{-7}$$

$$(\hat{\Delta}_\ell\,X)^i \equiv \hat{D}_j\,(\hat{\ell}X)^{ij} = -\hat{D}_j\hat{T}^{ij} + \frac{2}{3}\,\psi^6\,\hat{D}^i K$$

- As promised, these equations are a set of 4 quasi-linear, coupled *elliptic* PDE's for the 4 "gravitational potentials" $\{\psi, X^i\}$

- **Summary of Initial Data Procedure**

◦ 1) Freely specify:

$$\{\hat{g}^{ij}, K, \hat{T}^{ij}\}$$

◦ 2) Solve constraints for the potentials:

$$\{\psi, X^i\}$$

◦ 3) Construct *physical* initial data using

$$g_{ij} = \psi^4\,\hat{g}_{ij}$$

$$\begin{aligned} K^{ij} &= \psi^{-10}\,\left(\hat{T}^{ij} + (\hat{\ell}X)^{ij}\right) + \frac{1}{3}\psi^{-4}\,\hat{g}^{ij}K \\ &= \psi^{-10}\hat{A}^{ij} + \frac{1}{3}\psi^{-4}\,\hat{g}^{ij}K \end{aligned}$$

- **Canonical Simplifications for Solving the Constraints:**

  - 1) *Flat* base geometry:
  $$\hat{g}_{ij} = f_{ij}$$
  $f_{ij} = \mathtt{diag}(1,1,1)$, for example, if we adopt the usual Cartesian coordinates (x,y,z). Then
  $$\hat{R} = 0$$
  and the elliptic operators
  $$\hat{\Delta} \quad \text{and} \quad \hat{\Delta}_\ell$$
  take on a relatively simple form.

  - 2) *Maximality* of initial slice:
  $$K = 0$$
  which eliminates additional terms from the constraint equations, and more importantly, *decouples* the momentum constraint from the Hamiltonian constraint.

  - 3) *"Minimal radiation"* condition:
  $$\hat{T}^{ij} = 0$$

- With the above choices, the constraints become
$$\begin{aligned}\hat{\Delta}\psi &= -\frac{1}{8}\hat{A}^{ij}\hat{A}_{ij}\psi^{-7} \\ &= -\frac{1}{8}(\hat{\ell}X)^{ij}(\hat{\ell}X)^{ij}\psi^{-7}\end{aligned}$$

$$(\hat{\Delta}_\ell X)^i = 0$$

- Because the momentum constraint is decoupled from the Hamiltonian constraint and is *linear*, it has been possible to find analytic (and semi-analytic) solutions of the momentum constraints corresponding to one or more black holes with freely specified linear and angular momenta.

Ensuring the Initial Data Describe Black Hole Geometries

- How can we tell whether a particular choice of $\{g_{ij}, K_{ij}\}$ which satisfies the constraints describes a black hole?

- Although a black hole, by definition, can only be identified from the *global* 4-geometry, a *sufficient* condition is that the initial spacelike hypersurface, $\Sigma(0)$, contain one or more *trapped surfaces*

- A trapped surface is a 2-surface (topologically S^2) on which the divergence of the *outgoing null rays* vanishes. This can be show to be equivalent to the following condition on the outward-pointing, *spacelike* unit normal to the surface, s^i:

$$D_k\, s^k + s^i s^j K_{ij} - K = 0$$

- Over the past decade or so, York and coworkers have developed a strategy based on *imaging* techniques originally introduced in GR by `Misner` (*Phys. Rev.*, **118**, 1110). Can gain some insight into the basic idea behind the strategy by considering the special case of the *Schwarzschild geometry* which describes a single, neutral, non-spinning black hole with mass M.

- The Schwarzschild metric can be written in *isotropic* form:

$${}^{(4)}ds^2 = -\frac{\left(1 - \frac{M}{2r}\right)^2}{\left(1 + \frac{M}{2r}\right)^2} dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2\, d\Omega^2\right)$$

  where $dr^2 + r^2\, d\Omega^2$ is the flat metric in spherical coordinates (r, θ, ϕ) amd there is *no* time dependence, so that all of the spacelike hypersurface, "t = constant" have identical 3-geometries.

