# Convergence of Lattice Trees to Super-Brownian Motion above the Critical Dimension

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# Abstract

A *lattice tree* is a finite connected set of lattice bonds containing no cycles. Lattice trees are interesting combinatorial objects and an important model for branched polymers in polymer chemistry and physics. In addition they provide an interesting example of critical phenomena in statistical physics with similar properties to models such as *self-avoiding walks* and *percolation.*

We use the *lace expansion* to prove convergence of the Fourier transform of the *$r$-point functions* (quantities which count critically weighted trees containing $r$ fixed points) for a spread-out model of lattice trees in $\mathbb{Z}^d$ for $d > 8$. Our results therefore provide additional evidence in support of the critical dimension $d_c = 8$. The spread out model allows bonds between vertices $x, y \in \mathbb{Z}^d$ with $\|x - y\|_\infty \leq L$, providing a small parameter $L^{-\frac{d}{2}}$ needed for convergence of the lace expansion. We extend the inductive approach (to the lace expansion on an interval) of van der Hofstad and Slade [19] to prove convergence of the Fourier transform of the 2-point function ($r = 2$). We then proceed by induction on $r$, equipped with the lace expansion on a tree [21]. Convergence of the $r$-point functions implies convergence of certain expectations of the spread out lattice trees model formulated as a measure valued process, to those of the canonical measure of super-Brownian motion. Appealing to the hypothesis of *universality*, we expect that the results also hold for the nearest neighbour model. Our results together with the convergence of the survival probability would imply convergence of the finite-dimensional distributions of our process to those of the canonical measure of super-Brownian motion. Convergence of the survival probability remains an open problem.

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# Chapter 1

# Introduction

This chapter serves as an introduction to the terminology, ideas and context of this thesis. In Section 1.1 we discuss some of the motivation for this thesis and give a brief outline of some of the relevant existing results. In Sections 1.2 and 1.3 we introduce the model that we study and state the main result. We conclude this chapter by defining some quantities that are the main focus of this thesis, and briefly discuss how they are connected to the main result.

## 1.1 Background and motivation

A *lattice tree* in $\mathbb{Z}^d$ is a finite connected set of lattice bonds containing no cycles (see Figure 1.1). Lattice trees are an important model for branched polymers. They are inherently combinatorial objects, so are of interest in combinatorics and graph theory. As we shall discuss shortly, our model for lattice trees is relevant to statistical physicists as a lattice model that exhibits a phase transition, with the behaviour at criticality being of particular interest. We can also describe our model as an example of a non-Markovian measure-valued process which converges

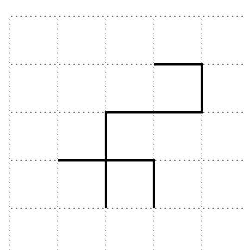

Figure 1.1: A nearest neighbour lattice tree in 2 dimensions.

(in dimensions $d > 8$) to a well known measure-valued Markov process in the scaling limit. Thus our results are also appealing to probabilists and researchers interested in stochastic processes.

### 1.1.1 Combinatorics and statistical physics

Lattice trees provide an interesting example of critical phenomena in statistical physics with similar properties to models such as *self-avoiding walks* (a model for linear polymers) and *percolation*. Let $l_n$ be the number of $n$-bond (nearest neighbour) lattice trees that contain the origin. An elementary question in combinatorics or graph theory would be "what is $l_n$?". Even in two dimensions the answer is not known for large values of $n$. However is known [24] that $\frac{l_{n+m}}{n+m} \geq \frac{l_n}{n}\frac{l_m}{m}$ in all dimensions, and a standard subadditivity argument then shows that $l_n^{\frac{1}{n}} \to \lambda > 0$ as $n \to \infty$. The bounds

$$c_1 n^{c_2 \log n} \lambda^n \leq l_n \leq c_3 n^{\frac{1}{d}} \lambda^n, \tag{1.1}$$

were proved in [23] and [26] respectively. Using the notation $f(x) \sim g(x)$ to mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, it is widely believed that

$$l_n \sim C\lambda^n n^{1-\theta}, \tag{1.2}$$

where $\theta$ is called a *critical exponent* for the model. Critical exponents convey information about the macroscopic or asymptotic properties of the model. The exponent $\theta$ is believed to depend on $d$, but not on the type of lattice or the type of bonds allowed (provided modest regularity conditions such as symmetry and finite range hold). An important example for our purposes is the *unrestricted 2-point function*, $\rho_p(x) = \sum_{T\in\mathcal{T}(x)} p^{\#T}$ where $\mathcal{T}(x)$ is the set of lattice trees containing the origin and $x$ and $\#T$ is the number of bonds in $T$. The function $\rho_p(x)$ is a power series with the coefficient of $p^N$ being the number of lattice trees containing 0 and $x$ consisting of $N$ bonds. This power series has nontrivial radius of convergence $p_c = \frac{1}{\lambda}$, at which it is believed that $\rho(x)$ changes from having exponential decay in $|x|$ for $p < p_c$ to power law decay

$$\rho_{p_c}(x) \approx \frac{C}{|x|^{d-2+\eta}}, \qquad \text{as } |x| \to \infty, \tag{1.3}$$

where $\approx$ represents some asymptotic behaviour that we do not state precisely at present. This kind of fundamental change in the properties of the model at $p = p_c$ is sometimes referred to as a phase transition.

The critical exponent $\eta$ in (1.3) is also thought to depend on $d$, but not on the type of lattice or bonds. This lack of dependence on the details of the model is called *universality*, and models with the same critical exponents are said to be in the same *universality class*. It should be pointed out that universality is a widely believed hypothesis in statistical physics rather than a rigorous mathematical theory. However there are many rigorous examples which give evidence in support of the hypothesis. Different critical exponents of a model are not independent of each other, and may obey a scaling or hyper-scaling relation (if the relation includes the dimension $d$) or inequality. A good source of information on critical exponents for lattice trees (self-avoiding branched polymers) is [9].

Lattice trees are self-avoiding objects by definition (since they contain no cycles). It is plausible that the self-avoidance constraint imposed by the model becomes less important as the dimension increases in the following sense. We might expect a randomly chosen branching lattice object in $d$ dimensions and containing $N$ bonds to be more likely to be self-avoiding as $d$ increases. In fact there is considerable evidence that for dimensions $d > 8$ the self-avoidance constraint is negligible in terms of the macroscopic view of the model. It is believed that for $d > 8$ the critical exponents cease to depend on the dimension and correspond to those of a simpler model, that does not have the constraint. The simpler model is called the *mean-field model*, and the dimension $d_c$ above which the constrained model has the same macroscopic properties as the mean-field model is called the *critical dimension*. Lubensky and Isaacson [25] proposed $d_c = 8$ as the critical dimension for lattice trees and animals.

There are few rigorous results for lattice trees for $d \leq 8$. The scaling limits of many models in statistical physics in 2 dimensions are believed to be described by a class of processes called *Stochastic Loewner Evolution (SLE)*, [30]. The SLE processes are candidates for the scaling limit of a model where the scaling limit is believed to have a property called *conformal invariance*. The scaling limit of lattice trees in 2 dimensions is not expected to have this property. Brydges and Imbrie [4] used a dimensional reduction approach to obtain strong results for a continuum (i.e. not lattice based) model for $d = 2, 3$. Appealing to universality, we would expect lattice trees to have the same critical exponents as the Brydges and Imbrie model. More is known in high dimensions, where the asymptotic behaviour should correspond to the mean-field model for lattice trees, *branching random walk*. Tasaki and Hara [29] showed in the context of lattice animals that the finiteness of the *square diagram* $\sum_{x,y,z} \rho_{p_c}(x)\rho_{p_c}(y-x)\rho_{p_c}(z-y)\rho_{p_c}(z)$ implies mean-field critical behaviour for the *susceptibility* $\chi(p) \equiv \sum_x \rho_p(x)$. The same methods and results apply to lattice trees. Hara and Slade [12], [13] proved the finiteness of the square diagram for sufficiently spread-out lattice trees (and animals) for $d > 8$, and for the nearest neighbour model for $d \gg 8$, as well as the mean-field critical behaviour of various quantities. Hara, van der Hofstad, and Slade [11] proved for a sufficiently spread out model that for $d > 8$, (1.3) holds with $\eta = 0$. This is the same exponent as for branching random walk. In [11] and this thesis the major tool of analysis is a technique known as the *lace expansion*, (introduced by Brydges and Spencer [5]). This technique is highly combinatorial in nature.

### 1.1.2 Probability and measure-valued processes

Most of the discussion in the following three paragraphs can be found in standard graduate level probability texts, for example [3].

Fix a probability space $(\Omega, \mathcal{F}, P)$ and suppose that $X_i$ are independent identically distributed real valued random variables with mean 0 and finite variance $\sigma^2$, and let $S_n = \sum_{i=1}^n X_i$. A fundamental result in probability theory, called the *central limit theorem* states that $\frac{S_n}{\sigma\sqrt{n}}$ converges weakly to a standard Gaussian random variable, $Z$. More precisely, defining probability measures $\mu_n(\bullet) \equiv P(\frac{S_n}{\sigma\sqrt{n}} \in \bullet)$, and

$$\mu(\bullet) \equiv P(Z \in \bullet) = \int_\bullet \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \tag{1.4}$$

then $\mu_n \stackrel{w}{\Longrightarrow} \mu$. Convergence takes place in a metric space of probability measures on $\mathbb{R}$ equipped with the weak topology (for example the Prohorov metric), $M_1(\mathbb{R})$, so that $\mu_n \stackrel{w}{\Longrightarrow} \mu$ if and only if for every bounded continuous $f : \mathbb{R} \to \mathbb{R}$, $\int f \; d\mu_n \to \int f \; d\mu$. We use the notation $E_\mu[f(X)] \equiv \int f \; d\mu$ where it is understood that $X$ is a random variable with distribution $\mu$. Therefore we can also write $\mu_n \stackrel{w}{\Longrightarrow} \mu \iff E_{\mu_n}[f(X_n)] \to E_\mu[f(X)]$, for every bounded continuous $f : \mathbb{R} \to \mathbb{R}$. To prove weak convergence in $\mathbb{R}$ (convergence in the space of probability measures on $\mathbb{R}$ with the weak topology) it is enough to show convergence of the Fourier transforms $E_{\mu_n}[e^{ikX_n}] \to E_\mu[e^{ikX}]$ (or more traditionally $E[e^{ik\frac{S_n}{\sigma\sqrt{n}}}] \to E[e^{ikZ}] = e^{-\frac{k^2}{2}}$) to that of the Gaussian. In other words the functions $\{f_k(x) \equiv e^{ikx} : k \in [-\pi, \pi]\}$ constitute a *convergence determining class*. These results can easily be generalised to $\mathbb{R}^d$-valued random variables. Note that the constraint that the $X_i$ be independent identically distributed may be relaxed (for example $X_i$ stationary, ergodic with $E[X_{n+1}|\mathcal{F}_n] = 0$) and the central limit theorem may still hold.

Setting $S_0 = 0$, the collection $\{S_n\}_{n\geq 0}$ is a random walk on $\mathbb{R}$, and writing $X_t^n = \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}}$, $t \geq 0$, defines a real-valued stochastic process $\{X_t^n\}_{t\geq 0}$ that is right continuous with left limits for each $n$, i.e. $\{X_t^n\}_{t\geq 0} \in D(\mathbb{R})$. Define probability measures $\mu_n$ on the Borel sets of $D(\mathbb{R})$ by $\mu_n(\bullet) \equiv P(\{X_t^n\}_{t\geq 0} \in \bullet)$. Another fundamental result in probability states that $\mu_n \stackrel{w}{\Longrightarrow} W$, where $W$ is *Wiener measure*. It is perhaps more commonly said that $\{X_t^n\}_{t\geq 0}$ converges weakly to a continuous, real valued stochastic process $B_t$ called Brownian motion, or that random walk converges to Brownian motion in the scaling limit. To prove convergence in the space of probability measures on $D(\mathbb{R})$ (weak topology) it is enough to prove convergence of the finite dimensional distributions and *tightness*. The $\{\mu_n\}$ are *tight* if for every $\epsilon > 0$ there exists a compact $K \subset D(\mathbb{R})$ such that $\sup_n \mu_n(K^c) < \epsilon$. Convergence of the finite dimensional distributions by definition means that for every $m \in \mathbb{N}$, $\tilde{\mathbf{t}} \in [0, \infty)^m$, and every bounded continuous $f : \mathbb{R}^m \to \mathbb{R}$ ,

$$E_{\mu_n}[f(X^n_{t_1}, \ldots, X^n_{t_m})] \to E_W[f(B_{t_1}, \ldots, B_{t_m})]. \tag{1.5}$$

To verify (1.5), it is enough to show convergence of the corresponding Fourier transforms $E_{\mu_n}[e^{i\vec{k}\cdot\vec{X}^n_{\vec{t}}}]$.

Due to the independence of the $X_i$, the process $\{X^n_t\}_{t\geq 0}$ is a *Markov process*. That is, the future of the process is independent of the past given the present. Since the increments (consider the $X_i$) of the process also have mean 0, $\{X^n_t\}_{t\geq 0}$ is a *martingale* ($E[X^n_{t+s}|\mathcal{F}^n_t] = X^n_t$ where $\mathcal{F}^n_t = \sigma(\{X^n_s\}_{s\leq t})$). Brownian motion is also a Markovian martingale. It has Hausdorff dimension $d \wedge 2$ and is almost surely self-avoiding in 4 or more dimensions. As such it is a sensible candidate for the scaling limit of self-avoiding walk (neither Markovian nor a martingale) for $d \geq 4$. A result of Hara and Slade shows that for $d > 4$, self-avoiding walk converges to Brownian motion in the scaling limit. In this case tightness follows from a negative correlation property of the model.

The following brief introduction to some important measure-valued processes is described in more mathematical detail in Chapter 7. Let $Y$ be a non-negative integer-valued random variable with mean 1. Critical branching random walk in $d$ dimensions is a process that starts with a single particle at time 0 located at the origin, and at each time $n \in \mathbb{N}$, each particle $\alpha$ alive at time $n$ independently gives birth to $Y_\alpha \overset{\mathcal{D}}{=} Y$ particles at independently and randomly chosen neighbouring vertices and then dies instantly. It can be described by a measure-valued process $X_n$ where for each fixed time $n$, $X_n$ is a finite measure ($X_n \in M_F(\mathbb{R}^d)$) on the Borel sets of $\mathbb{R}^d$ with $X_n(B)$ being the number of particles $\alpha$ alive at time $n$ whose spatial location is some $x \in B$. In this way, a realisation of a measure-valued process describes the evolution in time of the distribution of mass. The mean offspring number of 1 is critical. It can be shown that this process dies out almost surely, but that the expected time when this happens is infinite. The process is Markovian due to the independence conditions and with the critical birth rate the process is also a martingale.

In a similar way to what was done for the simple random walk case, we can define the branching random walk process for all $t \geq 0$, so that it is right continuous with left limits. With appropriate scaling of space and time, critical branching random walk converges weakly (i.e. convergence in the space of measures on $D(M_F(\mathbb{R}^d)))$ to a measure-valued process $X_t$ called super-Brownian motion (SBM). This is of course a statement that $\mu'_n \overset{w}{\Longrightarrow} \mathbb{N}_0$ for some measures $\mu'_n \in M_F(D(M_F(\mathbb{R}^d))$ and some other measure $\mathbb{N}_0$ called the canonical measure of super-Brownian motion (CSBM). Tightness of the measures $\mu'_n$ can be verified using martingale methods. Now the support process $\{A_t\}_{t\geq 0}$ ($A_t$ is the support of the measure $X_t$) of a SBM has Hausdorff dimension $4 \wedge d$ and has no self intersections in dimensions $d \geq 8$ ($\mathbb{N}_0$ almost everywhere). This is the appropriate way to say that SBM is self-avoiding for $d \geq 8$.

Intuitively, by comparison with the self-avoiding walk results we might expect that our critical lattice trees model (described as a measure-valued process with appropriate scaling) converges weakly to CSBM in the same sense as branching random walk, for $d > 8$. Studying a different but related limit conjectured by Aldous [2], it was shown in [7] that sufficiently spread out lattice trees in dimensions $d > 8$ converge to *integrated super-Brownian excursion* (ISE) as the total size of the tree goes to infinity. ISE is a probability measure on probability measures on $\mathbb{R}^d$, i.e. $\mathcal{I} \in M_1\left(M_1(\mathbb{R}^d)\right)$ which describes the distribution of the total mass of CSBM (conditioned to be 1). ISE contains no information about time evolution, however some results concerning ancestry were also proved in [7].

In this thesis, we prove convergence of the finite dimensional distributions of an appropriately defined lattice trees process to those of CSBM, for $d > 8$. This convergence is obtained by proving convergence of the Fourier transforms of relevant quantities and using the existence of a certain exponential moment of CSBM. The main tool used in the proof is the lace expansion, in the form of both (an extension of) the inductive approach of [19] and the lace expansion on a tree of [21].

Tightness remains an open problem. The processes in question are neither martingales nor Markovian, so many of the standard methods for proving tightness do not immediately apply.

## 1.2 The model

We now present the basic definitions of the quantities of interest. We restrict ourselves to the integer lattice $\mathbb{Z}^d$.

**Definition 1.2.1.**

1. *A* bond *is a pair of distinct vertices in the lattice.*
2. *A* cycle *is a set of distinct bonds* $\{v_1v_2, v_2v_3, \ldots, v_{l-1}v_l, v_lv_1\}$*, for some* $l \geq 3$.
3. *A* lattice tree *is a finite set of vertices and lattice bonds connecting those*

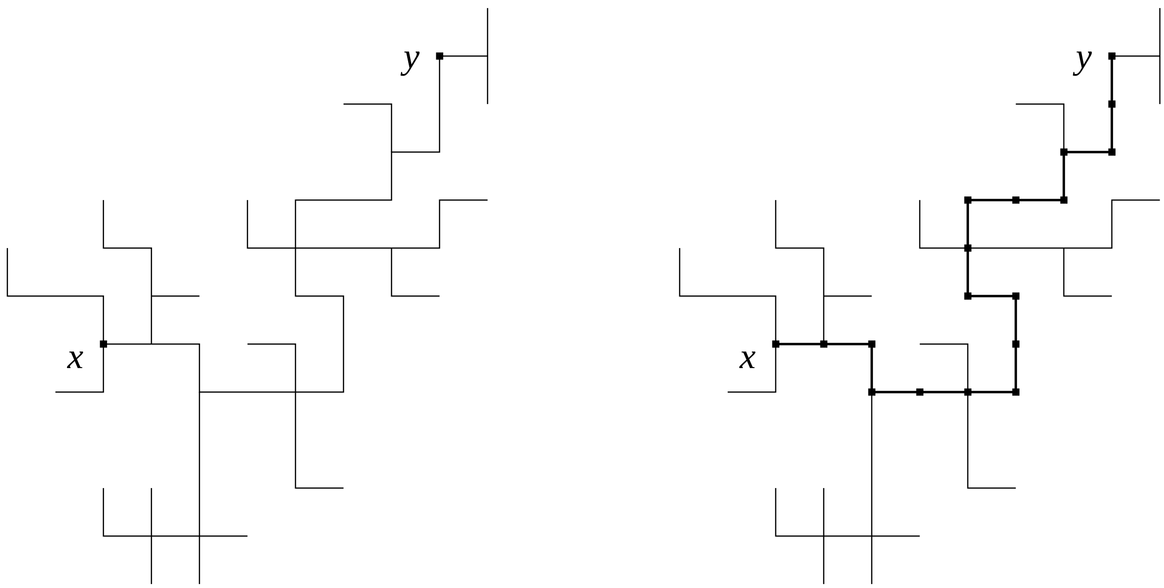


Figure 1.2: A nearest neighbour lattice tree in 2 dimensions. The backbone from $x$ to $y$ of length $n = 17$ is highlighted in the second figure.

*vertices, that contains no cycles. This includes the single vertex lattice tree that contains no bonds.*

4. *Let* $r \geq 2$ *and let* $x_i$, $i \in \{1, \ldots, r\}$ *be vertices in* $T$. *Since* $T$ *contains no cycles then there exists a minimal connected subtree containing all the* $x_i$, *called the* skeleton *connecting the* $x_i$. *If* $r = 2$ *we often refer to the skeleton connecting* $x_1$ *to* $x_2$ *as the* backbone.

**Remark 1.2.2.** *The nearest-neighbour model consists of nearest neighbour bonds* $\{x_1, x_2\}$ *with* $x_1, x_2 \in \mathbb{Z}^d$ *and* $|x_1 - x_2| = 1$. *Figure 1.2 shows an example of a nearest-neighbour lattice tree in* $\mathbb{Z}^2$.

We use $\mathbb{Z}_+$ to denote the nonnegative integers $\{0, 1, 2, \ldots\}$.

**Definition 1.2.3.**

1. *For* $x \in \mathbb{Z}^d$ *let* $\mathcal{T}_x = \{T : x \in T\}$. *Note that this set always includes the* single vertex lattice tree, $T = \{x\}$ *that contains no bonds. We also let* $\mathcal{T}_y(x) = \{T \in \mathcal{T}_Y : x \in T\}$, *and often write* $\mathcal{T}(x)$ *for* $\mathcal{T}_0(x)$, *the set of lattice trees containing the vertices* 0 *and* $x$.

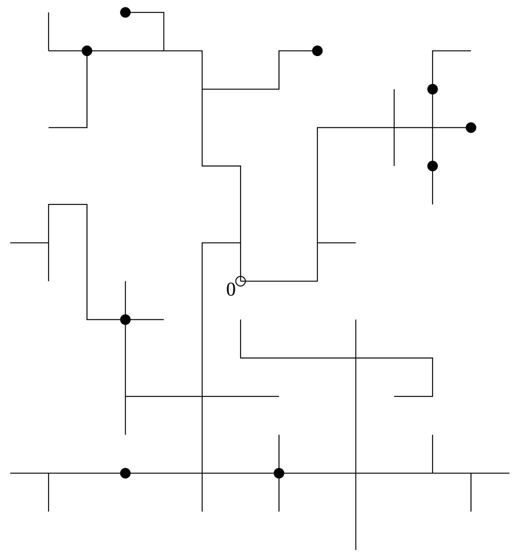


Figure 1.3: A nearest neighbour lattice tree $T$ in 2 dimensions with the set $T_i$ for $i = 10$.

2. *For $T \in \mathcal{T}_0$ we let $T_i$ be the set of vertices $x$ in $T$ such that the backbone from $0$ to $x$ consists of $i$ bonds. In particular for $T \in \mathcal{T}_0$ we have $T_0 = \{0\}$. A tree $T \in \mathcal{T}_0$ is said to* survive until time $n$ *if $T_n \neq \emptyset$.*

3. *For $\tilde{\mathbf{x}} = (x_1, \ldots, x_{r-1}) \in \mathbb{Z}^{d(r-1)}$ and $\tilde{\mathbf{n}} \in \mathbb{Z}_+^{r-1}$ we we write $\tilde{\mathbf{x}} \in T_{\tilde{\mathbf{n}}}$ if $x_i \in T_{n_i}$ for each $i$ and define $\mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}}) \equiv \{T \in \mathcal{T}_0 : \tilde{\mathbf{x}} \in T_{\tilde{\mathbf{n}}}\}$.*

If we think of $T \in \mathcal{T}_0$ as representing a migrating population in discrete time, then $T_i$ can be thought of as the set of locations of particles alive at time $i$. Figure 1.3 identifies the set $T_{10}$ for a fixed $T$. Similarly $\mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})$ can be thought of as the set of trees for which there is a particle at $x_i$ alive at time $n_i$ for each $i$.

In order to provide a small parameter needed for convergence of the lace expansion, we consider trees taking "steps" of size $\leq L$ for some large parameter $L$. The steps are weighted according to a function $D$ which is supported on $[-L, L]^d$ and which has total mass 1. Thus $D$ represents a kind of step probability function. We define this formally in the following subsection. The methods and results in this paper rely heavily on the main results of [11] and [19]. Since the assumptions on the model are stronger in [11], it is this finite range $L, D$ spread out model that we consider. The following definition and the subsequent remark are taken, almost verbatim from [11].

**Definition 1.2.4.** *Let $h$ be a non-negative bounded function on $\mathbb{R}^d$ which is piecewise continuous, symmetric under the $\mathbb{Z}^d$-symmetries of reflection in coordinate hyperplanes and rotation by $\frac{\pi}{2}$, supported in $[-1,1]^d$, and normalised ($\int_{[-1,1]^d} h(x)d^dx = 1$). Then for large $L$ we define*

$$D(x) = \frac{h(x/L)}{\sum_{x\in\mathbb{Z}^d} h(x/L)}. \tag{1.6}$$

**Remark 1.2.5.** *Since $\sum_{x\in\mathbb{Z}^d} h(x/L) \sim L^d$ using a Riemann sum approximation to $\int_{[-1,1]^d} h(x)d^dx$, the assumption that $L$ is large ensures that the denominator of (1.6) is non-zero. Since $h$ is bounded $\sum_{x\in\mathbb{Z}^d} h(x/L) \sim L^d$ also implies that*

$$\|D\|_\infty \le \frac{C}{L^d}. \tag{1.7}$$

*We define $\sigma^2 = \sum_x |x|^2 D(x)$. The sum $\sum_x |x|^r D(x)$ can be regarded as a Riemann sum and is asymptotic to a multiple of $L^r$ for $r > 0$. In particular $\sigma$ and $L$ are comparable. A basic example obeying the conditions of Definition 1.2.4 is given by the function $h(x) = 2^{-d} I_{[-1,1]^d}(x)$ for which $D(x) = (2L+1)^{-d} I_{[-L,L]^d\cap\mathbb{Z}^d}(x)$.*

**Definition 1.2.6 ($L, D$ spread out lattice trees).** *Let $\Omega_D = \{x \in \mathbb{Z}^d : D(x) > 0\}$. We define an $L, D$* spread out lattice tree *to be a lattice tree consisting of bonds $\{x, y\}$ such that $y - x \in \Omega_D$.*

The results of this thesis are for $L, D$ spread out lattice trees in dimensions $d > 8$. Appealing to the hypothesis of universality, we expect that the results also hold for nearest-neighbour lattice trees. However from this point on, unless otherwise stated, "lattice trees" and related terminology refers to $L, D$ spread out lattice trees.

**Definition 1.2.7 (Weight of a tree.).** *Given a finite set of bonds $B$ and a non-negative parameter $p$, we define the* weight *of $B$ to be*

$$W_{p,D}(B) = \prod_{\{x,y\}\in B} pD(y-x), \tag{1.8}$$

*with $W_{p,D}(\emptyset) = 1$. If $T$ is a lattice tree we define*

$$W_{p,D}(T) = W_{p,D}(B_T), \tag{1.9}$$

*where $B_T$ is the set of bonds of $T$.*

**Definition 1.2.8 ($\rho(x)$).** *Let*

$$\rho_p(x) = \sum_{T\in\mathcal{T}(x)} W_{p,D}(T). \tag{1.10}$$

Clearly we have $\rho_p(0) \geq 1$ for all $L, p$ since the single vertex lattice tree contains no bonds and therefore has weight 1. A standard subadditivity argument [24] shows that there is a finite, positive $p_c$ at which $\sum_x \rho_p(x)$ converges for $p < p_c$ and diverges for $p > p_c$. Hara, van der Hofstad and Slade [11] proved the following Theorem.

**Theorem 1.2.9.** *Let $d > 8$ and fix $\nu > 0$. There exists a constant $\overline{A}$ (depending on $d$ and $L$) and an $L_0$ (depending on $d$ and $\nu$) such that for $L \geq L_0$,*

$$\rho_{p_c}(x) = \frac{\overline{A}}{\sigma^2(|x| \vee 1)^{d-2}} \left[ 1 + \mathcal{O}\left( \frac{L^{(d-8)\wedge 2}}{(|x| \vee 1)^{((d-8)\wedge 2)-\nu}} \right) + \mathcal{O}\left( \frac{L^2}{(|x| \vee 1)^{2-\nu}} \right) \right]. \tag{1.11}$$

*Constants in the error terms are uniform in both $x$ and $L$, and $\overline{A}$ is bounded above uniformly in $L$.*

We henceforth take our trees at criticality and write

$$W(\cdot) = W_{p_c,D}(\cdot), \quad \text{and} \quad \rho(x) = \rho_{p_c}(x). \tag{1.12}$$

Hara, van der Hofstad and Slade [11] also proved that $p_c\rho(0) \leq 1 + \mathcal{O}\left(L^{-2+\nu}\right)$ and

$$\rho(x) \leq C\left( I_{x=0} + \frac{I_{x \neq 0}}{L^{2-\nu}\left(|x| \vee 1\right)^{d-2}} \right), \tag{1.13}$$

where the constants in the above statements depend on $\nu$ and $d$, but not $L$.

## 1.3 A measure-valued process

Let $M_F(\mathbb{R}^d)$ denote the space of finite measures on $\mathbb{R}^d$ with the weak topology. For each $i, n \in \mathbb{N}$ and each lattice tree $T$, we define a finite measure $X^{n,T}_{\frac{i}{n}} \in M_F(\mathbb{R}^d)$ by

$$X^{n,T}_{\frac{i}{n}} = \frac{C_1}{n} \sum_{x:\sqrt{C_2 n}x \in T_i} \delta_x, \tag{1.14}$$

where $\delta_x(B) = I_{x \in B}$ for all $B \in \mathcal{B}(\mathbb{R}^d)$. The constants $C_1, C_2$ depend on $L$ and $d$ and will be stated explicitly later. Figure 1.3 shows a fixed tree $T$ and the set $T_i$ for $i = 10$. For this $T$, the measure $X^{n,T}_{\frac{10}{n}}$ assigns measure $\frac{C_1}{n}$ to each vertex in the set $T_{10}/\sqrt{C_2 n} \equiv \{x : \sqrt{C_2 n}x \in T_{10}\}$.

We extend this definition to all $t \in \mathbb{R}^+$ by

$$X^{n,T}_t = X^{n,T}_{\frac{\lfloor nt \rfloor}{n}}. \tag{1.15}$$

Thus for fixed $n, T$, $\{X_t^{n,T}\}$ is constant on $[\frac{i}{n}, \frac{i+1}{n})$.

For a *Polish space* (complete, separable metric space) $E$ we let $D(E) = D([0,\infty), E)$ denote the space of right continuous paths with left limits taking values in $E$. Then $D(E)$ equipped with the *Skorokhod topology* is also Polish ([8], Theorem 5.6). Let $M_F(E)$ denote the space of finite measures on a Polish space $E$. Then $M_F(E)$ equipped with the *weak topology* is also Polish ([6], statement 3.1.1.). Since $X_t^{n,T} = X_{\frac{i}{n}}^{n,T}$ for all $t \in [\frac{i}{n}, \frac{i+1}{n})$, for each fixed $n, T$, $\{X_t^{n,T}\} \in D(M_F(\mathbb{R}^d))$. The above discussion says that $D(M_F(\mathbb{R}^d))$ (with the appropriate topologies) is a Polish space.

Next we must decide what we mean by a "random tree". We define a probability measure $\mathbb{P}$ on the countable set $\mathcal{T}_0$ by $\mathbb{P}(\{T\}) = \frac{W(T)}{\rho(0)}$, so that

$$\mathbb{P}(B) = \frac{\sum_{T \in B} W(T)}{\rho(0)}, \qquad B \subset \mathcal{T}_0. \tag{1.16}$$

Lastly we define the measures $\mu_n \in M_F(D(M_F(\mathbb{R}^d)))$ by

$$\mu_n(H) = C_3 n \mathbb{P}\left(\{T : \{X_t^{n,T}\}_{t \in \mathbb{R}_+} \in H\}\right), \qquad H \in \mathcal{B}(D(M_F(\mathbb{R}^d))), \tag{1.17}$$

where $\mathcal{B}(E)$ denotes the Borel $\sigma$-algebra on $E$ and $C_3$ is another constant that will be stated explicitly later. We expect that $\mu_n \overset{w}{\Longrightarrow} \mathbb{N}_0$, where $\mathbb{N}_0$ is the canonical measure of super-Brownian motion. Convergence as a stochastic process follows from convergence of the finite-dimensional distributions and tightness (see for example [3] Theorems 8.1 and 15.1). The precise definition of this convergence is technical, and thus we postpone its formalisation until Chapter 7. In particular, $\mathbb{N}_0(X_\epsilon \neq 0_M)$ (where $0_M$ denotes the zero measure) is finite but becomes infinite as $\epsilon \searrow 0$. Therefore it is natural to consider $\mathbb{N}_0$ on the set where extinction occurs after time $\epsilon$.

We note in Chapter 7 that to prove the usual statement of convergence of the finite-dimensional distributions we would require the asymptotics of the *survival probability* $\mathbb{P}(T_n > 0)$. However without the survival asymptotics we prove the Theorem 1.3.1, which is the main result of this thesis for probabilists (statistical physicists may be more interested in Theorems 1.4.3 and 1.4.5), in which $\{Y_t^n\}$ denotes a process chosen according to the finite measure $\mu_n$ and $\{Y_t\}$ denotes *super-Brownian excursion*, i.e. a measure-valued path chosen according to the $\sigma$-finite measure $\mathbb{N}_0$. We also use $\mathcal{D}_F$ to denote the set of discontinuities of a function $F$.

**Theorem 1.3.1.** *There exists $L_0 \gg 1$ such that for every $L \geq L_0$, with $\mu_n$ defined by (1.17) the following holds:*

*For every* $\epsilon, \lambda > 0$*,* $m \in \mathbb{N}$*,* $\vec{t} \in (\epsilon, \infty)^m$ *and every* $F : \big(M_F(\mathbb{R}^d)^m\big) \to \mathbb{R}$ *bounded by a polynomial and such that* $\mathbb{N}_0(\vec{X}_{\vec{t}} \in \mathcal{D}_F) = 0$*,*

$$(1) \qquad E_{\mu_n}\left[F(\vec{Y}^n_{\vec{t}})Y^n_\epsilon(1)\right] \to E_{\mathbb{N}_0}\left[F(\vec{Y}_{\vec{t}})Y_\epsilon(1)\right], \qquad \textit{and} \tag{1.18}$$

$$(2) \qquad E_{\mu_n}\left[F(\vec{Y}^n_{\vec{t}})I_{\{Y^n_\epsilon(1)>\lambda\}}\right] \to E_{\mathbb{N}_0}\left[F(\vec{Y}_{\vec{t}})I_{\{Y_\epsilon(1)>\lambda\}}\right]. \tag{1.19}$$

The factors in Theorem 1.3.1 involving the total mass at time $\epsilon$, are essentially two ways of ensuring that our convergence statements are about finite measures. In particular these factors ensure that there is no contribution from the no contribution from processes with arbitrarily small lifetime.

As we have already noted in Section 1.1, it is often sufficient to prove results such as (1.18-1.19) for a suitable class of test functions. For any measure $\mu$ on $\mathbb{R}^d$ and $\phi : \mathbb{R}^d \to \mathbb{C}$ we define $\mu(\phi) = \int_{\mathbb{R}^d} \phi d\mu$. In particular,

$$\begin{aligned} X^{n,T}_t(\phi) \equiv \int_{\mathbb{R}^d} \phi dX^{n,T}_t &= \frac{C_1}{n} \sum_{x:\sqrt{nC_2}x \in T_{\lfloor nt \rfloor}} \int_{\mathbb{R}^d} \phi(y) d\delta_x \\ &= \frac{C_1}{n} \sum_{x:\sqrt{nC_2}x \in T_{\lfloor nt \rfloor}} \phi(x). \end{aligned} \tag{1.20}$$

For $k \in [-\pi, \pi]^d$, let $\phi_k(x) : \mathbb{Z}^d \to \mathbb{C}$ be defined by $\phi_k(x) = e^{ik\cdot x}$. We indirectly prove the following Lemma in Chapter 7.

**Lemma 1.3.2.** *Suppose that for every* $r \geq 2$*, every* $\tilde{\mathbf{k}} \in \mathbb{R}^{(r-1)d}$*, and every* $\tilde{\mathbf{t}} \in (0, \infty)^{r-1}$*,*

$$E_{\mu_n}\left[\prod_{j=1}^{r-1} Y^n_{t_j}(\phi_{k_j})\right] \to E_{\mathbb{N}_0}\left[\prod_{j=1}^{r-1} Y_{t_j}(\phi_{k_j})\right]. \tag{1.21}$$

*Then Theorem 1.3.1 holds.*

Note that for $\tilde{\mathbf{t}} \in (0,\infty)^{r-1}$ we have,

$$\begin{aligned}
\mathbb{E}_{\mu_n}\left[\prod_{j=1}^{r-1} Y_{t_j}^n(\phi_{k_j})\right] &= C_3 n \mathbb{E}_{\mathbb{P}}\left[\prod_{j=1}^{r-1} X_{t_j}^{n,T}(\phi_{k_j})\right] \\
&= \frac{C_3 C_1^{r-1}}{\rho(0) n^{r-2}} \sum_{T\in\mathcal{T}_0} W(T) \prod_{j=1}^{r-1} \sum_{x_j : \sqrt{nC_2}x_j \in T_{\lfloor nt_j \rfloor}} \phi_{k_j}(x_j) \\
&= \frac{C_3 C_1^{r-1}}{\rho(0) n^{r-2}} \sum_{\tilde{\mathbf{x}} \in \frac{\mathbb{Z}^{d(r-1)}}{\sqrt{nC_2}}} \left(\prod_{j=1}^{r-1} \phi_{k_j}(x_j)\right) \sum_{T\in\mathcal{T}_{\lfloor n\tilde{\mathbf{t}}\rfloor}(\tilde{\mathbf{x}})} W(T) \\
&= \frac{C_3 C_1^{r-1}}{\rho(0) n^{r-2}} \sum_{\tilde{\mathbf{x}} \in \mathbb{Z}^{d(r-1)}} e^{\frac{i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}}{\sqrt{nC_2}}} \sum_{T\in\mathcal{T}_{\lfloor n\tilde{\mathbf{t}}\rfloor}(\tilde{\mathbf{x}})} W(T).
\end{aligned} \tag{1.22}$$

This suggests that it might prove useful to examine the quantities $\sum_{T\in\mathcal{T}_{\lfloor n\tilde{\mathbf{t}}\rfloor}(\tilde{\mathbf{x}})} W(T)$.