- In this coordinate system, the Schwarzschild solution *trivially* satisfies the momentum constraints since

$$K_{ij} = A_{ij} = \hat{A}_{ij} = K = \cdots = 0$$

- The Hamiltonian constraint reduces to

$$\hat{\Delta}\psi = 0$$

and since the 3-metric is conformally flat

$$g_{ij} = \left(1 + \frac{M}{2r}\right)^4 f_{ij} \equiv \psi^4 f_{ij} = \psi^4\, \hat{g}_{ij}$$

it is easy to see that

$$\psi = 1 + \frac{M}{2r}$$

is a solution of $\hat{\Delta}\psi = 0$. Note that

$$\lim_{r\to\infty} \psi = 1$$

which, physically, expresses the fact that the 3-geometry is asymptotically flat (Euclidean); mathematically, the above relation is a *boundary condition*—some such condition *must* be imposed in order that the *elliptic* Hamiltonian constraint has a unique solution.

- Now, consider an $r =$ constant 2-sphere on a slice of our Schwarzschild geometry. From $g_{ij}\, s^i s^j = 1$, the unit normal, s^i, has components

$$s^i = (\psi^{-2}, 0, 0)$$

and since $K_{ij} = K = 0$, the trapped surface condition reads:

$$D_k\, s^k = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g} s^k) = 0$$

or, in the (flat) base geometry ($\sqrt{g} = \psi^6 \sqrt{\hat{g}}$):

$$\frac{\partial}{\partial x^k} \left(\psi^6 \sqrt{\hat{g}} s^k\right) = 0$$

which in spherical coordinates ($\sqrt{\hat{g}} \sim r^2$) becomes

$$\frac{\partial}{\partial r} \left(r^2 \psi^4\right) = 0$$

or

$$\frac{\partial\psi}{\partial r} + \frac{\psi}{2r} = 0.$$

- Easy to show that $\psi = 1 + M/(2r)$ satisfies this equation at $r = 2M$. Thus, the surface $r = 2M$ is (marginally) trapped and we can conclude that the hypersurface is embedded in a spacetime containing a black hole.

The "Imaging" Technique

- The $r = 2M \equiv a$ surface in the Schwarzschild example has another interesting property.

- Consider the "Kelvin" transformation ("inversion through a sphere"):

$$\bar{r} = \frac{a^2}{r}$$

or in Cartesian coordinates:

$$\bar{x} = \frac{a^2}{r^2}x \qquad \bar{y} = \frac{a^2}{r^2}y \qquad \bar{z} = \frac{a^2}{r^2}z.$$

This transformation maps points outside the 2-sphere, $r = a$, to points inside, and vice-versa: the surface itself is a fixed-set of the transformation.

- It can be shown that this transformation is an *isometry* (see `Weinberg`, *Gravitation and Cosmology*, §13.1) of the 3-geometry of the "t = constant" slice of the Schwarzschild spacetime—the physical 3-geometry is form-invariant under the transformation, i.e.:

$$g_{ij}(x) = J_l^k\, J_j^l\, g_{kl}(\bar{x})$$

where $J_i^k \equiv \partial\bar{x}^k/\partial x^i$ is the *Jacobian* of the transformation

- The inversion symmetry tells us that a "t = constant" slice of Schwarzschild in these particular coordinates can be interpreted as two identical, asymptotically flat universes, joined by a *"throat"* at $r = 2M$.

- York and his co-workers have advocated a general approach to generating black hole initial data which is based on the idea of *demanding* inversion isometry.

- For an n-hole calculation this involves the specification of n inversion centres, c^i_α and inversion ("hole") radii, a_α, together with the requirement that the geometry of the hypersurface be isometric w.r.t inversion through *any* and *all* of the 2-spheres $r_\alpha \equiv |x^i - c^i_\alpha| = a_\alpha$.

- This procedure has the dual effect of
  - 1) generating initial geometries which *do* contain trapped surfaces
  - 2) providing *inner boundary conditions* at the $r_\alpha = a_\alpha$ surfaces for ψ (and possibly the vector potential), thus eliminating the need to deal with singular behaviour as $r \to 0$ when solving the constraint equations.