## 1.4 The $r$-point functions

**Definition 1.4.1 (2-point function).** *For $\zeta \geq 0$, $n \in \mathbb{N}$, and $x \in \mathbb{R}^d$ we define,*

$$t_n(x;\zeta) = \zeta^n \sum_{T\in\mathcal{T}_n(x)} W(T). \tag{1.23}$$

*We also define $t_n(x) = t_n(x;1)$.*

**Definition 1.4.2 (Fourier Transform).** *Given an absolutely summable function $f : \mathbb{Z}^l \to \mathbb{R}$, we let $\widehat{f}(k) = \sum_x e^{ik\cdot x} f(x)$ ($k \in [-\pi,\pi]^l$) denote the Fourier transform of $f$.*

In [19] the authors show that if a recursion relation of the form

$$f_{n+1}(k;z) = \sum_{m=1}^{n+1} g_m(k;z) f_{n+1-m}(k;z) + e_{n+1}(k;z) \tag{1.24}$$

holds, and certain assumptions $S$, $D$, $E$, and $G$ on the functions $f_\bullet$, $g_\bullet$ and $e_\bullet$ hold then there exists a critical value $z_c$ of $z$ such that $f_n(k,z_c)$ (appropriately scaled) converges (up to a constant factor) to the Fourier transform of the Gaussian density as $n \longrightarrow \infty$. In Appendix A we extend this result (based on the ideas of [18]) by generalizing assumptions $E$ and $G$ according to a parameter $p \geq 1$, where the $p = 1$

case is that which is proved in [19]. In Section 3.2 we show that $\widehat{t}_n(k;\zeta)$ obeys the recursion relation

$$\widehat{t}_{n+1}(k;\zeta) = \sum_{m=1}^{n+1} \widehat{\pi}_{m-1}(k;\zeta)\zeta p_c \widehat{D}(k)\widehat{t}_{n+1-m}(k;\zeta) + \widehat{\pi}_{n+1}(k;\zeta), \tag{1.25}$$

where $\pi_m(x;\zeta)$ is a function that is defined in Section 3.2. After massaging this relation somewhat, the important ingredients in verifying assumptions $E$ and $G$ for our lattice trees model are bounds on $\widehat{\pi}_m$ using information about $\rho(x)$ and $\widehat{t}_l(k;\zeta)$ for $l < m$. The quantities $\widehat{\pi}_{m-1}(k;\zeta)$ are defined using a technique known as the *lace expansion.* The lace expansion is discussed in Chapter 2 and it enables us to express $\pi_{m-1}$ in terms of Feynman diagrams, that can be bounded using (1.13) and bounds on $\widehat{t}_l(k;\zeta)$ for $l < m$. As in previous work already discussed, the critical dimension $d_c = 8$ appears in this analysis as the dimension above which the *square diagram*

$$\rho^{(4)}(0) = \sum_{x,y,z} \rho(x)\rho(y-x)\rho(z-y)\rho(z) \tag{1.26}$$

converges.

Ultimately we verify assumptions $E_p$ and $G_p$ for our lattice trees model with $p = 2$ and thus the results of Appendix A are valid. The parameter $\zeta$ appears in (1.4.1) as an additional weight on bonds in the backbone of trees $T \in \mathcal{T}_n(x)$. Those trees are already critically weighted by $p_c$ (a weight present on *every* bond in the tree) as described by Definition 1.2.7 and (1.12) and exhibit mean-field behaviour in the form of Theorem 1.2.9. One might therefore expect a Gaussian limit for $\widehat{t}_n$ with $\zeta = 1$.

The following theorem follows from the induction approach of Appendix A, together with a short argument showing that the critical value of $\zeta$ obtained from the induction is $\zeta_c = 1$.

**Theorem 1.4.3.** *Fix $d > 8$, $t > 0$, $\gamma \in (0, 1 \wedge \frac{d-8}{2})$ and $\delta \in (0, (1 \wedge \frac{d-8}{2}) - \gamma)$. There exists a positive $L_0 = L_0(d)$ such that: For every $L \geq L_0$ there exist positive $A, v$ depending on $d$ and $L$ such that*

$$\widehat{t}_{\lfloor nt \rfloor}\left(\frac{k}{\sqrt{v\sigma^2 n}}\right) = Ae^{-\frac{k^2}{2d}t} + \mathcal{O}\left(\frac{k^2}{n}\right) + \mathcal{O}\left(\frac{k^2 t^{1-\delta}}{n^\delta}\right) + \mathcal{O}\left(\frac{1}{(nt \vee 1)^{\frac{d-8}{2}}}\right), \tag{1.27}$$

*with the error estimate uniform in $\left\{k \in \mathbb{R}^d : k^2 \leq \frac{C\log(\lfloor nt \rfloor)}{t}\right\}$, where $C = C(\gamma)$ and the constants in the second and third error terms may depend on $L$.*

Based on Theorem 1.4.3 and (1.22), we choose $C_2 = v\sigma^2$ in (1.14).

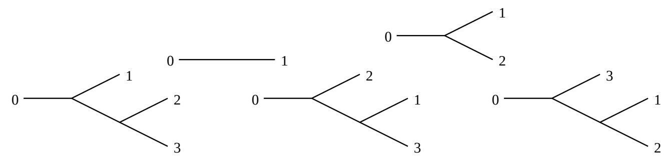


Figure 1.4: The unique shape $\alpha(r)$ for $r = 2, 3$ and the 3 shapes for $r = 4$.

**Definition 1.4.4 ($r$-point function).** *For $r \geq 3$, $\tilde{\mathbf{n}} \in \mathbb{N}^{(r-1)}$ and $\tilde{\mathbf{x}} \in \mathbb{R}^{d(r-1)}$ we define*

$$t^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}}) = \sum_{T \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})} W(T). \tag{1.28}$$

To state a version of Theorem 1.4.3 for $r$-point functions for $r > 3$ we need the notion of *shapes.*

A *shape* is an abstract set of vertices and edges connecting those vertices. The *degree* of a vertex $v$ is the number of edges incident to $v$. Vertices of degree 1 are called *leaves.* Vertices of degree $\geq 3$ are called *branch points.* We are primarily concerned with shapes that have a binary tree topology as follows. There is a unique shape for $r = 2$ consisting of 2 vertices (labelled 0, 1) connected by a single edge. The vertex labelled 0 is called the *root.* For $r \geq 3$ we have $\prod_{j=3}^{r}(2j-5)$ $r$-shapes obtained by adding a vertex to any of the $2(r-1)-3$ edges of each $(r-1)$-shape, and a new edge to that vertex. The leaf of this new edge is labelled $r-1$. Each $r$-shape has $2r-3$ edges, labelled in a fixed but arbitrary manner as $1, \ldots, 2r-3$. This is illustrated in figure 1.4 which shows the shapes for $r = 2, 3, 4$. Let $\Sigma_r$ denote the set of $r$-shapes. We make the edges in $\alpha \in \Sigma_r$ directed by directing them away from the root.

By construction each $r$-shape has $r-2$ branch points, each of degree 3. Thus the unique shape for $r = 3$ (Figure 1.4) has 3 leaves and 1 branch point.

Given a shape $\alpha \in \Sigma_r$ and $\tilde{\mathbf{k}} \in \mathbb{R}^{(r-1)d}$ we define $\vec{\kappa}(\alpha) \in \mathbb{R}^{(2r-3)d}$ as follows. For each leaf $j$ in $\alpha$ (other than 0) we let $E_j$ be the set of edges in $\alpha$ of the unique path in $\alpha$ from 0 to $j$. For $l = 1, \ldots, 2r-3$, we define

$$\kappa_l(\alpha) = \sum_{j=1}^{r-1} k_j I_{\{l \in E_j\}}. \tag{1.29}$$

Next, given $\alpha$ and $\vec{s} \in \mathbb{R}_+^{(2r-3)}$ we define $\tilde{\varsigma}(\alpha) \in \mathbb{R}_+^{(r-1)}$ by

$$\varsigma_j(\alpha) = \sum_{l \in E_j} s_l. \tag{1.30}$$

Finally we define

$$R_{\tilde{\mathbf{t}}}(\alpha) = \{\vec{s} : \tilde{\varsigma}(\alpha) = \tilde{\mathbf{t}}\}. \tag{1.31}$$

This is an $r-2$-dimensional subset of $\mathbb{R}_+^{(2r-3)}$. For $r=3$ we simply have

$$R_{\tilde{\mathbf{t}}}(\alpha) = \{(s, t_1 - s, t_2 - s) : s \in [0, t_1 \wedge t_2]\}. \tag{1.32}$$

It is known [1] that for $r \geq 2$, $0 < t_1 < t_2 \cdots < t_{r-1}$ and $\phi_k(x) = e^{ik\cdot x}$,

$$E_{\mathbb{N}_0}\left[\prod_{j=1}^{r-1} X_{t_j}(\phi_{k_j})\right] = \sum_{\alpha\in\Sigma_r} \int_{R_{\tilde{\mathbf{t}}}(\alpha)} \prod_{l=1}^{2r-3} e^{-\frac{\kappa_l(\alpha)^2 s_l}{2d}} d\vec{s}. \tag{1.33}$$

For $r=3$ this reduces to

$$\int_0^{t_1\wedge t_2} e^{-\frac{(k_1+k_2)^2 s}{2d}} e^{-\frac{k_1^2(t_1-s)}{2d}} e^{-\frac{k_2^2(t_2-s)}{2d}} ds. \tag{1.34}$$

**Theorem 1.4.5.** *Fix $d > 8$, $\gamma \in (0, 1 \wedge \frac{d-8}{2})$ and $\delta \in (0, (1 \wedge \frac{d-8}{2}) - \gamma)$. There exists $L_0 = L_0(d) \gg 1$ such that: for each $L \geq L_0$ there exists $V = V(d, L) > 0$ such that for every $\tilde{\mathbf{t}} \in (0,\infty)^{(r-1)}$, $r \geq 3$, $R > 0$, and $\|\vec{k}\|_\infty \leq R$,*

$$\hat{t}^{r}_{\lfloor n\tilde{\mathbf{t}}\rfloor}\left(\frac{\tilde{\mathbf{k}}}{\sqrt{v\sigma^2 n}}\right) = n^{r-2}V^{r-2}A^{2r-3}\left[\sum_{\alpha\in\Sigma_r}\int_{R_{\tilde{\mathbf{t}}}(\alpha)} \prod_{l=1}^{2r-3} e^{-\frac{\kappa_l(\alpha)^2 s_l}{2d}} d\vec{s} + \mathcal{O}\left(\frac{1}{n^\delta}\right)\right], \tag{1.35}$$

*where the constant in the error term depends on $\tilde{\mathbf{t}}, R$ and $L$.*

Based on Theorem 1.4.5 we choose $C_1 = V^{-1}A^{-2}$ and $C_3 = VA\rho(0)$ in (1.14) and (1.17). Theorem 1.4.5 is proved in Chapter 4 using the lace expansion on a tree of [21]. The proof proceeds by induction on $r$, with Theorem 1.4.3 as the initializing case. Lattice trees $T \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})$ can be classified according to their skeleton (recall Definition 1.2.1). Such trees typically have a skeleton with the topology of some $\alpha \in \Sigma_r$ and the lace expansion and induction hypothesis combine to give the main contribution to (1.35). The relatively few trees that do not have the topology of any $\alpha \in \Sigma_r$ are considered separately and are shown to contribute only to the error term of (1.35).

Theorems 1.4.3 and 1.4.5, combined with the observations (1.22) and (1.33) verify the conditions of Lemma 1.3.2. Thus assuming Lemma 1.3.2, Theorems 1.4.3 and 1.4.5 are sufficient to prove the main result, Theorem 1.3.1. Lemma 1.3.2 and Theorem 1.3.1 are proved in Chapter 7.

# Chapter 2

# The lace expansion

The lace expansion on an interval was introduced in [5] for weakly self-avoiding walk, and was applied to lattice trees in [12, 13, 7, 11]. It has also been applied to various other models such as strictly self-avoiding walk, oriented and unoriented percolation and the contact process. The lace expansion on a tree was introduced in [21] and was applied to networks of mutually avoiding, SAW joined with the topology of a tree. Our analysis requires some modifications to the definitions of connected graph and lace given in [21]. In this chapter we follow [21] with some small modifications and define the notion of a lace on a star-shaped network. In Section 2.1 we introduce our terminology and define and construct laces on star shaped networks of degree 1 or 3. In Section 2.2 we analyse products of the form $\prod_{st\in\mathcal{N}}[1+U_{st}]$ and perform the lace expansion in a general setting. Such products will appear in formulas for the $r$-point functions in Chapters 3 and 4.

## 2.1 Graphs and Laces

Given a shape $\alpha \in \Sigma_r$, and $\vec{n} \in \mathbb{N}^{2r-3}$ we define $\mathcal{N} = \mathcal{N}(\alpha, \vec{n})$ to be the *skeleton network* formed by inserting $n_i - 1$ vertices into edge $i$ of $\alpha$, $i = 1, \ldots, 2r-3$. Thus edge $i$ in $\alpha$ becomes a path of length $n_i$ in $\mathcal{N}$.

Fix a connected subnetwork $\mathcal{M} \subseteq \mathcal{N}$. The *degree* of a vertex $v$ in $\mathcal{M}$ is the number of edges in $\mathcal{M}$ incident to $v$. A vertex of $\mathcal{M}$ is a *leaf* (resp. *branch point*) of $\mathcal{M}$ if it is of degree 1 (resp. 3) in $\mathcal{M}$. A *path* in $\mathcal{M}$ is any connected subnetwork $\mathcal{M}_1 \subset \mathcal{M}$ such that $\mathcal{M}_1$ has no branch points. A *branch* of $\mathcal{M}$ is a path of $\mathcal{M}$ containing at least two vertices, whose two endvertices are both leaves or branch points of $\mathcal{M}$, and whose interior vertices (if they exist) are not leaves or branch points of $\mathcal{M}$. Note that if $b' \in \mathcal{M}_1 \subset \mathcal{M}$ is a branch point of $\mathcal{M}_1$ then it is also a branch point of $\mathcal{M}$ but the reverse implication does not hold in general. Similarly if $v \in \mathcal{M}_1$ is a leaf of $\mathcal{M}$ then it is also a leaf of $\mathcal{M}_1$ but the reverse implication

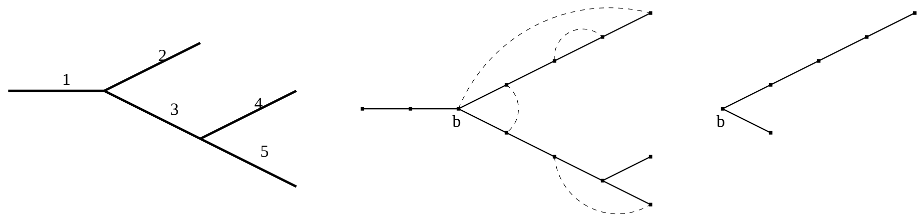


Figure 2.1: A shape $\alpha \in \Sigma_r$ for $r = 4$ with fixed branch labellings, followed by a graph $\Gamma$ on $\mathcal{N}(\alpha, (2,4,3,1,1))$, and the subnetwork $\mathcal{A}_b(\Gamma)$.

does not hold in general. Two vertices $s, t$ are *neighbours* in $\mathcal{M}$ if there exists some branch in $\mathcal{M}$ of which $s, t$ are the two endvertices (this forces $s$ and $t$ to be of degree 1 or 3). Two vertices $s, t$ of $\mathcal{M}$ are said to be *adjacent* if there is an edge in $\mathcal{M}$ that is incident to both $s$ and $t$.

For $r \geq 3$, let $b$ denote the unique branch point of $\mathcal{N}$ neighbouring the root. If $r = 2$, let $b$ be one of the leaves of $\mathcal{N}$. Without loss of generality we assume that the edge in $\alpha$ (and hence the branch in $\mathcal{N}$) containing the root is labelled 1 and we assume that the other two branches incident to $b$ are labelled $2, 3$. Vertices in $\mathcal{N}$ may be relabelled according to branch and distance along the branch, with branches oriented away from the root. For example the vertices on branch 1 from the root 0 to the branch point (or leaf if $r = 2$) $b$ neighbouring the root would be labelled $0 = (1,0), (1,1), \dots, (1,n_1) = b$.

Examples illustrating some of the following definitions appear in Figures 2.1-2.2.

**Definition 2.1.1.**

1. *A* bond *is a pair $\{s,t\}$ of vertices in $\mathcal{M}$ with the vertex labelling inherited from $\mathcal{N}$. Let $\mathbf{E}_{\mathcal{M}}$ denote the set of bonds of $\mathcal{M}$. The set of edges and vertices of the unique minimal path in $\mathcal{M}$ joining (and including) $s$ and $t$ is denoted by $[s,t]$. The bond $\{s,t\}$ is said to* cover *$[s,t]$. We often abuse the notation and write $st$ for $\{s,t\}$.*

2. *A* graph *on $\mathcal{M}$ is a set of bonds. Let $\mathcal{G}_{\mathcal{M}}$ denote the set of graphs on $\mathcal{M}$. The graph containing no bonds will be denoted by $\emptyset$.*

3. *Let $\mathcal{R} = \mathcal{R}_{\mathcal{M}}$ denote the set of bonds which cover more than one branch point of $\mathcal{M}$. If $r \leq 3$ then $\mathcal{R} = \emptyset$ since in this case $\mathcal{M} \subseteq \mathcal{N}$ cannot have more than one branch point. Let $\mathcal{G}_{\mathcal{M}}^{-\mathcal{R}} = \{\Gamma \in \mathcal{G}_{\mathcal{M}} : \Gamma \cap \mathcal{R}_{\mathcal{M}} = \emptyset\}$, i.e. the set of graphs on $\mathcal{M}$ containing no bonds in $\mathcal{R}$.*

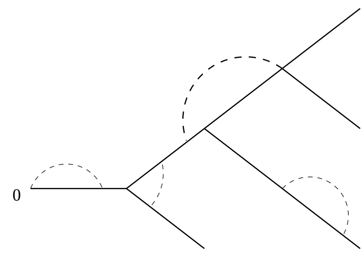


Figure 2.2: A graph $\Gamma \in \mathcal{G}(\mathcal{N})$ that contains a bond in $\mathcal{R}$. The bond in $\mathcal{R}$ appears darker. For simplicity, only the leaves and branch points of $\mathcal{N}$ are explicit.

4. *A graph $\Gamma \in \mathcal{G}_{\mathcal{M}}$ is a* connected graph *on $\mathcal{M}$ if, as sets of edges, $\cup_{st\in\Gamma}[s,t] = \mathcal{M}$ (i.e. if every edge of $\mathcal{M}$ is covered by some $st \in \Gamma$). Let $\mathcal{G}_{\mathcal{M}}^{con}$ denote the set of connected graphs on $\mathcal{M}$, and $\mathcal{G}_{\mathcal{M}}^{-\mathcal{R},con} = \mathcal{G}_{\mathcal{M}}^{con} \cap \mathcal{G}_{\mathcal{M}}^{-\mathcal{R}}$.*

5. *A connected graph $\Gamma \in \mathcal{G}_{\mathcal{M}}^{con}$ is said to be* minimal *or* minimally connected *if the removal of any of its bonds results in a graph that is not connected (i.e. for any $st \in \Gamma$, $\Gamma \setminus st \notin \mathcal{G}_{\mathcal{M}}^{con}$).*

6. *Given $\Gamma \in \mathcal{G}_{\mathcal{M}}$ and a subnetwork $\mathcal{A} \subset \mathcal{M}$ we define $\Gamma|_{\mathcal{A}} = \{st \in \Gamma : s,t \in \mathcal{A}\}$.*

7. *Given a vertex $v \in \mathcal{M}$ and $\Gamma \in \mathcal{G}_{\mathcal{M}}$ we let $\mathcal{A}_v(\Gamma)$ be the largest connected subnetwork $\mathcal{A}$ of $\mathcal{M}$ containing $v$ such that $\Gamma|_{\mathcal{A}}$ is a connected graph on $\mathcal{A}$. Note that $\mathcal{A}$ could be a single vertex. In particular $\mathcal{A}_v(\emptyset) = v$.*

8. *Let $\mathcal{E}_{\mathcal{N}}^b$ be the set of graphs $\Gamma \in \mathcal{G}_{\mathcal{N}}^{-\mathcal{R}}$ such that $\mathcal{A}_b(\Gamma)$ contains a vertex adjacent to some branch point $b' \neq b$ of $\mathcal{N}$. Note that this set is empty if $r \leq 3$, since then $\mathcal{N}$ contains at most one branch point. Note also that if $b$ is adjacent to another branch point of $\mathcal{N}$, then even $\emptyset \in \mathcal{E}_{\mathcal{N}}^b$, since $\mathcal{A}_b(\emptyset) = b$.*

The existence of $\mathcal{A}_v(\Gamma)$ is clear since if $\mathcal{A}_1$ and $\mathcal{A}_2$ are connected subnetworks of $\mathcal{M}$ containing $v$ such that $\Gamma|_{\mathcal{A}_i}$ is a connected graph on $\mathcal{A}_i$, then $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ also has this property.

For $\Delta \in \{0,1,2,3\}$, $\vec{n} \in \mathbb{N}^{\Delta}$ let $\mathcal{S}^{\Delta}(\vec{n})$ denote the network consisting of $\Delta$ paths meeting at a common vertex $v$, where path $i$ is of length $n_i > 0$ (contains $n_i$ edges). This is called a star-shaped network of degree $\Delta$. By definition of our networks $\mathcal{N}(\alpha, \vec{n})$, with $\vec{n} \in \mathbb{N}^{2r-3}$, for any $\Gamma \in \mathcal{G}_{\mathcal{N}}^{-\mathcal{R}} \setminus \mathcal{E}_{\mathcal{N}}^b$, $\mathcal{A}_b(\Gamma)$ contains at most one branch point and is therefore a star-shaped subnetwork of degree 3 (if it contains a branch point), 2, 1, or 0 (if $\mathcal{A}_b(\Gamma)$ is a single vertex). Since it contains no branch point, a star shaped network $\mathcal{S}^1(n)$ of degree 1 may be identified with the interval $[0,n]$, and we can write $\mathcal{S}[0,n]$ for $\mathcal{S}^1(n)$. Similarly a star-shaped network $\mathcal{S}^2(n_1,n_2)$

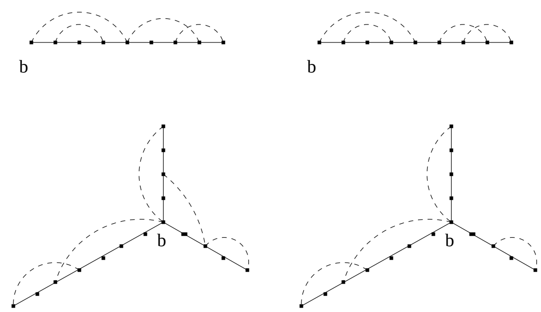


Figure 2.3: Two graphs on each of $\mathcal{S}^1(8)$ and $\mathcal{S}^3(4,4,7)$. The first graph for each star is connected. The second is disconnected. The connected graph on $\mathcal{S}^3(4,4,7)$ is a lace while the connected graph on $\mathcal{S}^1(8)$ is not a lace.

of degree 2 may be identified with the interval $[-n_2, n_1]$ and we can write $\mathcal{S}[-n_2, n_1]$ for $\mathcal{S}^2(n_1, n_2)$. Our main interest will be connected graphs on star-shaped networks.

Figure 2.3 shows graphs on each of $\mathcal{S}^1(8)$ and $\mathcal{S}^3(4,4,7)$. The first graph in each case is connected, while the second is disconnected.

**Definition 2.1.2.** *Fix a connected subnetwork $\mathcal{M} \subseteq \mathcal{N}$. Let $\Gamma \in \mathcal{G}_{\mathcal{M}}^{-\mathcal{R},con}$ be given and let $v$ be a branch point of $\mathcal{M}$. If $\mathcal{M}$ contains no branch points then we let $v$ be one of the leaves of $\mathcal{M}$.*

*Let $\Gamma_e^b \subset \Gamma$ be the set of bonds $s_i t_i$ in $\Gamma$ which cover the vertex $v$ and which have an endpoint (without loss of generality $t_i$) strictly on branch $\mathcal{M}_e$ (i.e. $t_i$ is a vertex of branch $\mathcal{M}_e$ and $t_i \neq v$). By definition of connected graph, $\Gamma_e^v$ will be nonempty. From $\Gamma_e^v$ we select the set $\Gamma_e^{v,max}$ for which the network distance from $t_i$ to $v$ is maximal. We choose the* bond associated to branch $\mathcal{M}_e$ at $v$ *as follows:*

1. *If there exists a unique element of $\Gamma_e^{v,max}$ whose network distance from $s_i$ to $v$ is maximal, then this $s_i t_i$ is the bond associated to branch $\mathcal{M}_e$ at $v$.*

2. *If not then the bond associated to branch $\mathcal{M}_e$ at $v$ is chosen (from the elements $\Gamma_e^{v,max}$ whose network distances from $s_i$ to $v$ are maximal) to be the bond $s_i t_i$ with $s_i$ on the branch of highest label.*

**Definition 2.1.3 (Lace).** *A* lace *on a star shape $\mathcal{S} = \mathcal{S}^\Delta(\vec{n})$, with $\vec{n} \in \mathbb{N}^\Delta$, $\Delta \in \{1,3\}$ is a connected graph $L \in \mathcal{G}_{\mathcal{S}}^{con}$ such that:*

- *If $st \in L$ covers a branch point $v$ of $\mathcal{S}$ then $st$ is the bond in $L$ associated to some branch $\mathcal{S}_e$ at $v$.*

- *If $st \in L$ does not cover such a branch point then $L \setminus st$ is not connected.*

*We write $\mathcal{L}(\mathcal{S})$ for the set of laces on $\mathcal{S}$, and $\mathcal{L}^N(\mathcal{S})$ for the set of laces on $\mathcal{S}$ consisting of exactly $N$ bonds.*

Note also that the definition of a lace can be extended to star-shapes of higher degree (e.g. see [21]) and even to more complex networks (for example networks with general tree topology). However we do not require such generality for our analysis.

See Figure 2.3 for some examples of connected graphs and laces. We now describe a method of constructing a lace $\mathbf{L}_\Gamma$ on a star-shaped network $\mathcal{S}$ of degree 1,2 or 3. Note that the only (connected) graph on a star-shape of degree 0 (i.e. a single vertex) is the graph $\Gamma = \emptyset$ containing no bonds, and we define $\mathbf{L}_\emptyset = \emptyset$.

**Definition 2.1.4 (Lace construction).** *Let $\mathcal{S}$ be a star-shaped network of degree 1, 2, or 3. In the latter case, $b$ is the branch point, otherwiseformer $b$ denotes one of the leaves of $\mathcal{S}$. Fix $\Gamma \in \mathcal{G}_{\mathcal{S}}^{-\mathcal{R},con}$. Let $F$ be the set of branch labels for branches incident to $b$. For each $e$ in $F$,*

- *Let $s_1^e t_1^e$ be the bond in $\Gamma$ associated to branch $\mathcal{S}_e$ at $b$, and let $b_e$ be the other endvertex of $\mathcal{S}_e$.*
- *Suppose we have chosen $\{s_1^e t_1^e, \ldots, s_l^e t_l^e\}$ and that $\cup_{i=1}^l [s_i^e t_i^e]$ does not cover $b_e$. Then we define*

$$\begin{aligned} t_{l+1}^e &= \max\{t \in \mathcal{S}_e : \exists\ s \in \mathcal{S}_e, s \leq_b t_l^e \text{ such that } st \in \Gamma\}, \\ s_{l+1}^e &= \min\{s \in \mathcal{S}_e : st_{l+1}^e \in \Gamma\}, \end{aligned} \tag{2.1}$$

  *where* max *(*min*) refers to choosing $t$ $(s)$ of maximum (minimum) network distance from $b$. Similarly $s \leq_b t$ if the network distance from $t$ to $b$ is greater than the network distance of $s$ from $b$.*
- *We terminate this procedure as soon as $b_e$ is covered by $\cup_{i=1}^l [s_i^e t_i^e]$, and set $\mathbf{L}_\Gamma(e) = \{s_1^e t_1^e, \ldots, s_l^e t_l^e\}$.*

*Next we define*

$$\mathbf{L}_\Gamma = \cup_{e \in F} \mathbf{L}_\Gamma(e), \tag{2.2}$$

*and given a lace $L \in \mathcal{L}(\mathcal{S})$ we define*

$$\mathcal{C}(L) = \{st \in \mathbf{E}_{\mathcal{S}} \setminus L : \mathbf{L}_{L \cup st} = L\} \tag{2.3}$$

*to be the set of bonds compatible with $L$.*

In particular if $L \in \mathcal{L}(\mathcal{S})$ and if there is a bond $s't' \in L$ (with $s't' \neq st$) which covers both $s$ and $t$, then $st$ is compatible with $L$.

The following results are proved for star-shaped networks in [21] for the different notion of connectivity. The proofs presented here are very similar.

**Proposition 2.1.5.** *Given a star shaped network $\mathcal{S} = \mathcal{S}^{\Delta}(\vec{n})$, $\Delta \in \{1, 3\}$, and a connected graph $\Gamma \in \mathcal{G}^{con}(\mathcal{S})$, the graph $\boldsymbol{L}_\Gamma$ is a lace on $\mathcal{S}$.*

*Proof.* By construction, every branch of $\mathcal{S}$ is covered by $\boldsymbol{L}_\Gamma$ so $\boldsymbol{L}_\Gamma$ is a connected graph on $\mathcal{S}$. Now suppose $st \in \boldsymbol{L}_\Gamma$ covers the branch point (or leaf if $\Delta = 1$) $b$ of $\mathcal{S}$, with $s \in \mathcal{S}_e$, $t \in \mathcal{S}_{e'}$ (where $e' = e$ if $s = b$ or $t = b$). Then $st$ was chosen as the bond in $\Gamma$ associated to $\mathcal{S}_e$ or $\mathcal{S}_{e'}$, so in particular it is the bond in $\boldsymbol{L}_\Gamma$ associated to $\mathcal{S}_e$ or $\mathcal{S}_{e'}$. Now if $st \in \boldsymbol{L}_\Gamma$ does not cover $b$ then $s$ and $t$ are on the same branch $\mathcal{S}_e$ for some $e$ and so $st = s_i^e t_i^e$ for some $i$. Now observe that if $\boldsymbol{L}_\Gamma \setminus st$ is a connected graph on $\mathcal{S}$ then we would not have chosen $s_i^e t_i^e = st$ in the construction of $\boldsymbol{L}_\Gamma$. □

**Proposition 2.1.6.** *Let $\Gamma \in \mathcal{G}_{\mathcal{S}}^{-\mathcal{R},con}$. Then $\boldsymbol{L}_\Gamma = L$ if and only if $L \subseteq \Gamma$ is a lace and $\Gamma \setminus L \subseteq \mathcal{C}(L)$.*

*Proof.* If $\boldsymbol{L}_\Gamma = L$, then $L$ is a lace by Proposition 2.1.5. By definition any bond $st \in \Gamma \setminus L$ that covers $b$ is compatible with $L$ since $\boldsymbol{L}_\Gamma$ contains the bond $s't'$ in $\Gamma$ associated to each branch $\mathcal{S}_e$ at $b$, and $s't'$ is therefore also the bond in $L \cup st$ associated to $\mathcal{S}_e$ at $b$. Similarly if $st \in \Gamma \setminus L$ does not cover $b$ then there are bonds in $\boldsymbol{L}_\Gamma$ chosen from all bonds $\Gamma$ to satisfy the optimal covering criteria (2.1). Therefore these same bonds satisfy those criteria when choosing from bonds in $\boldsymbol{L}_\Gamma \cup st$, so that $\boldsymbol{L}_{L \cup st} = L$ and $st$ is compatible with $L$.

For the reverse direction, let $L \subseteq \Gamma$ be a lace and $\Gamma \setminus L \subseteq \mathcal{C}(L)$. Assume that $\boldsymbol{L}_\Gamma \neq L$. Then

(a) there exists $st \in \boldsymbol{L}_\Gamma \cap (\Gamma \setminus L)$ or

(b) there exists $st \in L \cap (\Gamma \setminus \boldsymbol{L}_\Gamma)$.

For (a), if $st \in \boldsymbol{L}_\Gamma \cap (\Gamma \setminus L)$ covers the branch point then by definition of $\boldsymbol{L}_\Gamma$ it is the bond in $\Gamma$ associated to some branch $\mathcal{S}_e$. Therefore for any lace $L' \subseteq \Gamma$, $st$ is the bond in $L' \cup st$ associated to $\mathcal{S}_e$ so $st$ is not compatible with any lace $L' \subset \Gamma$. Since $st \in \Gamma \setminus L$ we have a contradiction. If $st \in \boldsymbol{L}_\Gamma \cap (\Gamma \setminus L)$ does not cover the branch point then $st = s_i^e t_i^e$ for some $e, i$. Then for this fixed $e$ there is a smallest $i$ such that $s_i^e t_i^e \in \boldsymbol{L}_\Gamma \cap (\Gamma \setminus L)$. Then this bond is not compatible with $L$ and we again have a contradiction.

For (b), since $\Gamma \setminus L \subseteq \mathcal{C}(L)$, we must have that every bond in $\Gamma$ associated to a branch $\mathcal{S}_e$ is in $L$. Since $L$ is a lace, these are the only bonds in $L$ which cover $b$ and they are also in $\boldsymbol{L}_\Gamma$ by definition. Therefore the $st \in L \cap (\Gamma \setminus \boldsymbol{L}_\Gamma)$ must satisfy $s, t \in \mathcal{S}_e$, $s, t \neq b$. Since $L$ is a lace, $L \setminus st$ is not connected, and therefore since $\boldsymbol{L}_\Gamma$ is a connected graph and $st \notin \boldsymbol{L}_\Gamma$ there must exist $s't'$ in $\boldsymbol{L}_\Gamma \cap (\Gamma \setminus L)$ and by case (a) we have the result. □

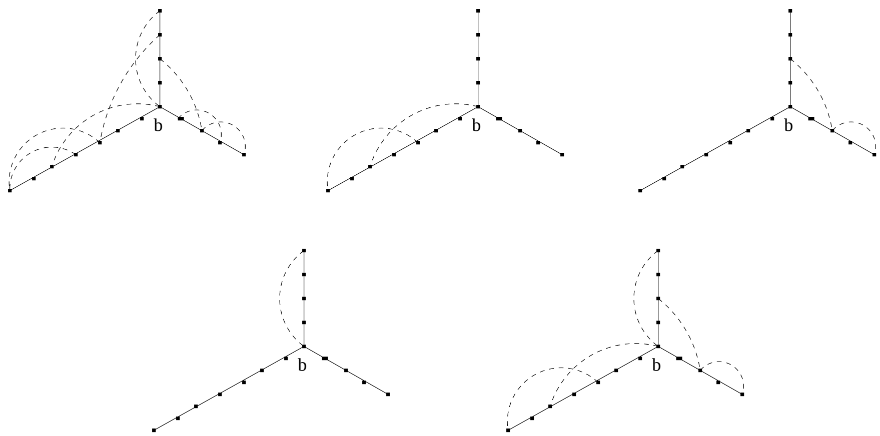


Figure 2.4: An illustration of the construction of a lace from a connected graph. The first figure shows a connected graph $\Gamma$ on a star $S^3_{(n_1,n_2,n_3)}$. The intermediate figures show each of the $\boldsymbol{L}_\Gamma(e)$ for $e \in F_b$, while the last figure shows the lace $\boldsymbol{L}_\Gamma$.


Figure 2.5: Basic examples of a minimal and a non-minimal lace for $\Delta = 3$. For the non-minimal lace, a removable edge is highlighted.

See Figure 2.4 for an example of a connected graph $\Gamma$ on a star-shaped network of degree 3, and its corresponding lace $\boldsymbol{L}_\Gamma$.

### 2.1.1 Classification of laces

**Definition 2.1.7 (Minimal).** *A lace on $\mathcal{S}$ is said to be* minimal *if the removal of any bond from the lace results in a disconnected graph on $\mathcal{S}$.*

A lace $L$ on a star shape $\mathcal{S}$ of degree 1 or 2 is necessarily minimal by Definitions 2.1.3 and 2.1.1. For a lace on a star shape of degree 3 this need not be true. See Figure 2.5 for an example of a minimal and a non-minimal lace for $\Delta = 3$. There is a more general version of the following Lemma for laces on star-shaped networks of higher degree, but we present only the results needed for our analysis.

**Lemma 2.1.8.** *(a) For a star shaped network $\mathcal{S}$ of degree $\Delta \in \{1, 2, 3\}$, any minimally connected graph $\Gamma \in \mathcal{G}^{con}(\mathcal{S})$ is a lace.*

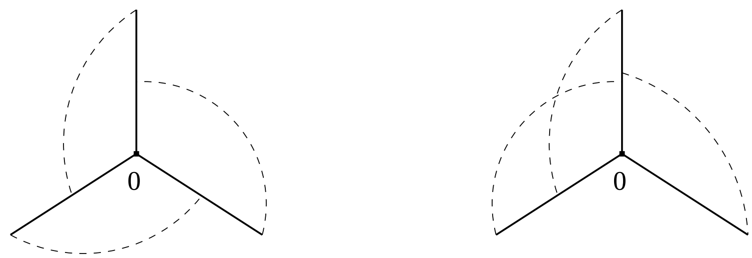


Figure 2.6: Basic examples of a cyclic and an acyclic lace.

*(b) For any non-minimal lace $L \in \mathcal{L}(\mathcal{S}^3)$, there exists a bond $st \in L$ (that covers the branch point) such that $L \setminus st \in \mathcal{L}(\mathcal{S})$ and $L \setminus st$ is minimal.*

*Proof.* For (a), let $\Gamma \in \mathcal{G}^{con}(\mathcal{S})$, and let $b$ be as in Definition 2.1.4. Let $st \in \Gamma$ cover $b$ and suppose $s \in \mathcal{S}_{e_1}$ and $t \in \mathcal{S}_{e_2}$ where $\mathcal{S}_{e_i}$ are branches of $\mathcal{S}$ (we may have $e_1 = e_2$). If $st$ is not the bond in $\Gamma$ associated to $\mathcal{S}_{e_i}$ then $\Gamma \setminus st$ covers $\mathcal{S}_{e_i}$. Therefore if $st$ is not the bond associated to either $\mathcal{S}_{e_1}$ or $\mathcal{S}_{e_2}$ then $\Gamma \setminus st$ covers $\mathcal{S}$ so that $\Gamma$ is not minimal. By Definition 2.1.3 this is enough to prove (a).

For (b), let $L \in \mathcal{L}(\mathcal{S}^3)$ be non-minimal. Then there exists $st \in L$ such that $L \setminus st$ is connected. By Definition 2.1.3, $st$ must be the edge in $L$ associated to some branch $e$, and in particular it covers the branch point. Since $\mathcal{S}^3$ is a star shape of degree 3 this means that $L$ contains exactly 3 bonds covering the branch point. Now observe that $L \setminus st$ satisfies the definition of a lace, and contains exactly 2 bonds covering the branch point. It follows immediately that $L \setminus st$ is minimal since a graph $\Gamma$ with only 1 bond covering the branch point of $\mathcal{S}^3$ cannot be a connected graph on $\mathcal{S}^3$. $\square$

As in part (b) of Lemma 2.1.8, a non-minimal lace contains a bond $st$ that is "removable" in the sense that $L \setminus st$ is still a lace. In general such a bond is not unique. One can easily construct a lace on a star shaped network of degree 3 for which each of the bonds $s_1t_1, \ldots, s_3t_3$ covering the branch point satisfy $L \setminus s_it_i \in \mathcal{L}(\mathcal{S})$.

**Definition 2.1.9 (Cyclic).** *A lace on a star shaped network $\mathcal{S}^3$ is* cyclic *if the edges covering the branch point can be ordered as $\{s_kt_k : k = 1, \ldots, 3\}$, with $t_k$ and $s_{k+1}$ on the same branch for each $k$ (with $s_4$ identified with $s_1$). A lace that is not cyclic is called* acyclic.