- For example, the isometry condition results in the following *general* inner boundary conditions for ψ:

$$\left. \frac{\partial \psi}{\partial r_\alpha} + \frac{\psi}{2 r_\alpha} \right|_{r_\alpha = a_\alpha} = 0$$

- Should be emphasized that the imaging approach is only *one* technique for facilitating the computation of black hole initial data—alternate approaches exist, such as that advocated by Unruh and implemented by Thornburg (*Class. Quant. Grav.*, **4**, 1119) where one demands that the $r_\alpha = a_\alpha$ surfaces be *trapped*, resulting in a different inner boundary condition for ψ.

- The imaging approach gets a little tricky for 2 or more black holes (have to deal with infinite series of images), but details and implementation have been worked out by York and co-workers (most recently and most extensively by Cook).

Solutions of the Momentum Constraints

- With canonical assumptions, momentum constraints are *linear* and *decoupled* from the Hamiltonian constraint.

- `Bowen and York` (*Phys Rev D*, **21**, 2047) constructed the following "fundamental" solutions, where P^i and S^i are freely specifiable 3-vectors:

$$\hat{A}^{\pm}_{ij}[P^i] = \frac{3}{2r^2}\left(P_i n_j + P_j n_i - (f_{ij} - n_i n_j)\, P^k n_k\right)$$
$$+ \text{ terms to enusre correct behaviour under isometry condition.}$$
$$\hat{A}_{ij}[S^i] = \frac{3}{r^3}\left(\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i\right)$$

  and showed that they correspond to single black holes having linear momentum P^i (as measured at spatial infinity) or angular momentum (spin), S^i respectively.

- Due to linearity of the momentum constraints, different fundamental solutions:

$$\hat{A}^{ij}_{\alpha} \equiv \hat{A}^{ij}_{\alpha}[c_\alpha, a_\alpha, P^i_\alpha, S^i_\alpha]$$

  can be superimposed as a first step towards a solution of the momentum constraints for two or more black holes.

- Straightforward sum of $\hat{A}^{ij}_{\alpha}$ will not satisfy inversion-symmetry conditions. `Kulkarni` *et al* (*Phys. Lett.* **96A**, 228) showed how to take a non-inversion-symmetric $\hat{A}^{ij}_{\alpha}$ and convert it to one which *does* respect the symmetry.

- `Cook` (*Phys. Rev. D* **44**, 2983), and `Matzner` have implemented numerical algorithms which (approximately) sum the infinite series implied by the formal solution given by Kulkarni *et al*

- **Summary:** We can construct semi-analytic solutions of the momentum constraints for the case of multiple holes, where each hole can (more or less) have freely specified linear and angular momenta.

- Then, to complete the black hole IVP, we need only solve the quasi-linear Hamiltonian constraint for the conformal factor ψ

$$\hat{\Delta}\psi = -\frac{1}{8}\hat{A}^{ij}\hat{A}_{ij}\psi^{-7}$$

  where the quantity $\hat{A}^{ij}\hat{A}_{ij}$ is a "known function" evaluated from a particular solution of the momentum constraints.

Outer ("large r") Boundary Condition for ψ

- Recall that for Schwarzschild geometry in isotropic coordinates we have

$$\psi = 1 + \frac{M}{2r}$$

- In general, for conformally flat solutions of the Hamiltonian constraint, *asymptotic flatness* requires

$$\lim_{r\to\infty} \psi = 1 + \frac{M}{2r} + O(r^{-2})$$

- In solving the Hamiltonian constraint, we might ideally impose

$$\psi = 1 \qquad \text{at} \qquad r = \infty$$

  but in practice, computational domain may only extend to some finite maximum radius, $r_{\max}$. Following `York and Piran`, we observe that

$$\frac{\partial \psi}{\partial r} = -\frac{M}{2r^2} + O(r^{-3}) = -\frac{\psi - 1}{r} + O(r^{-3})$$

  Thus we impose

$$\left.\frac{\partial \psi}{\partial r} + \frac{\psi - 1}{r}\right|_{r=r_{\max}} = 0$$

- Important to note that this is only an *approximate* boundary condition. If we consider two calculations where the condition is imposed at different values of $r_{\max}$, then the generated solutions of the Hamiltonian constraint must necessarily differ.