See Figure 2.6 for an example of this classification.

## 2.2 The Expansion

Here we examine products of the form $\prod_{st\in\mathbf{E}_{\mathcal{N}}}[1+U_{st}]$. Following the method of [22] we can express such a product as

$$\prod_{st\in\mathbf{E}_{\mathcal{N}}}[1+U_{st}]=\prod_{st\in\mathbf{E}_{\mathcal{N}}\setminus\mathcal{R}}[1+U_{st}]-\left(\prod_{st\in\mathbf{E}_{\mathcal{N}}\setminus\mathcal{R}}[1+U_{st}]\right)\left(1-\prod_{st\in\mathcal{R}}[1+U_{st}]\right). \quad (2.4)$$

Define $K(\mathcal{M})=\prod_{st\in\mathbf{E}_{\mathcal{M}}\setminus\mathcal{R}}[1+U_{st}]$. Expanding such a product we obtain, for each possible subset of $\mathbf{E}_{\mathcal{M}}\setminus\mathcal{R}$, a product of $U_{st}$ for $st$ in that subset. The subsets of $\mathbf{E}_{\mathcal{M}}\setminus\mathcal{R}$ are precisely the graphs on $\mathcal{M}$ which contain no elements of $\mathcal{R}$, hence

$$K(\mathcal{M})=\sum_{\Gamma\in\mathcal{G}_{\mathcal{M}}^{-\mathcal{R}}}\prod_{st\in\Gamma}U_{st}, \quad (2.5)$$

where the empty product $\prod_{st\in\emptyset}U_{st}=1$ by convention. Similarly we define

$$J(\mathcal{M})=\sum_{\Gamma\in\mathcal{G}_{\mathcal{N}}^{-\mathcal{R},con}}\prod_{st\in\Gamma}U_{st}. \quad (2.6)$$

If $\mathcal{M}$ is a single vertex then $J(\mathcal{M})=1$. If $\mathcal{S}$ is a star-shaped network of degree 1 or 3 then

$$\begin{aligned}J(\mathcal{S})&=\sum_{L\in\mathcal{L}(\mathcal{S})}\sum_{\substack{\Gamma\in\mathcal{G}_{\mathcal{S}}^{con}:\\ \boldsymbol{L}_\Gamma=L}}\prod_{st\in\Gamma}U_{st}=\sum_{L\in\mathcal{L}(\mathcal{S})}\prod_{st\in L}U_{st}\sum_{\substack{\Gamma\in\mathcal{G}_{\mathcal{S}}^{con}:\\ \boldsymbol{L}_\Gamma=L}}\prod_{s't'\in\Gamma\setminus L}U_{s't'}\\ &=\sum_{L\in\mathcal{L}(\mathcal{S})}\prod_{st\in L}U_{st}\sum_{\Gamma'\subset\mathcal{C}(L)}\prod_{s't'\in\Gamma'}U_{s't'}=\sum_{N=1}^{\infty}\sum_{L\in\mathcal{L}^N(\mathcal{S})}\prod_{st\in L}U_{st}\prod_{s't'\in\mathcal{C}(L)}[1+U_{s't'}],\end{aligned} \quad (2.7)$$

where the second to last equality holds since for fixed $L$, $\{\Gamma\in\mathcal{G}_{\mathcal{S}}^{con}:\boldsymbol{L}_\Gamma=L\}=\{L\cup\Gamma':\Gamma'\subseteq\mathcal{C}(L)\}$ by Proposition 2.1.6. The last equality holds as in the discussion preceding (2.5) since expanding $\prod_{s't'\in\mathcal{C}(L)}[1+U_{s't'}]$ we obtain for each possible subset of $\mathcal{C}(L)$, a product of $U_{st}$ for $st$ in that subset.

**Recursion type expression for $K(\mathcal{N})$**

Recall that $\mathcal{N}=\mathcal{N}(\alpha,\vec{n})$ where $\alpha\in\Sigma_r$ and $\vec{n}\in\mathbb{N}^{2r-3}$, for some $r\geq 2$. If $r=2$ then let $b$ be the root of $\mathcal{N}$. Otherwise let $b$ be the branch point neighbouring the root of $\mathcal{N}$. In each case let $\mathcal{S}_{\mathcal{N}}^-$ be the largest connected subnetwork of $\mathcal{N}$ containing $b$ and no vertices that are adjacent to any other branch points of $\mathcal{N}$ ($\mathcal{S}_{\mathcal{N}}^-$ could be empty or a single vertex). Observe that for any graph $\Gamma\in\mathcal{G}_{\mathcal{N}}^{-\mathcal{R}}\setminus\mathcal{E}_{\mathcal{N}}^b$, the subnetwork $\mathcal{A}_b(\Gamma)$ contains no branch point of $\mathcal{N}$ other than $b$ (if $r\geq 3$) and hence is a star shape of degree 0, 1 or 3.

**Definition 2.2.1.** *If $\mathcal{M}$ is a connected subnetwork of $\mathcal{N}$ then we define $\mathcal{N}\setminus\mathcal{M}$ to be the set of vertices of $\mathcal{N}$ that are not in $\mathcal{M}$ together with the edges of $\mathcal{N}$ connecting them. In general $(\mathcal{N}\setminus\mathcal{M})\cup\mathcal{M}$ contains fewer edges than $\mathcal{N}$, and $\mathcal{N}\setminus\mathcal{M}$ need not be connected. However if $\mathcal{M}\subset\mathcal{S}^-_{\mathcal{N}}$ then $\mathcal{N}\setminus\mathcal{M}$ has at most 3 connected components (at most 1 if $r=2$) and we write $(\mathcal{N}\setminus\mathcal{M})_i$, $i=1,2,3$ for these components, where we allow $(\mathcal{N}\setminus\mathcal{M})_i=\emptyset$.*

Definition 2.2.1 allows us to write

$$\begin{aligned} K(\mathcal{N}) &= \sum_{\Gamma\in\mathcal{G}^{-\mathcal{R}}_{\mathcal{N}}\setminus\mathcal{E}^b_{\mathcal{N}}}\prod_{st\in\Gamma}U_{st}+\sum_{\Gamma\in\mathcal{E}^b_{\mathcal{N}}}\prod_{st\in\Gamma}U_{st} \\ &= \sum_{\substack{\mathcal{A}\subset\mathcal{S}^-_{\mathcal{N}}:\\ b\in\mathcal{A}}}\sum_{\Gamma\in\mathcal{G}^{con}_{\mathcal{A}}}\prod_{st\in\Gamma}U_{st}\prod_{i=1}^{3}\sum_{\Gamma_i\in\mathcal{G}^{-\mathcal{R}}_{(\mathcal{N}\setminus\mathcal{A})_i}}\prod_{s^it^i\in\Gamma_i}U_{s^it^i}+\sum_{\Gamma\in\mathcal{E}^b_{\mathcal{N}}}\prod_{st\in\Gamma}U_{st}, \end{aligned} \tag{2.8}$$

where the sum over $\mathcal{A}$ is a sum over connected subnetworks of $\mathcal{N}$ containing $b$ and no vertices adjacent to any other branch points of $\mathcal{N}$. Some of the $(\mathcal{N}\setminus\mathcal{A})_i$ may be a single vertex or empty and we define $\sum_{\Gamma_i\in\mathcal{G}_\emptyset}\prod_{s^it^i\in\Gamma_i}U_{s^it^i}=1$. Defining $E^{(b)}(\mathcal{N})=\sum_{\Gamma\in\mathcal{E}^b_{\mathcal{N}}}\prod_{st\in\Gamma}U_{st}$, we have

$$K(\mathcal{N})\quad\sum_{\substack{\mathcal{A}\subset\mathcal{S}^-_{\mathcal{N}}:\\ b\in\mathcal{A}}}J(\mathcal{A})\prod_{i=1}^{3}K((\mathcal{N}\setminus\mathcal{A})_i)+E^{(b)}(\mathcal{N}). \tag{2.9}$$

Depending on $\mathcal{N}$, the first term of (2.9) may be zero since $\mathcal{S}^-_{\mathcal{N}}$ may be empty. The fact that for any $\mathcal{A}$ contributing to this first term, the subtrees $(\mathcal{N}\setminus\mathcal{A})_i$ are of degree $r_i<r$ is what allows for an inductive proof of Theorem 1.4.5.

If $r=2$ then $\mathcal{N}$ contains no branch point. In this case we may identify the star-shaped network $\mathcal{S}^1(m)$ with the interval $[0,m]$ and (2.8)-(2.9) reduce to

$$K([0,n])=\sum_{m\le n}J([0,m])K([m+1,n]), \tag{2.10}$$

which is the usual relation for the expansion of $K(\cdot)$ on an interval for this notion of connectivity (see for example [11]). Otherwise $b$ is a branch point of $\mathcal{N}$ and we let $K(\emptyset)\equiv 1$, and $I_i=I_i(\mathcal{N})$ be the indicator function that the branch $i$ is incident to $b$ and another branch point $b_i$. Therefore for a fixed network $\mathcal{N}$, $n_i-2I_2=n_i-2I_2(\mathcal{N})$ is equal to either $n_2-2$ (if branch 2 is incident to $b$ and another branch point $b_i$) or $n_i$. Then (2.8)-(2.9) give

$$K(\mathcal{N})=\sum_{m_1\le n_1}\sum_{\substack{m_2\le n_2-2I_2\\ m_3\le n_3-2I_3}}J(\mathcal{S}^\Delta(\vec{m}))\prod_{i=1}^{3}K((\mathcal{N}\setminus\mathcal{S}^\Delta(\vec{m}))_i)+E^{(b)}(\mathcal{N}), \tag{2.11}$$

where $\mathcal{S}^{\Delta}(\vec{m})$ is a star-shaped network satisfying

$$\mathcal{S}^{\Delta}(\vec{m}) = \begin{cases} \{b\} & , \text{ if } \vec{m} = \vec{0} \\ \mathcal{S}^3(\vec{m}) & , \text{ if } m_i \neq 0 \text{ for all } i \\ \mathcal{S}[0, m_i] & , \text{ if } m_i \neq 0, \text{ and } m_j = 0 \text{ for } j \neq i \\ \mathcal{S}[-m_j, m_i] & , \text{ if } j > i, \ m_j \neq 0, \ m_i \neq 0, \text{ and } m_k = 0 \text{ for } k \neq i, j. \end{cases} \tag{2.12}$$

In the case where there is another branch point $b_e$ that is adjacent to $b$ in $\mathcal{N}$ (so that $n_2$ or $n_3$ is 1), the sum over at least one of $m_2, m_3$ in (2.11). However note that this case contributes to the term $E^{(b)}(\mathcal{N})$, as required.

The combinatorial analysis of

- $E^{(b)}(\mathcal{N})$ and
- the contribution to (2.4) from graphs containing a bond in $\mathcal{R}$

is difficult and we postpone it until Chapter 6. Neither term appears in our analysis of the 2-point function in Chapter 3.

# Chapter 3

# The 2-point function

## 3.1 Organisation

In this chapter we prove Theorem 1.4.3 using an extension of the inductive approach to the lace expansion of [19]. The extension of the induction approach is described and proved in a general setting in Appendix A. Broadly speaking there are two main ingredients involved in applying the results of Appendix A. Firstly we must obtain a *recursion relation* for the quantity of interest, the Fourier transform of the 2-point function, and massage this relation so that it takes the form

$$\begin{aligned} &f_{n+1}(k;z) = \sum_{m=1}^{n+1} g_m(k;z) f_{n+1-m}(k;z) + e_{n+1}(k;z), \text{ with} \\ &f_0(k;z) = 1, \qquad f_1(k;z) = z\widehat{D}(k), \qquad e_1(k;z) = 0. \end{aligned} \tag{3.1}$$

Secondly we must verify the hypotheses that certain bounds on the quantities $f_m$ for $1 \leq m \leq n$ appearing in (3.1) imply further bounds on the quantities $g_m, e_m$, for $2 \leq m \leq n+1$. This second ingredient consists of reducing the bounds required to diagrammatic estimates, and then estimating the relevant diagrams.

In Section 3.2 we prove a recursion relation of the form (3.1) for a quantity closely related to the Fourier transform of the 2-point function. In Section 3.3 we state the assumptions of the inductive approach for a specific choice of parameters corresponding to our particular model. In Section 3.4 we reduce the verification of these assumptions to proving a single result, Proposition 3.4.1. Assuming Proposition 3.4.1, the induction approach then yields Theorem 3.4.3, which we show in Section 3.5 implies Theorem 1.4.3.

The diagrammatic estimates involved in proving Proposition 3.4.1 provide the most model dependent aspect of the analysis and these are postponed until Chapter 5.

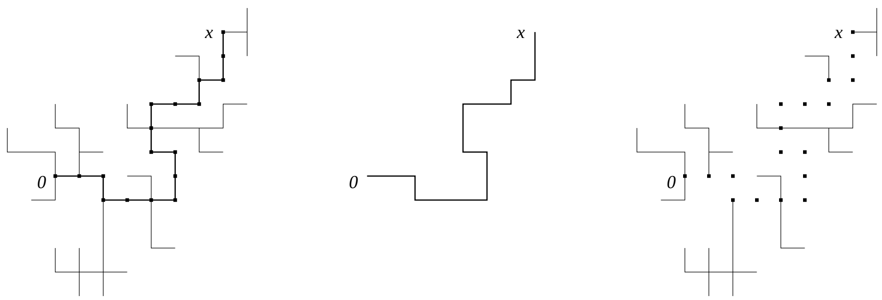


Figure 3.1: The first figure is of a lattice tree $T \in \mathcal{T}_n(0,x)$ for $n = 17$. The second figure shows the backbone which is also a (self-avoiding) walk $\omega$, while the third shows the branches emanating from the backbone, which are also mutually avoiding lattice trees $R_0, \ldots, R_n$.

## 3.2 Recursion relation for the 2-point function

Recall Definitions 1.2.4, 1.2.6, and 1.2.8. Also recall from Definition 1.4.1 that the two point function is defined as

$$t_n(x) = \zeta^n \sum_{T \in \mathcal{T}_n(x)} W(T). \tag{3.2}$$

Every tree $T \in \mathcal{T}_n(x)$ consists of a unique backbone (which is a self-avoiding walk) $\omega$ connecting $0 = \omega(0)$ to $x = \omega(n)$ that contains $n$ bonds, together with branches emanating from each vertex in the backbone. The branches emanating from the backbone vertices are themselves lattice trees $R_0, \ldots, R_n$, and by the definition of lattice tree (applied to $T$) they must be mutually avoiding. Since each $R_i$ contains the vertex $\omega(i)$, the mutual avoidance of the $R_i$ incorporates the self-avoidance of the backbone $\omega$. See Figure 3.1 for a pictorial view of this discussion. Let

$$U_{st} = U(R_s, R_t) = \begin{cases} -1, & \text{if } R_s \cap R_t \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then $\prod_{0 \leq s < t \leq n}[1 + U_{st}]$ is the indicator function that all the $R_i$ avoid each other. Summarising the above discussion and using the fact that the weight $W(T)$ of a tree factorises into (bond) disjoint components (see Definition 1.2.7) we can write,

$$\begin{aligned} t_n(x;\zeta) = \zeta^n & \sum_{\substack{\omega : 0 \to x, \\ |\omega| = n}} W(\omega) \times \\ & \sum_{R_0 \in \mathcal{T}_{\omega(0)}} W(R_0) \sum_{R_1 \in \mathcal{T}_{\omega(1)}} W(R_1) \cdots \sum_{R_n \in \mathcal{T}_{\omega(n)}} W(R_n) \prod_{0 \leq s < t \leq n} [1 + U_{st}], \end{aligned} \tag{3.4}$$

where the first sum is over *simple random walks* of length $n$ from 0 to $x$. To simplify this expression, we abuse notation and replace (3.4) with

$$t_n(x;\zeta) = \zeta^n \sum_{\substack{\omega:0\to x,\\ |\omega|=n}} W(\omega) \prod_{i=0}^{n} \sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i) \prod_{0\le s<t\le n} [1+U_{st}]\,. \tag{3.5}$$

Recall Definition 2.1.1 and the discussion following it. The set of vertices $[0,n]$ corresponds to the set of vertices of $\mathcal{N}(\alpha,n)$, where $\alpha$ is the unique shape in $\Sigma_2$. Since this $\mathcal{N}$ contains no branch points, we have $\mathcal{R}=\emptyset$ and therefore from Section 2.2 we have $\prod_{0\le s<t\le n}[1+U_{st}] = K(\mathcal{N}) = K([0,n])$. Hence

$$t_n(x;\zeta) = \zeta^n \sum_{\substack{\omega:0\to x\\ |\omega|=n}} W(\omega) \prod_{i=0}^{n} \sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i) K([0,n]). \tag{3.6}$$

**Definition 3.2.1.** *For $m\ge 0$ we define*

$$\pi_m(x;\zeta) = \zeta^m \sum_{\substack{\omega:0\to x\\ |\omega|=m}} W(\omega) \prod_{i=0}^{m} \sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i) J([0,m]). \tag{3.7}$$

*Note that for $m=0$ this is simply $\sum_{R_0\in\mathcal{T}_0} W(R_i) = \rho(0)$ if $x=0$ and zero otherwise.*

**Definition 3.2.2.** *Let $f,g$. We define the convolution of absolutely summable functions $f$ and $g$ to be the function*

$$(f*g)(x) = \sum_{y\in\mathbb{Z}^d} f(y)g(x-y). \tag{3.8}$$

*Clearly, by the substitution $u=x-y$ we have $(f*g)=(g*f)$. Moreover since $\sum_{y,z\in\mathbb{Z}^d} |f(y)g(z-y)h(x-z)| < \infty$ by Fubini, $(f*(g*h))(x) = ((f*g)*h)(x)$, and we can do pairwise convolutions in any order.*

The following recursion relation is the starting point for obtaining a relation of the form (3.1).

**Proposition 3.2.3.**

$$t_{n+1}(x;\zeta) = \sum_{m=1}^{n} (\pi_m * \zeta p_c D * t_{n-m})(x;\zeta) + \pi_{n+1}(x;\zeta) + \rho(0)(\zeta p_c D * t_n)(x;\zeta). \tag{3.9}$$

*Proof.* By definition

$$t_{n+1}(x;\zeta) = \zeta^{n+1} \sum_{\substack{\omega : 0 \to x, \\ |\omega| = n+1}} W(\omega) \prod_{i=0}^{n+1} \sum_{R_i \in \mathcal{T}_{\omega(i)}} W(R_i) K([0, n+1]). \tag{3.10}$$

Equation (2.10) gives

$$K([0,n+1]) = K([1,n+1]) + \sum_{m=1}^{n} J([0,m]) K([m+1,n+1]) + J([0,n+1]). \tag{3.11}$$

Putting this expression into equation (3.10) gives rise to three terms which we consider separately.

1. The contribution from graphs for which 0 is not covered by any bond: For this term we break the backbone from 0 to $x$ (a walk of length $n+1$) into a single step walk and the remaining $n$-step walk as follows.

$$\begin{aligned} &\zeta^{n+1} \sum_{\substack{\omega : 0 \to x, \\ |\omega| = n+1}} W(\omega) \prod_{i=0}^{n+1} \sum_{R_i \in \mathcal{T}(\omega(i))} W(R_i) K[1, n+1] \\ &= \sum_{R_0 \in \mathcal{T}_0} W(R_0) \sum_{y \in \Omega_D} \sum_{\substack{\omega_1 : 0 \to y, \\ |\omega_1| = 1}} \zeta W(\omega_1) \times \\ &\quad \sum_{\substack{\omega_2 : y \to x, \\ |\omega_2| = n}} \zeta^n W(\omega_2) \prod_{i=1}^{n+1} \sum_{R_i \in \mathcal{T}(\omega_2(i-1))} W(R_i) K[1, n+1], \end{aligned} \tag{3.12}$$

where $K[1, n+1]$ depends on $R_1, \ldots, R_{n+1}$ but not $R_0$. Therefore using the substitutions $R'_j = R_{j+1}$ this is equal to

$$\begin{aligned} &\rho(0) \sum_{y \in \Omega_D} \sum_{\substack{\omega_1 : 0 \to y, \\ |\omega_1| = 1}} \zeta W(\omega_1) \times \\ &\quad \sum_{\substack{\omega_2 : y \to x, \\ |\omega_2| = n}} \zeta^n W(\omega_2) \prod_{j=0}^{n} \sum_{R'_j \in \mathcal{T}(\omega_2(j))} W(R'_j) K[0, n] \\ &= \rho(0) \sum_{y \in \Omega_D} p_c \zeta D(y) t_n(x-y;\zeta) \\ &= \rho(0) p_c \zeta (D * t_n)(x). \end{aligned} \tag{3.13}$$

2. The contribution from graphs which are connected on $[0, n+1]$:

$$\zeta^{n+1} \sum_{\substack{\omega : 0 \to x, \\ |\omega| = n+1}} W(\omega) \prod_{i=0}^{n+1} \sum_{R_i \in \mathcal{T}_{\omega(i)}} W(R_i) J([0, n+1]) = \pi_{n+1}(x; \zeta) \quad (3.14)$$

3. The contribution from graphs which are connected on $[0, m]$ for some $m \in \{1, \ldots, n\}$: For this term we break the backbone from 0 to $x$ (a walk of length $n+1$) up into three walks, of lengths $m, 1, n-m$ respectively

$$\zeta^{n+1} \sum_{\substack{\omega : 0 \to x, \\ |\omega| = n+1}} W(\omega) \prod_{i=0}^{n+1} \sum_{R_i \in \mathcal{T}_{\omega(i)}} W(R_i) \sum_{m=1}^{n} J[0, m] K[m+1, n+1]$$

$$= \sum_{m=1}^{n} \sum_{u} \sum_{v} \sum_{\substack{\omega_1 : 0 \to u, \\ |\omega_1| = m}} \zeta^m W(\omega_1) \left( \prod_{i=0}^{m} \sum_{R_i \in \mathcal{T}_{\omega_1(i)}} W(R_i) \right) J[0, m] \times$$

$$\sum_{\substack{\omega_2 : u \to v, \\ |\omega_2| = 1}} \zeta W(\omega_2) \times$$

$$\sum_{\substack{\omega_3 : v \to x \\ |\omega_3| = n-m}} \zeta^{n-m} W(\omega_3) \left( \prod_{i=m+1}^{n+1} \sum_{R_i \in \mathcal{T}_{\omega_3(i-(m+1))}} W(R_i) \right) K[m+1, n+1]. \quad (3.15)$$

Now $[0, m]$ and $[m+1, n+1]$ are disjoint, so $J([0, m])$ and $K([m+1, n+1])$ contain information about disjoint subsets of $\{R_i : i \in \{0, \ldots, n+1\}\}$. Using the substitutions $R'_j = R_{j+m+1}$ this is equal to:

$$\sum_{m=1}^{n}\sum_{u}\sum_{v} \sum_{\substack{\omega_1 : 0 \to u, \\ |\omega_1| = m}} \zeta^m W(\omega_1) \left( \prod_{i=0}^{m} \sum_{R_i \in \mathcal{T}_{\omega_1(i)}} W(R_i) \right) J[0,m] \times$$

$$p_c \zeta D(v-u) \sum_{\substack{\omega_3 : v \to x \\ |\omega_3| = n-m}} \zeta^{n-m} W(\omega_3) \left( \prod_{j=0}^{n-m} \sum_{R'_j \in \mathcal{T}_{\omega_3(j)}} W(R_j) \right) K[0, n-m]$$

$$= \sum_{m=1}^{n}\sum_{u}\sum_{v} \pi_m(u;\zeta) p_c \zeta D(v-u) t_{n-m}(x-v;\zeta)$$

$$= \sum_{m=1}^{n} (\pi_m * p_c \zeta D * t_{n-m})(x;\zeta). \tag{3.16}$$

□

Dividing both sides of Equation (3.9) by $\rho(0)$ and taking Fourier transforms we get

$$\frac{\widehat{t}_{n+1}(k;\zeta)}{\rho(0)} = \sum_{m=1}^{n} \frac{\widehat{\pi}_m(k;\zeta)}{\rho(0)} \rho(0)\zeta p_c \widehat{D}(k) \frac{\widehat{t}_{n-m}(k;\zeta)}{\rho(0)} + \frac{\widehat{\pi}_{n+1}(k;\zeta)}{\rho(0)} + \rho(0)\zeta p_c \widehat{D}(k) \frac{\widehat{t}_n(k;\zeta)}{\rho(0)}. \tag{3.17}$$

We now massage (3.17) into the form (3.1) required for the analysis of Appendix A.

**Definition 3.2.4.** *For fixed $\zeta \geq 0$, define*

*1) $z = \rho(0)\zeta p_c$.*

*2) $f_0(k;z) = 1$, $f_1(k;z) = g_1(k;z) = z\widehat{D}(k)$, and $e_1(k;z) = 0$.*

*3) For $n \geq 2$,*

$$\begin{aligned} f_n(k;z) &= \frac{\widehat{t}_n(k;\zeta)}{\rho(0)}, \qquad g_n(k;z) = \frac{\widehat{\pi}_{n-1}(k;\zeta)}{\rho(0)} z\widehat{D}(k) \\ e_n(k;z) &= g_{n-1}(k;z) \left[ \frac{\widehat{t}_1(k;\zeta)}{\rho(0)} - z\widehat{D}(k) \right] + \frac{\widehat{\pi}_n(k;\zeta)}{\rho(0)}. \end{aligned} \tag{3.18}$$

We note from (3.17) with $n = 0$ that since $t_0(x) = \rho(0) I_{x=0}$,we have $\widehat{t}_0(k) = \rho(0)$ and

$$\frac{\widehat{t}_1(k;\zeta)}{\rho(0)} - z\widehat{D}(k) = \frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)}. \tag{3.19}$$

Therefore for $n \geq 2$

$$e_n(k;z) = g_{n-1}(k;z)\frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)} + \frac{\widehat{\pi}_n(k;\zeta)}{\rho(0)}. \tag{3.20}$$

For $n \geq 3$ this is

$$e_n(k;z) = \frac{\widehat{\pi}_{n-2}(k;\zeta)}{\rho(0)} z\widehat{D}(k)\frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)} + \frac{\widehat{\pi}_n(k;\zeta)}{\rho(0)}. \tag{3.21}$$

**Lemma 3.2.5.** *The choices of $f_m$, $g_m$, $e_m$ above satisfy Equation (3.1).*

*Proof.* The case $n = 0$ is trivially true by definition of $f_0$, $f_1$, $g_1$ and $e_1$. We use (3.19-3.20) for the case $n = 1$ so that,

$$\begin{aligned}
&\sum_{m=1}^{n+1} g_m(k;z)f_{n+1-m}(k;z) + e_{n+1}(k;z) \\
&= g_1(k;z)f_1(k;z) + g_2(k;z)f_0(k;z) + e_2(k;z) \\
&= z\widehat{D}(k)z\widehat{D}(k) + \frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)} z\widehat{D}(k) + z\widehat{D}(k)\left[\frac{\widehat{t}_1(k;\zeta)}{\rho(0)} - z\widehat{D}(k)\right] + \frac{\widehat{\pi}_2(k;\zeta)}{\rho(0)} \\
&= \frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)} z\widehat{D}(k) + z\widehat{D}(k)\frac{\widehat{t}_1(k;\zeta)}{\rho(0)} + \frac{\widehat{\pi}_2(k;\zeta)}{\rho(0)} \\
&= \frac{\widehat{t}_2(k;\zeta)}{\rho(0)},
\end{aligned} \tag{3.22}$$

by (3.17) for $n = 1$. For $n \geq 2$,

$$\begin{aligned}
&\sum_{m=1}^{n+1} g_m(k;z)f_{n+1-m}(k;z) + e_{n+1}(k;z) \\
&= g_1(k;z)f_n(k;z) + g_n(k;z)f_1(k;z) + g_{n+1}(k;z)f_0(k;z) + e_{n+1}(k;z) + \\
&\quad \sum_{m=2}^{n-1} g_m(k;z)f_{n+1-m}(k;z) \\
&= z\widehat{D}(k)\frac{\widehat{t}_n(k;\zeta)}{\rho(0)} + \frac{\widehat{\pi}_{n-1}(k;\zeta)}{\rho(0)} z\widehat{D}(k)z\widehat{D}(k) + \frac{\widehat{\pi}_n(k;\zeta)}{\rho(0)} z\widehat{D}(k) + \\
&\quad \frac{\widehat{\pi}_{n-1}(k;\zeta)}{\rho(0)} z\widehat{D}(k)\left[\frac{\widehat{t}_1(k;\zeta)}{\rho(0)} - z\widehat{D}(k)\right] + \frac{\widehat{\pi}_{n+1}(k;\zeta)}{\rho(0)} + \\
&\quad \sum_{m=2}^{n-1} \frac{\widehat{\pi}_{m-1}(k;\zeta)}{\rho(0)} z\widehat{D}(k)\frac{\widehat{t}_{n+1-m}(k;\zeta)}{\rho(0)}.
\end{aligned} \tag{3.23}$$

The second term cancels with the second part of the fourth term. The last term added to the third term and the first part of the fourth term gives

$$\sum_{m=1}^{n} \frac{\widehat{\pi}_m(k;\zeta)}{\rho(0)} z\widehat{D}(k) \frac{\widehat{t}_{n-m}(k;\zeta)}{\rho(0)}, \tag{3.24}$$

which appears on the right side of (3.17). The remaining terms here are the remaining terms on the right side of (3.17), hence by (3.17) the entire quantity is equal to $\frac{\widehat{t}_{n+1}(k;\zeta)}{\rho(0)} = f_{n+1}(k;z)$ as required. □

## 3.3 Assumptions of the induction method

The induction approach to the lace expansion of [19] is extended in Appendix A with the introduction of two parameters $\theta$ and $p^*$ and a set $B \subset [1, p^*]$. In this chapter we apply the extension with the choices $\theta = \frac{d-4}{2}$, $p^* = 2$, $B = \{2\}$ and we define $\beta = L^{-\frac{d}{p^*}} = L^{-\frac{d}{2}}$. The induction method is discussed thoroughly in Appendix A, and so we simply restate the assumptions in this section, and verify them in the next section.

We have already shown in Section 3.2 that for our choices of $f_m, g_m, e_m$ as given in Definition 3.2.4,

$$f_{n+1}(k;z) = \sum_{m=1}^{n+1} g_m(k;z) f_{n+1-m}(k;z) + e_{n+1}(k;z) \qquad (n \geq 0), \tag{3.25}$$

with $f_0(k;z) = 1$.

**Assumption S**. For every $n \in \mathbb{N}$ and $z > 0$, the mapping $k \mapsto f_n(k;z)$ is symmetric under replacement of any component $k_i$ of $k$ by $-k_i$, and under permutations of the components of $k$. The same holds for $e_n(\cdot;z)$ and $g_n(\cdot;z)$. In addition, for each $n$, $|f_n(k;z)|$ is bounded uniformly in $k \in [-\pi,\pi]^d$ and $z$ in a neighbourhood of 1 (which may depend on $n$).

**Assumption D**. We assume that

$$f_1(k;z) = z\hat{D}(k), \quad e_1(k;z) = 0. \tag{3.26}$$

In particular, this implies that $g_1(k;z) = z\hat{D}(k)$. Define $a(k) = 1 - \hat{D}(k)$. As part of Assumption D, we also assume:

(i) $D$ is normalised so that $\hat{D}(0) = 1$, and has $2+2\epsilon$ moments for some $\epsilon > 0$, i.e.,

$$\sum_{x\in\mathbb{Z}^d} |x|^{2+2\epsilon} D(x) < \infty. \tag{3.27}$$

(ii) There is a constant $C$ such that, for all $L \geq 1$,

$$\|D\|_\infty \leq CL^{-d}, \qquad \sigma^2 = \sigma_L^2 \leq CL^2, \tag{3.28}$$

(iii) There exist constants $\eta, c_1, c_2 > 0$ such that

$$c_1 L^2 k^2 \leq a(k) \leq c_2 L^2 k^2 \quad (\|k\|_\infty \leq L^{-1}), \tag{3.29}$$

$$a(k) > \eta \quad (\|k\|_\infty \geq L^{-1}), \tag{3.30}$$

$$a(k) < 2 - \eta \quad (k \in [-\pi, \pi]^d). \tag{3.31}$$

For $h : [-\pi, \pi]^d \to \mathbb{C}$, we define

$$\nabla^2 h(k') = \sum_{j=1}^{d} \frac{\partial^2}{\partial k_j^2} h(k) \Bigg]_{k=k'}. \tag{3.32}$$

The relevant bounds on $f_m$, which *a priori* may or may not be satisfied, are that

$$\|\widehat{D}^2 f_m(\cdot; z)\|_2 \leq \frac{K}{L^{\frac{d}{2}} m^{\frac{d}{4}}}, \quad |f_m(0; z)| \leq K, \quad |\nabla^2 f_m(0; z)| \leq K\sigma^2 m, \tag{3.33}$$

for some positive constant $K$. We define

$$\beta = L^{-\frac{d}{2}}. \tag{3.34}$$

The bounds in (3.33) are identical to the ones in (A.13), with our choices if $p^* = 2$, $B = \{2\}$, and $\theta = \frac{d-4}{2}$.

**Assumption E**. There is an $L_0$, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0, 1)$, and a function $K \mapsto C_e(K)$, such that if (3.33) holds for some $K > 1$, $L \geq L_0$, $z \in I$ and for all $1 \leq m \leq n$, then for that $L$ and $z$, and for all $k \in [-\pi, \pi]^d$ and $2 \leq m \leq n+1$, the following bounds hold:

$$|e_m(k; z)| \leq C_e(K)\beta m^{-\frac{d-4}{2}}, \quad |e_m(k; z) - e_m(0; z)| \leq C_e(K) a(k) \beta m^{-\frac{d-6}{2}}. \tag{3.35}$$

**Assumption G**. There is an $L_0$, an interval $I \subset [1 - \alpha, 1 + \alpha]$ with $\alpha \in (0, 1)$, and a function $K \mapsto C_g(K)$, such that if (3.33) holds for some $K > 1$, $L \geq L_0$, $z \in I$ and for all $1 \leq m \leq n$, then for that $L$ and $z$, and for all $k \in [-\pi, \pi]^d$ and $2 \leq m \leq n+1$, the following bounds hold:

$$|g_m(k; z)| \leq C_g(K)\beta m^{-\frac{d-4}{2}}, \quad |\nabla^2 g_m(0; z)| \leq C_g(K)\sigma^2 \beta m^{-\frac{d-6}{2}}, \tag{3.36}$$

$$|\partial_z g_m(0; z)| \leq C_g(K)\beta m^{-\frac{d-6}{2}}, \tag{3.37}$$

$$|g_m(k; z) - g_m(0; z) - a(k)\sigma^{-2}\nabla^2 g_m(0; z)| \leq C_g(K)\beta a(k)^{1+\epsilon'} m^{-\frac{d-6}{2}+\epsilon'}, \tag{3.38}$$

with the last bound valid for any $\epsilon' \in [0, 1 \wedge (\frac{d-8}{2}))$.

## 3.4 Verifying assumptions

**Assumption S:** The quantities $f_n(k;z)$, $n = 0, 1, \ldots$ are (up to constants), Fourier transforms of $t_n(x,\zeta)$, which are symmetric by symmetry of $D$. Hence the $f_n$ have all required symmetries. Similarly $\pi_m(x,\zeta)$ are symmetric by symmetry of $D$, so that the quantities $g_n, e_n$ also have the required symmetries. Now $f_0 = 1$ is trivially uniformly bounded in $k$ and $z \leq 2$. Furthermore for $n \geq 1$, using the bound $\prod[1 + U_{st}] \leq 1$ in (3.5) we obtain $\sum_x t_n(x;\zeta) \leq (\zeta p_c)^n \rho(0)^{n+1} \sum_x D^{(n)}(x) = (\zeta p_c)^n \rho(0)^{n+1}$, where $D^{(n)}$ denotes the $n$-fold convolution of $D(\bullet)$. Therefore for $n \geq 1$, $|f_n(k,z)| \leq \frac{\sum_x t_n(x;\zeta)}{\rho(0)} \leq (\zeta p_c \rho(0))^n = z^n$ so that $f_n$ is bounded uniformly in $k \in [-\pi,\pi]^d$ and $z$ in a neighbourhood of 1 and therefore satisfies the weak bound of Assumption S.

**Assumption D:** By Definition 3.2.4 we have $f_1(k,z) = z\widehat{D}(k)$ and $e_1 = 0$. Additionally, all moments of $D$ are finite, so choosing $\epsilon = 1$ ensures that (3.27) and 3.28) hold trivially (see Remark 1.2.5). The remaining conditions (iii) are verified by van der Hofstad and Slade in [19].

We therefore turn our attention to verifying assumptions $E$ and $G$. Recall from Definition 3.2.4 and (3.20) that for $n \geq 2$, $g_n$ and $e_n$ could be expressed in terms of the quantities $\widehat{\pi}_m$ for $m \leq n$. In Chapter 5 we will prove the following proposition.

**Proposition 3.4.1 ($\pi_m$ bounds).** *Suppose the bounds (3.33) hold for some $z^* \in (0,2)$, $K > 1$, $L \geq L_0$ and every $m \leq n$. Then for that $K, L$, and for all $z \in [0,z^*]$, $m \leq n+1$ and $q \in \{0,1,2\}$,*

$$\sum_x |x|^{2q} |\pi_m(x;\zeta)| \leq \frac{C(K)\sigma^{2q}\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}-q}}, \tag{3.39}$$

*where $\zeta = \frac{z}{\rho(0)p_c}$, the constant $C = C(K,d)$ does not depend on $L$, $m$ and $z$, and $\nu > 0$ is the constant appearing in Theorem 1.2.9.*

We choose $\nu < 1$ in (1.13) so that $2 - \frac{6\nu}{d} > 1$ and therefore $\beta^{2-\frac{6\nu}{d}} \leq L^{-\frac{d}{2}}$. The proof of Proposition 3.4.1 involves reformulating $\pi_m$ in terms of laces and estimating Feynmann diagrams corresponding to those laces. For now we concentrate our efforts on verifying assumptions $E$ and $G$ assuming Proposition 3.4.1.