"The Battle of the Codes"

- There exist (at least) three codes for solving the Hamiltonian constraint for two-black hole initial data: **CZ**, **CT**, and **MQ**.

$$\{\mathbf{CZ}, \mathbf{CT}, \mathbf{MQ}\} : [\, c_\alpha, a_\alpha, P^i_\alpha, S^i_\alpha \,;\, \{\mathrm{aux}_\beta\} \,] \mapsto \hat{\psi}$$

"ˆ" now denotes "approximate" as in "approximate numerical solution"

Issues in Numerical Solution of (3d) Elliptic PDE's

- (Analytic) coordinate system.
- Fundamental numerical variable(s).
- (Numerical) coordinate system/Computational domain.
- Discretization technique.
- Algorithm for solution of N, discrete, non-linear equations.
- Accuracy (mathematical, physical, computational).
- Cost: (machine, human).

CZ: The Čadež code. (`G. Cook, Cornell`)

- (Analytic) coordinate system.

  Čadež coordinates: (η, ξ, ϕ)

- Fundamental numerical variable.

  $$\psi$$

- (Numerical) coordinate system/Computational domain.

  Uniform or "effectively uniform" grid in η, ξ and ϕ. ($\xi = \xi(s)$; grid uniform in s). Single discretization scale, h.

- Discretization technique.

  Linearization, followed by $O(h^2)$ *centred, conservative* finite differencing.

- Algorithm for solution of discrete, non-linear equations.

  FAS (Full Approximation Storage), Block-Multigrid.

- Accuracy.

  $O(h^2)$.

- Cost.

  $O(N)$ at best.

CT: The Cartesian Code. (`Klasky, Matzner & Choptuik, UT`)

- (Analytic) coordinate system.

$$\text{Cartesian coordinates:} \quad (x, y, z)$$

- Fundamental numerical variable.

$$\psi$$

- (Numerical) coordinate system/Computational domain.

  Subset of cubical, uniform mesh $\{(ih, jh, kh)\}$; $n+1$ grid points on edge of computational cube ($N \sim n^3$).

- Discretization technique.

  Linearization, followed by *centred*, *forward* and *backward*, "$O(h^2)$" finite differencing.

- Algorithm for solution of discrete, non-linear equations.

  Variant of Line-SOR (Successive OverRelaxation)

- Accuracy.

  $O(h)$.

- Cost.

  $O(N^{4/3})$ at best.

MQ: The Multiquadric Code. (`Dubal, Matzner & Oliveira, UT`)

- (Analytic) coordinate system.

$$\text{Cartesian coordinates:} \quad (x, y, z)$$

- Fundamental numerical variable.

$$\Psi \equiv \left[\left(\frac{a_1}{r_1}\right)^3 + \left(\frac{a_2}{r_2}\right)^3 + \left(\frac{a_1 a_2}{r_1 r_2}\right)^{3/2} \right]^{-1} (\psi - 1)$$

- (Numerical) coordinate system/Computational domain.

  Set of N *grid points* (*basis points*), $\{x_q^i\} = \{(x_q, y_q, z_q)\}$

- Discretization technique (after linearization).

$$\hat{\Phi}[\{x_q^i\}, s_{\min}, s_{\max}] = c_1 + \sum_{q=2}^{N} c_q \, \tilde{g}_q \qquad \tilde{g}_q \equiv g_q - g_1$$

$$g_q = [(x-x_q)^2+(y-y_q)^2+(z-z_q)^2+s_q^2]^{1/2} \qquad s_q^2 = s_{\min}^2 \left(\frac{s_{\max}^2}{s_{\min}^2}\right)^{(q-1)/(N-1)}$$

- Algorithm for solution of discrete, non-linear equations.

  Singular-value decomposition (SVD) of *full, ill-conditioned* linear systems.

- Accuracy.

  ??

- Cost.

  $O(N^3)$.

Comparing the Results of the Codes (`Cook` *et al*)

- Performed series of "numerical experiments" designed to assess relative accuracy of code.

- **Basic Technique:** *Convergence Testing*

- For fixed problem-specific parameters, ($c_\alpha, a_\alpha, P^i_\alpha, S^i_\alpha, \cdots$), run each code at varying *resolutions* which for all 3 codes can be identified with N, although for FD codes, $h \sim N^{1/3}$ is more natural.