**Assumption E:** Suppose there is some $z^* \in (0,2)$, $K > 1$, $L \geq L_0$ such that (3.33) holds for all $m \leq n$. Let $z \in [0,z^*]$. Recall that $e_1(k;z) = 0$ and observe from (3.20) that

$$
\begin{aligned}
|e_2(k;z)| &= \left| z\widehat{D}(k)\frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)} + \frac{\widehat{\pi}_2(k;\zeta)}{\rho(0)} \right| \le z\left|\frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)}\right| + \left|\frac{\widehat{\pi}_2(k;\zeta)}{\rho(0)}\right| \\
&\le \frac{zC(K)\beta^{2-\frac{6\nu}{d}}}{\rho(0)} + \frac{C(K)\beta^{2-\frac{6\nu}{d}}}{\rho(0)2^{\frac{d-4}{2}}} \le \frac{C'(K)\beta^{2-\frac{6\nu}{d}}}{2^{\frac{d-4}{2}}},
\end{aligned} \tag{3.40}
$$

where we have applied Proposition 3.4.1 with $|\widehat{\pi}_m(k;\zeta)| \le \sum_x |\pi_m(x;\zeta)|$, and have also used $\rho(0) \ge 1$. Similarly for $3 \le m \le n+1$,

$$
\begin{aligned}
|e_m(k;z)| &= \left| \widehat{\pi}_{m-2}(k;\zeta)z\widehat{D}(k)\frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)^2} + \frac{\widehat{\pi}_m(k;\zeta)}{\rho(0)} \right| \\
&\le \frac{C(K)\beta^{2-\frac{6\nu}{d}}}{\rho(0)^2(m-2)^{\frac{d-4}{2}}} zC(K)\beta^{2-\frac{6\nu}{d}} + \frac{C(K)\beta^{2-\frac{6\nu}{d}}}{\rho(0)m^{\frac{d-4}{2}}} \le \frac{C'(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}}.
\end{aligned} \tag{3.41}
$$

Thus we have obtained the first bound of Assumption E. It follows immediately that

$$
|e_m(k;z) - e_m(0;z)| \le (|e_m(k;z)| + |e_m(0;z)|) \le \frac{C'(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}}, \tag{3.42}
$$

for all $m \ge 2$. By (3.30) this satisfies the second bound of Assumption E for $\|k\|_\infty \ge L^{-1}$. Thus it remains to establish the second bound of Assumption E for $\|k\|_\infty \le L^{-1}$, for which we use the method of [21].

Let $h : \mathbb{Z}^d \to \mathbb{R}$ be absolutely summable, and symmetric in each coordinate and under permutations of coordinates. Now

$$
\begin{aligned}
\left|\widehat{h}(k) - \widehat{h}(0)\right| &\le \left|\widehat{h}(k) - \widehat{h}(0) - \frac{|k|^2}{2d}\nabla^2\widehat{h}(0)\right| + \left|\frac{|k|^2}{2d}\nabla^2\widehat{h}(0)\right| \\
&= \left|\sum_x \left(\cos(k\cdot x) - 1 - \frac{|k|^2}{2d}\sum_{i=1}^d x_i^2\right) h(x)\right| + \left|\frac{|k|^2}{2d}\nabla^2\widehat{h}(0)\right|.
\end{aligned} \tag{3.43}
$$

By symmetry we have that

$$
\frac{1}{d}\sum_x |x|^2 h(x) = \frac{1}{d}\sum_{i=1}^d \sum_x x_i^2 h(x) = \sum_x x_j^2 h(x), \tag{3.44}
$$

which implies $-\frac{|k|^2}{d}\nabla^2\widehat{h}(0) = \sum_x \sum_{j=1}^d (k_j x_j)^2 h(x)$. On the other hand if $i \ne j$ then $\sum_x x_i x_j h(x) = 0$, so that $\sum_x (k\cdot x)^2 h(x)$ also equals $\sum_x \sum_{i=1}^d (k_i x_i)^2 h(x)$. Thus we can rewrite (3.43) as

$$
\left|\widehat{h}(k) - \widehat{h}(0)\right| \le \left|\sum_x \left(\cos(k\cdot x) - 1 + \frac{1}{2}(k\cdot x)^2\right) h(x)\right| + \left|\frac{|k|^2}{2d}\nabla^2\widehat{h}(0)\right|. \tag{3.45}
$$

We claim that there exists a constant $c$, such that for all $\eta \in [0,1]$, $|\cos(t) - 1 + \frac{1}{2}t^2| \leq ct^{2+2\eta}$. To see this note that for $|t| \geq 1$ the left hand side is bounded above by $2 + \frac{1}{2}t^2 \leq \frac{5}{2}t^2 \leq \frac{5}{2}t^{2+2\eta}$. For $|t| < 1$ the left hand side is bounded above by

$$\sum_{n=2}^{\infty} \frac{t^{2n}}{(2n)!} = t^{2+2\eta} \sum_{n=2}^{\infty} \frac{t^{2(n-1-\eta)}}{(2n)!} \leq t^{2+2\eta} \sum_{n=2}^{\infty} \frac{1}{(2n)!} \leq ct^{2+2\eta}, \tag{3.46}$$

where the constant is independent of $\eta$. This verifies the claim. Putting this result into (3.45) we get

$$\left|\widehat{h}(k) - \widehat{h}(0)\right| \leq C \sum_x |(k \cdot x)^{2+2\eta} h(x)| + \left|\frac{|k|^2}{2d} \nabla^2 \widehat{h}(0)\right|. \tag{3.47}$$

In particular if we choose $\eta = 0$ then (3.47) becomes

$$\begin{aligned} \left|\widehat{h}(k) - \widehat{h}(0)\right| &\leq C \sum_x \sum_{j=1}^{d} (k_j x_j)^2 |h(x)| + \frac{|k|^2}{2d} \nabla^2 \widehat{h}(0) \\ &\leq C|k|^2 \sum_x x_j^2 |h(x)|. \end{aligned} \tag{3.48}$$

Now $e_m(k;z) - e_m(0;z)$ is equal to

$$(g_{n-1}(k;z) - g_{n-1}(0;z)) \frac{\widehat{\pi}_1(k;\zeta)}{\rho(0)} + g_{n-1}(0;z) \frac{\widehat{\pi}_1(k;\zeta) - \widehat{\pi}_1(0;\zeta)}{\rho(0)} + \frac{\widehat{\pi}_m(k;\zeta) - \widehat{\pi}_m(0;\zeta)}{\rho(0)}. \tag{3.49}$$

By (3.47) with $\eta = 0$, and Proposition 3.4.1 with $q = 1$ we have that

$$|\widehat{\pi}_m(k;\zeta) - \widehat{\pi}_m(0;\zeta)| \leq C(K) k^2 \frac{\sigma^2 \beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}}. \tag{3.50}$$

Therefore $|e_m(k;z) - e_m(0;z)|$ is bounded above by

$$\begin{aligned} &|g_{m-1}(k;z) - g_{m-1}(0;z)| \frac{|\widehat{\pi}_1(k;\zeta)|}{\rho(0)} + |g_{m-1}(0;z)| C(K) k^2 \frac{\sigma^2 \beta^{2-\frac{6\nu}{d}}}{\rho(0)} + C(K) k^2 \frac{\sigma^2 \beta^{2-\frac{6\nu}{d}}}{\rho(0) m^{\frac{d-6}{2}}} \\ &\leq \frac{C(K) \beta^{2-\frac{6\nu}{d}}}{\rho(0)} \left( |g_{m-1}(k;z) - g_{m-1}(0;z)| + |g_{m-1}(0;z)| k^2 \sigma^2 + \frac{k^2 \sigma^2}{m^{\frac{d-6}{2}}} \right). \end{aligned} \tag{3.51}$$

Thus recalling that $g_1(k;z) = z\widehat{D}(k)$ we have

$$|e_2(k;z) - e_2(0;z)| \leq \frac{C(K) \beta^{2-\frac{6\nu}{d}}}{\rho(0)} \left( za(k) + zk^2 \sigma^2 + \frac{k^2 \sigma^2}{2^{\frac{d-6}{2}}} \right). \tag{3.52}$$

For $m \geq 3$, recall that $g_{m-1}(k;z) = \frac{\widehat{\pi}_{m-2}(k;\zeta)}{\rho(0)} z\widehat{D}(k)$ which gives

$$|g_{m-1}(k;z) - g_{m-1}(0;z)| \leq \frac{z}{\rho(0)} \left[ |\widehat{\pi}_{m-2}(k;\zeta) - \widehat{\pi}_{m-2}(0;\zeta)|\widehat{D}(0) + a(k)|\widehat{\pi}_{m-2}(0;\zeta)| \right]$$
$$\leq \frac{C(K)k^2\sigma^2\beta^{2-\frac{6\nu}{d}}}{(m-2)^{\frac{d-6}{2}}} + \frac{C(K)a(k)\beta^{2-\frac{6\nu}{d}}}{(m-2)^{\frac{d-4}{2}}}. \tag{3.53}$$

Therefore for $m \geq 3$,

$$|e_m(k;z) - e_m(0;z)| \leq C(K)\beta^{2-\frac{6\nu}{d}} \left( \frac{k^2\sigma^2\beta^{2-\frac{6\nu}{d}}}{(m-2)^{\frac{d-6}{2}}} + \frac{a(k)\beta^{2-\frac{6\nu}{d}}}{(m-2)^{\frac{d-4}{2}}} + \frac{zk^2\sigma^2\beta^{2-\frac{6\nu}{d}}}{(m-2)^{\frac{d-4}{2}}} + \frac{k^2\sigma^2}{m^{\frac{d-6}{2}}} \right). \tag{3.54}$$

Both (3.52) for $m = 2$ and (3.54) for $m \geq 3$ are bounded above by $\frac{C'(K)a(k)\beta}{m^{\frac{d-6}{2}}}$ for $\|k\|_\infty \leq L^{-1}$ by (3.29) and the fact that $\sigma^2 \sim L^2$ (see Remark 1.2.5).

**Assumption G:** Suppose there is some $z^* \in (0,2)$, $K > 1$, $L \geq L_0$ such that (3.33) holds for all $m \leq n$. Let $z \in [0, z^*]$. As for Assumption E, we may apply Proposition 3.4.1 to obtain for $2 \leq m \leq n+1$

$$|g_m(k;z)| = \left| z\widehat{D}(k) \frac{\widehat{\pi}_{m-1}(k;\zeta)}{\rho(0)} \right| \leq \frac{zC(K)\beta^{2-\frac{6\nu}{d}}}{\rho(0)(m-1)^{\frac{d-4}{2}}} \leq \frac{C'(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}}, \tag{3.55}$$

which gives the first bound of Assumption G.

For the second bound we note that by symmetry the first derivatives of $\widehat{\pi}_m$ and $\widehat{D}$ vanish at 0. Hence for $m \geq 2$

$$|\nabla^2 g_m(0;z)| = \left| \nabla^2 \left[ z\widehat{D}(k) \frac{\widehat{\pi}_{m-1}(k;\zeta)}{\rho(0)} \right]_{k=0} \right| = \frac{z}{\rho(0)} |\nabla^2 \widehat{\pi}_{m-1}(0) + \widehat{\pi}_{m-1}(0)\nabla^2 \widehat{D}(0)|$$
$$\leq \frac{z}{\rho(0)} \left( \frac{C(K)\beta^{2-\frac{6\nu}{d}}\sigma^2}{m^{\frac{d-6}{2}}} + \frac{C(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}}\sigma^2 \right) \leq \frac{C'(K)\beta^{2-\frac{6\nu}{d}}\sigma^2}{m^{\frac{d-6}{2}}}. \tag{3.56}$$

This verifies the second bound of Assumption G.

Next for $m \geq 2$, we have that

$$g_m(k;z) = \widehat{\pi}_{m-1}(k;\zeta) \frac{z\widehat{D}(k)}{\rho(0)} = z^m \left( \frac{\widehat{\pi}_{m-1}(k;\zeta)}{z^{m-1}} \right) \frac{\widehat{D}(k)}{\rho(0)} \tag{3.57}$$

where $\frac{\widehat{\pi}_{m-1}(k;\zeta)}{z^{m-1}}$ does not depend on $z$ (or $\zeta$). Therefore

$$|\partial_z g_m(k;z)| = \left| m z^{m-1} \left( \frac{\widehat{\pi}_{m-1}(k;\zeta)}{z^{m-1}} \right) \frac{\widehat{D}(k)}{\rho(0)} \right|$$
$$= \left| m\widehat{\pi}_{m-1}(k;\zeta) \frac{\widehat{D}(k)}{\rho(0)} \right| \leq \frac{C'(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}}, \tag{3.58}$$

which proves the third part of assumption G.

Now for $\|k\|_\infty \geq L^{-1}$, (3.30) applies and we have that for $m \geq 2$,

$$\begin{aligned} &|g_m(k;z) - g_m(0;z) - a(k)\sigma^{-2}\nabla^2 g_m(0;z)| \\ &\leq \frac{C'(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}} + \frac{C'(K)\beta^{2-\frac{3\nu}{d}}}{m^{\frac{d}{4}}} + a(k)\frac{C'(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}} \\ &\leq a(k)^2 \frac{C'_\eta(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}}, \end{aligned} \tag{3.59}$$

since $a(k) > \eta$, and where the constant depends on $\eta$. This satisfies the final part of assumption G for $\|k\|_\infty \geq L^{-1}$.

For $\|k\|_\infty \leq L^{-1}$, we again use the method of [21]. By the triangle inequality we bound $|g_m(k;z) - g_m(0;z) - a(k)\sigma^{-2}\nabla^2 g_m(0;z)|$ by

$$\left| g_m(k;z) - g_m(0;z) - \frac{|k|^2}{2d}\nabla^2 g_m(0;z) \right| + \left| (1-\widehat{D}(k))\sigma^{-2} - \frac{|k|^2}{2d} \right| |\nabla^2 g_m(0;z)|. \tag{3.60}$$

Recall that for $m \geq 2$, $g_m(k;z) = \frac{z}{\rho(0)}(\widehat{\pi_m * D})(k)$. On the first term we apply the analysis of the first term of (3.43), to the symmetric function $\pi_m * D$. Choosing $\eta = \epsilon'$ we see that the first term of (3.60) is bounded by

$$zC|k|^{2+2\epsilon'} \sum_x |x|^{2+2\epsilon'} |(\pi_{m-1} * D)(x)|, \tag{3.61}$$

with the constant independent of $\epsilon'$. We claim that

$$\sum_x |x|^{2+2\epsilon'}|(\pi_{m-1}*D)(x)| \leq \left( \sum_x |(\pi_{m-1} * D)(x)| \right)^{\frac{1-\epsilon'}{2}} \left( \sum_x |x|^4 |(\pi_{m-1} * D)(x)| \right)^{\frac{1+\epsilon'}{2}}. \tag{3.62}$$

If $\epsilon' = 1$ then the bound (3.62) holds trivially. If $\epsilon' < 1$ then (3.62) is Hölder's inequality with

$$f(x) = |x|^{2+2\epsilon'}|(\pi_{m-1}*D)(x)|^{\frac{1+\epsilon'}{2}}, \quad g(x) = |(\pi_{m-1}*D)(x)|^{\frac{1-\epsilon'}{2}}, \quad \frac{1+\epsilon'}{2} + \frac{1-\epsilon'}{2} = 1. \tag{3.63}$$

Applying Proposition 3.4.1 with $q = 0$ gives

$$\sum_x |(\pi_{m-1} * D)(x)| \leq \sum_y |\pi_{m-1}(y)| \sum_x D(x-y) \leq \frac{C(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}}. \tag{3.64}$$

We now apply Proposition 3.4.1 with $q = 0, 2$ together with the inequality $(a+b)^4 \leq 8(a^4+b^4)$ (obtained by squaring the inequality $(a+b)^2 \leq 2(a^2+b^2)$ and applying the same inequality again) to get

$$
\begin{aligned}
\sum_x |x|^4 |(\pi_{m-1} * D)(x)| \leq & 8 \left( \sum_y |y|^4 |\pi_{m-1}(y)| \sum_x D(x-y) + \right. \\
& \left. \sum_y |\pi_{m-1}(y)| \sum_x |x-y|^4 D(x-y) \right) \\
\leq & C \left( \sum_y |y|^4 |\pi_{m-1}(y)| + \sum_y |\pi_{m-1}(y)| \sigma^4 \right) \\
\leq & \frac{\sigma^4 C(K) \beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-8}{2}}} + \frac{\sigma^4 C(K) \beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}} \leq \frac{\sigma^4 C(K) \beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-8}{2}}}.
\end{aligned}
\tag{3.65}
$$

Note that we have used Remark 1.2.5 to obtain $\sum_x |x|^r D(x) \leq C\sigma^r$ with the constant independent of $L$ (it may depend on $r$). Putting (3.64) and (3.65) back into (3.62) we get

$$
\begin{aligned}
\sum_x |x|^{2+2\epsilon'} |\pi_{m-1}(x)| \leq & \left( \frac{C(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-4}{2}}} \right)^{\frac{1-\epsilon'}{2}} \left( \frac{\sigma^4 C(K)\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-8}{2}}} \right)^{\frac{1+\epsilon'}{2}} \\
\leq & \frac{\sigma^{2(1+\epsilon')} C(K) \beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}-\epsilon'}}.
\end{aligned}
\tag{3.66}
$$

Combining (3.66) with (3.61) gives

$$
\begin{aligned}
\left| g_m(k;z) - g_m(0;z) - \frac{|k|^2}{2d} \nabla^2 g_m(0;z) \right| \leq & \frac{C(K)\beta^{2-\frac{6\nu}{d}} (\sigma^2 |k|^2)^{1+\epsilon'}}{m^{\frac{d-6}{2}-\epsilon'}} \\
\leq & \frac{C(K)\beta^{2-\frac{6\nu}{d}} a(k)^{1+\epsilon'}}{m^{\frac{d-6}{2}-\epsilon'}},
\end{aligned}
\tag{3.67}
$$

when $\|k\| \leq L^{-1}$. This satisfies the required final bound of Assumption G.

It remains to verify this bound for the term inside the second absolute value in expression (3.60). For this term we write

$$
\frac{1-\widehat{D}(k)}{\sigma^2} - \frac{|k|^2}{2d} = \frac{1}{\sigma^2} \left( \widehat{D}(k) - \widehat{D}(0) - \frac{|k|^2}{2d} \nabla^2 \widehat{D}(0) \right), \tag{3.68}
$$

and proceed as for the first term to obtain

$$
\left| \frac{1-\widehat{D}(k)}{\sigma^2} - \frac{|k|^2}{2d} \right| \leq c|k|^{2+2\epsilon'} \sum_x |x|^{2+2\epsilon'} |D(x)| \leq c|k|^{2+2\epsilon'} L^{2(1+\epsilon')}. \tag{3.69}
$$

Together with Proposition 3.4.1 with $q = 1$ this gives

$$\left|(1 - \widehat{D}(k))\sigma^{-2} - \frac{|k|^2}{2d}\right| |\nabla^2 g_m(0;z)| \le \frac{C(K)\beta^{2-\frac{6\nu}{d}}\sigma^2}{m^{\frac{d-6}{2}}} \frac{(|k|^2 L^2)^{1+\epsilon'}}{\sigma^2}, \tag{3.70}$$

which satisfies the required final bound of Assumption G for $\|k\| \le L^{-1}$.

**Remark 3.4.2.** *We have actually verified slightly stronger statements than those of Assumptions E and G. For the purposes of proving Theorem 3.4.3 we were only required to verify the bounds of Assumptions E and G for $z = z^*$, however we proved that if the bounds (3.33) hold for some $z^*$ then the bounds of Assumptions E and G hold* uniformly *in $z \in [0, z^*]$.*

We have now verified that Assumptions S,D,E,G all hold provided Proposition 3.4.1 holds. Thus subject to proving Proposition 3.4.1, we may apply the induction method of Appendix A and obtain Theorem A.2.1 which for our model is the following.