- In principle, at least for FD codes, estimates of

$$\text{"solution error"} \equiv \hat{\psi}[N] - \psi$$

$$\text{"convergence rate"} \equiv \frac{\partial(\hat{\psi}[N] - \psi)}{\partial N}$$

can be made from (numerically determined) approximations of

$$\frac{\partial \hat{\psi}[N]}{\partial N} \quad , \quad \frac{\partial^2 \hat{\psi}[N]}{\partial N^2} \quad , \quad \ldots$$

plus assumptions about convergence behaviour.

- Convergence checking is an inherently *intrinsic* technique—do not need exact/reference solution against which to check results.

Generating Reference Solutions Using Richardson Extrapolation

- Čadež results have very regular $O(h^2)$ convergence:

$$\hat{\psi}^{\mathbf{CZ}}[h] = \psi - h^2\, e_2^{\mathbf{CZ}} + O(h^4)$$

so that *Richardson extrapolation* techniques such as

$$\begin{aligned}\frac{1}{3}\left(4\hat{\psi}^{\mathbf{CZ}}[h] - \hat{\psi}^{\mathbf{CZ}}[2h]\right) &= \frac{1}{3}\left(4\psi - 4h^2 e_2^{\mathbf{CZ}} - \psi + 4h^2 e_2^{\mathbf{CZ}}\right) + O(h^4)\\ &= \psi + O(h^4)\end{aligned}$$

dramatically improves the accuracy of the numerical results and provides us with effectively exact solutions which we use to measure the level of error and convergence rate for the various codes.

Some Details of the Comparison

- Computed numerical solutions for 3 models.
- Parameters for a particular model, A1B8:

$$a_1 = 1 \qquad c_1 = (0,0,4) \qquad P_1^i = (14,0,0) \qquad S_1^i = (-280,280,0)$$

$$a_2 = 1 \qquad c_2 = (0,0,-4) \qquad P_2^i = (-14,0,0) \qquad S_2^i = (0,280,280)$$

  Produces a "genuinely 3d" solution representing 2 highly relativistic, near maximally-spun-up black holes. Reasonably confident that subsequent evolution will describe collision/merger of the two holes.

- Ran each code at *three* resolutions ("low", "medium" and "high" resolution).
- Compared solutions at two sets of *reference points* to Čadež solutions which were Richardson extrapolated to those points
  - Points used in low resolution Čadež calculation.
  - Set of points along z-axis and interior to Cartesian computational domain.
- Performed extra set of runs where Čadež $r_{\max}$ was "pulled in" to investigate level of error in Cartesian calculations due to relatively small $r_{\max}$.

Summary of Results of Comparison

- Overall level of agreement—$\sim 1\%$ for high resolution runs—quite good.

- Evidence indicates that all codes *do* converge to the same solution for fixed problem parameters (including $r_{\rm max}$)

- **The Čadež Code**: Accuracy for given amount of computational work and convergence rate clearly superior to other two methods.

◦ Combination of boundary conforming/exponentially stretched coordinate system, uniform $O(h^2)$ differencing, and multigrid solution yield near-optimal solution technique.

- **The Cartesian Code**: Yields reasonably accurate solutions with somewhat indeterminate convergence rate which is to be expected given the nature of the discretization scheme.

◦ Error dominated by treatment of boundary conditions, both inner and outer. Adaptive methods (local refinement) should help, and application of the multigrid technique should lower computational cost. Both of these issues currently and under investigation.

- **The Multiquadrics Code**: Solution error is comparable to that in the Cartesian code. Error analysis hampered by lack of theory. For idealized test problems *exponential* convergence typically associated with spectral methods can be achieved. For current calculations, error/convergence rate also depends on adjustable parameters $s_{\rm min}$, $s_{\rm max}$, distribution of grid points ...

◦ Method has several attractive features:
1) grid-free, easy to deal with complex solution geometries
2) natural handling of boundary conditions
3) relatively low N
4) easy to implement
... so seems worthy of continued investigation.

Conclusions

- Task of determining initial data for black hole collisions can now be considered "routine."
    - However, continued development of techniques for solving elliptic equations in numerical relativity may be crucial for efficiently evolving the data.
- **"All that remains is the evolution problem."**