**Theorem 3.4.3.** *Fix $d > 8$, $\gamma \in (0, 1 \wedge \frac{d-8}{2})$ and $\delta \in (0, (1 \wedge \frac{d-8}{2}) - \gamma)$. There exists a positive $L_0 = L_0(d)$ such that:*

*For every $L \ge L_0$ there exist $A', v, z_c$ depending on $d$ and $L$ such that the following statements hold:*

**(a)**

$$\frac{\widehat{t}_n\left(\frac{k}{\sqrt{v\sigma^2 n}}; \frac{z_c}{\rho(0)p_c}\right)}{\rho(0)} = A' e^{-\frac{k^2}{2d}} \left[1 + \mathcal{O}\left(\frac{k^2}{n^\delta}\right) + \mathcal{O}\left(\frac{1}{n^{\frac{d-8}{2}}}\right)\right], \tag{3.71}$$

*with the error estimate uniform in $\{k \in \mathbb{R}^d : 1 - \widehat{D}(k/\sqrt{v\sigma^2 n}) \le \gamma n^{-1} \log n\}$.*

**(b)**

$$-\frac{\nabla^2 \widehat{t}_n\left(0; \frac{z_c}{\rho(0)p_c}\right)}{\widehat{t}_n\left(0; \frac{z_c}{\rho(0)p_c}\right)} = v\sigma^2 n \left[1 + \mathcal{O}\left(\frac{1}{L^{\frac{d}{2}} n^\delta}\right)\right]. \tag{3.72}$$

**(c)** *for every $p \ge 1$,*

$$\left\|\widehat{D}^2 \widehat{t}_n\left(\cdot; \frac{z_c}{\rho(0)p_c}\right)\right\|_p \le \frac{C}{L^{\frac{d}{p}} n^{\frac{d}{2p} \wedge \frac{d-8}{2}}}. \tag{3.73}$$

**(d)** *The constants $z_c$, $A'$ and $v$ obey*

$$\begin{aligned} 1 &= \sum_{m=1}^{\infty} g_m(0; z_c), \\ A' &= \frac{1 + \sum_{m=1}^{\infty} e_m(0; z_c)}{\sum_{m=1}^{\infty} m g_m(0; z_c)}, \\ v &= -\frac{\sum_{m=1}^{\infty} \nabla^2 g_m(0; z_c)}{\sigma^2 \sum_{m=1}^{\infty} m g_m(0; z_c)}. \end{aligned} \tag{3.74}$$

*The constants* $A', v$ *satisfy* $A' = 1 + \mathcal{O}\left(L^{-\frac{d}{2}}\right)$, $v = 1 + \mathcal{O}\left(L^{-\frac{d}{2}}\right)$. *Also* $z_c = 1 + \mathcal{O}(L^{-\frac{d}{2}})$.

To reiterate the induction method shows that (3.33) holds for all $m$, provided Proposition 3.4.1 holds.

## 3.5 Proof of Theorem 1.4.3

In this section we show that Theorem 1.4.3 follows from Theorem 3.4.3(a). Comparing the two Theorems and setting $A = A'\rho(0)$ (recall that $\zeta_c = \frac{z_c}{\rho(0)p_c}$), it is clear that to prove Theorem 1.4.3 it is sufficient to prove the following two Lemmas

**Lemma 3.5.1.** *For $d$, $\gamma$, $\delta$ and $L_0$ as in Theorem 3.4.3, there exists a constant $C_0 = C_0(d, \gamma)$ such that*

$$\widehat{t}_{\lfloor nt \rfloor}\left(\frac{k}{\sqrt{v\sigma^2 n}}; \zeta_c\right) = Ae^{-\frac{k^2}{2d}t} + \mathcal{O}\left(\frac{k^2}{n}\right) + \mathcal{O}\left(\frac{k^2 t^{1-\delta}}{n^\delta}\right) + \mathcal{O}\left(\frac{1}{(nt \vee 1)^{\frac{d-8}{2}}}\right), \quad (3.75)$$

*with the error estimates uniform in* $\left\{k \in \mathbb{R}^d : |k|^2 \leq C_0 \log(\lfloor nt \rfloor \vee 1)\right\}$.

**Lemma 3.5.2.** *The critical value $\zeta_c = \frac{z_c}{\rho(0)p_c}$ in Theorem 3.4.3 is 1.*

The significance of Lemma 3.5.1 is to incorporate the continuous time variable $t$ into the asymptotic formula (3.71) and to present a more palatable region of $\mathbb{R}^d$ on which the error estimates in are uniform.

*Proof of Lemma 3.5.1.* The statement is trivial for $\lfloor nt \rfloor = 0$, so we assume that $\lfloor nt \rfloor \geq 1$. Incorporating a time variable by $n \mapsto \lfloor nt \rfloor$, $k \mapsto k\sqrt{\frac{\lfloor nt \rfloor}{n}}$ into (3.71), and using $A = A'\rho(0)$ we have that $\widehat{t}_{\lfloor nt \rfloor}\left(\frac{k}{\sqrt{v\sigma^2 n}}; \zeta_c\right)$ is equal to

$$\widehat{t}_{\lfloor nt \rfloor}\left(\frac{k\sqrt{\frac{\lfloor nt \rfloor}{n}}}{\sqrt{v\sigma^2 \lfloor nt \rfloor}}; \zeta_c\right) = Ae^{-\frac{k^2 \lfloor nt \rfloor}{2dn}}\left[1 + \mathcal{O}\left(\frac{k^2 \lfloor nt \rfloor^{1-\delta}}{n}\right) + \mathcal{O}\left(\frac{1}{\lfloor nt \rfloor^{\frac{d-8}{2}}}\right)\right], \quad (3.76)$$

where the error estimate is uniform in

$$\mathbf{H}_{n,t} \equiv \left\{k \in \mathbb{R}^d : 1 - D\left(\frac{k\sqrt{\frac{\lfloor nt \rfloor}{n}}}{\sqrt{v\sigma^2 \lfloor nt \rfloor}}\right) \leq \gamma \lfloor nt \rfloor^{-1} \log \lfloor nt \rfloor\right\}. \quad (3.77)$$

We claim that there exists a constant $C_0$ such that $\{k : |k|^2 \leq C_0 \log(\lfloor nt \rfloor)\} \subset \mathbf{H}_{n,t}$.

Define

$$\mathbf{G}_{n,t} \equiv \left\{ k : \|k\|_\infty \leq \frac{\sqrt{v\sigma^2 \lfloor nt \rfloor}}{L} \right\}. \tag{3.78}$$

By (3.29) and using the fact that $\sigma \sim L$, there exists $C_1 > 0$ such that for $k \in \mathbf{G}_{n,t}$,

$$1 - D\left( \frac{k\sqrt{\frac{\lfloor nt \rfloor}{n}}}{\sqrt{v\sigma^2 \lfloor nt \rfloor}} \right) \leq \frac{C_1 k^2}{\lfloor nt \rfloor}. \tag{3.79}$$

Now since $\|k\|_\infty^2 \leq |k|^2$ and using the fact that $\sigma \sim L$, there exists a constant $C_0 < \frac{\gamma}{C_2}$ such that

$$\{k : |k|^2 \leq C_0 \log(\lfloor nt \rfloor)\} \subset \mathbf{G}_{n,t}. \tag{3.80}$$

Then for $k^2 \leq C_0 \log(\lfloor nt \rfloor)$ we have

$$\begin{aligned} 1 - D\left( \frac{k\sqrt{\frac{\lfloor nt \rfloor}{n}}}{\sqrt{v\sigma^2 \lfloor nt \rfloor}} \right) &\leq \frac{C_1 k^2}{\lfloor nt \rfloor} \leq \frac{C_1 C_0 \log(\lfloor nt \rfloor)}{\lfloor nt \rfloor} \\ &\leq \frac{\gamma \log(\lfloor nt \rfloor)}{\lfloor nt \rfloor}. \end{aligned} \tag{3.81}$$

Thus verifies the claim, and thus (3.76) holds with the error estimate is uniform in $\{k : |k|^2 \leq C_0 \log(\lfloor nt \rfloor)\}$. Since $\lfloor nt \rfloor \leq nt$ in the first error term of (3.76), and

$$\left| e^{-\frac{k^2 \lfloor nt \rfloor}{2dn}} - e^{-\frac{k^2 t}{2d}} \right| \leq \frac{k^2}{2d}\left( t - \frac{\lfloor nt \rfloor}{n} \right) = \mathcal{O}\left( \frac{k^2}{n} \right), \tag{3.82}$$

we have proved Lemma 3.5.1. □

The significance of Lemma 3.5.2 was discussed immediately before the statement of Theorem 1.4.3 in Section 1.4. Essentially, $\zeta$ was a weight introduced so that we could apply the induction method of Appendix A. That $\zeta_c$ should be 1 is intuitive since the lattice trees are already critically weighted (by $p_c$) and this idea is the basis of the following proof.

*Proof of Lemma 3.5.2.* The susceptibility, $\chi(z)$ is defined as

$$\begin{aligned} \chi(z) &\equiv \sum_n f_n(0; z) = \sum_n \frac{\widehat{t}_n(0; \zeta)}{\rho(0)} \\ &= \sum_n \zeta^n \frac{1}{\rho(0)} \sum_x \sum_{T \in \mathcal{T}_n(0,x)} W(T) \equiv \bar{\chi}(\zeta), \end{aligned} \tag{3.83}$$

where $\zeta = \frac{z}{\rho(0) p_c}$.

Let $z_c'$ denote the radius of convergence of $\chi(z)$. By Theorem 3.4.3 there exists a $z_c > 0$ (resp. $\zeta_c$) such that

$$\zeta_c^n \frac{1}{\rho(0)} \sum_x \sum_{T \in \mathcal{T}_n(x)} W(T) \to A, \tag{3.84}$$

so that

$$\left( \frac{1}{\rho(0)} \sum_x \sum_{T \in \mathcal{T}_n(x)} W(T) \right)^{\frac{1}{n}} \to \frac{1}{\zeta_c}. \tag{3.85}$$

Thus the radius of convergence of $\bar{\chi}(\zeta)$ (resp. $\chi(z)$) is $\zeta_c > 0$ (resp. $z_c$).

Write $\sum_x \rho(x) = \lim_{M \to \infty} \sum_{|x|<M} \rho(x)$ and observe that $\sum_{|x|<M} \frac{1}{(|x| \vee 1)^{d-a}} \simeq M^a$. It follows from Theorem 1.2.9 that $\sum_x \rho(x) = \infty$. Thus

$$\bar{\chi}(1) = \frac{1}{\rho(0)} \sum_n \sum_x \sum_{T \in \mathcal{T}_n(0,x)} W(T) = \frac{1}{\rho(0)} \sum_x \rho(x) = \infty, \tag{3.86}$$

which implies that $\zeta_c \leq 1$.

Recall from (1.16) that $P(T \in \mathcal{T}_n(0,x)) = \frac{\sum_{T \in \mathcal{T}_n(0,x)} W(T)}{\rho(0)}$. Then Theorem 3.4.3 states that for every $k$,

$$\zeta_c^n \sum_x e^{i \frac{k \cdot x}{\sqrt{\sigma^2 v n}}} P(T \in \mathcal{T}_n(0,x)) \to A e^{-\frac{k^2}{2d}}. \tag{3.87}$$

Setting $k = 0$ we have

$$\frac{\zeta_c^n}{A} \sum_x P(T \in \mathcal{T}_n(0,x)) \to 1, \tag{3.88}$$

and dividing (3.87) by (3.88) gives

$$\sum_x e^{i \frac{k \cdot x}{\sqrt{\sigma^2 v n}}} \frac{P(T \in \mathcal{T}_n(0,x))}{\sum_u P(T \in \mathcal{T}_n(0,u))} \to e^{-\frac{k^2}{2d}}. \tag{3.89}$$

Let $Z_n$ be $\mathbb{Z}^d$-valued random variables defined by $P(Z_n = x) = \frac{P(T \in \mathcal{T}_n(0,x))}{\sum_u P(T \in \mathcal{T}_n(0,u))}$. Then (3.89) is the statement that

$$\mathbb{E}\left[ e^{ik \cdot \frac{Z_n}{\sqrt{\sigma^2 v n}}} \right] \to \mathbb{E}\left[ e^{ik \cdot Z} \right], \tag{3.90}$$

for every $k$, where $Z \sim \mathcal{N}(0, I_d)$. This is equivalent to $\frac{Z_n}{\sqrt{\sigma^2 v n}} \xrightarrow{\mathcal{D}} Z$, and thus for every $R \geq 0$ we have

$$P\left( \frac{Z_n}{\sqrt{\sigma^2 v n}} \in B(0,R) \right) \to P\left( Z \in B(0,R) \right), \tag{3.91}$$

where $B(0,R)$ denotes the ball with centre 0 and radius $R$ in $(\mathbb{R}^d, |\bullet|)$. Choose $R_0$ such that $P(Z \in B(0,R_0)) \geq \frac{2}{3}$. Then there exists an $N_0 = N_0(R_0)$ such that for every $n \geq N_0$,

$$P\left(\frac{Z_n}{\sqrt{\sigma^2 vn}} \in B(0,R_0)\right) \geq \frac{1}{2}. \tag{3.92}$$

Therefore for every $n \geq N_0$,

$$\sum_{|x| \leq R_0\sqrt{\sigma^2 vn}} \frac{P(T \in \mathcal{T}_n(0,x))}{\sum_u P(T \in \mathcal{T}_n(0,u))} = P\left(Z_n \in B(0, R_0\sqrt{\sigma^2 vn})\right) \geq \frac{1}{2}. \tag{3.93}$$

Applying (3.88) to the denominator, we find that there exists $N_1 \geq N_0$ such that for every $n \geq N_1$,

$$\frac{\zeta_c^n}{A} \sum_{|x| \leq R_0\sqrt{\sigma^2 vn}} P(T \in \mathcal{T}_n(0,x)) \geq \frac{1}{3}, \qquad \text{i.e.} \quad \sum_{|x| \leq R_0\sqrt{\sigma^2 vn}} P(T \in \mathcal{T}_n(0,x)) \geq \frac{C}{\zeta_c^n}. \tag{3.94}$$

Bounding $\sum_{T \in \mathcal{T}_n(0,x)} W(T)$ by $\rho(x) = \sum_m \sum_{T \in \mathcal{T}_m(0,x)} W(T)$, it follows that

$$\sum_{|x| \leq R_0\sqrt{\sigma^2 vn}} \rho(x) \geq \frac{C\rho(0)}{\zeta_c^n}. \tag{3.95}$$

We also have from (1.13) that,

$$\sum_{|x| < R_0\sqrt{\sigma^2 vn}} \rho(x) \leq \sum_{|x| < R_0\sqrt{\sigma^2 vn}} \frac{C(L)}{(|x| \vee 1)^{d-2}} \leq C(L,R_0)n. \tag{3.96}$$

Thus from (3.96) and (3.94), $\frac{C}{\zeta_c^n} \leq Cn$ for every $n \geq n_0$. This requires that $\zeta_c \geq 1$ and we have the result. □

Assuming that Proposition 3.4.1 holds, we have now verified Lemmas 3.5.1 and 3.5.2, and hence we have proved Theorem 1.4.3. We postpone the proof of Proposition 3.4.1 to Chapter 5.

# Chapter 4

# The $r$-point functions

We have shown Gaussian behaviour (Theorem 1.4.3) of the 2-point function with appropriate scaling in Chapter 3. We now wish to prove the analogous result for $r$-point functions, Theorem 1.4.5. The proof is by induction on $r$, with Chapter 2 already having verified the initializing case $r = 2$. We use the technology of the lace expansion on a tree of van der Hofstad and Slade [21] as expressed in Chapter 2, and prove the result, assuming certain diagrammatic bounds. The diagrammatic estimates are again postponed until Chapter 6.

## 4.1 Preliminaries

Recall from Definitions 1.2.3 and 1.4.4 that for fixed $r \geq 2$, $\tilde{\mathbf{n}} \in \mathbb{Z}_+^{r-1}$ and $\tilde{\mathbf{x}} \in \mathbb{R}^{r-1}$, we have

$$\mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}}) = \{T \in \mathcal{T}_0 : \mathbf{x_i} \in T_{\mathbf{n_i}},\ \ i = 1, \ldots, r-1\} \tag{4.1}$$

and

$$t^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}}) = \sum_{T \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})} W(T). \tag{4.2}$$

For $T \in \mathcal{T}(x)$, let $T_{\rightsquigarrow x}$ be the backbone in $T$ from 0 to $x$.

**Definition 4.1.1.** *A lattice tree $B$ is said to be an $(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$* bare tree *if*

*1) $B \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})$ and*

*2) $\cup_{i=1}^{r-1} B_{\rightsquigarrow x_i} = B$.*

*We let $\mathbf{B}(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$ denote the set of $(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$ bare trees. If $B \in \mathbf{B}(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$ then we write $\mathcal{T}_B = \{T \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}}) : T_{\rightsquigarrow x_i} = B_{\rightsquigarrow x_i},\ i \in 1, \ldots, r-1\}$ for the set of lattice trees containing $B$ as a subtree.*

Since every $T \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})$ has a unique minimal connected subtree $(\cup_{i=1}^{r-1} T_{\rightsquigarrow x_i})$ connecting 0 to the $x_i$, $i = 1, \dots, r-1$, we have

$$t^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}}) = \sum_{B \in \mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}})} \sum_{T \in \mathcal{T}_B} W(T). \tag{4.3}$$

**Definition 4.1.2 (Branch point).** *Let $B \in \mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$. A vertex $x \in B$ is a* branch point *of $B$ if there exist $i, j \in \{1, \dots, r-1\}$, $i \neq j$ such that $x_i$ and $x_j$ are distinct leaves (vertices of degree 1) of $B$ and $B_{\rightsquigarrow x_i} \cap B_{\rightsquigarrow x_j} = B_{\rightsquigarrow x}$. The* degree *of a branch point $x \in B$ is the number of bonds $\{a, b\} \in B$ such that either $a = x$ or $b = x$.*

As they are defined in terms of the leaves of $B \in \mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$, branch points of $B$ depend on $B$ but not the set $\mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$ of which $B$ is a member. In particular if $B$ is also in $\mathbf{B}(\tilde{\mathbf{n}}',\tilde{\mathbf{x}}')$ then our definition gives rise to the same set of branch points. By definition, a branch point that is not the origin must have degree $\geq 3$.

**Definition 4.1.3 (Degenerate bare tree).** *For fixed $r$, a bare tree $B \in \mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$ is said to be* non-degenerate *if $B$ contains exactly $r-2$ distinct branch points, each of degree 3, none of which is the origin. Otherwise $B$ is said to be* degenerate. *We write $\mathbf{B}_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$ for the set of degenerate trees in $\mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$ and set $\mathbf{B}^c_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}}) = \mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}}) \setminus \mathbf{B}_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$.*

Clearly from (4.3) we have

$$t^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}}) = \sum_{B \in \mathbf{B}^c_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})} \sum_{T \in \mathcal{T}_B} W(T) + \sum_{B \in \mathbf{B}_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})} \sum_{T \in \mathcal{T}_B} W(T). \tag{4.4}$$

**Definition 4.1.4.** *Let $B \in \mathbf{B}(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$. Two distinct vertices $y$, $y^*$ in $B$ are said to be* net-neighbours *in $B$ if the unique path in $B$ from $y$ to $y^*$ contains no other branch points of $B$ other than $y$, $y^*$. A* net-path *in $B$ is a path in $B$ connecting the origin or a branch point in $B$ to a net-neighbouring branch point or leaf in $B$.*

**Lemma 4.1.5.** *Fix $r \geq 2$, $\tilde{\mathbf{n}} \in \mathbb{N}^{r-1}$, $\tilde{\mathbf{x}} \in \mathbb{Z}^{d(r-1)}$.*

1. *If $B \in \mathbf{B}^c_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$ then $B$ consists of $2r-3$ net-paths joined together with the topology of $\alpha$ for some $\alpha \in \Sigma_r$.*

2. *If $B \in \mathbf{B}_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})$ then $B$ contains fewer than $2r-3$ nonempty netpaths and fewer than $r-2$ branch points that are not the origin.*

*Proof.* Induction on $r$. For $r = 2$, there are no degenerate bare trees and the result is trivial.

Suppose the result holds for all $r' < r$.

1. Let $B \in \mathbf{B}_D^c(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$. Then $B$ contains $r-2$ branch points, each of which is of degree 3, none of which is the origin. Let $x \neq 0, x_{r-1}$ be the unique branch point in $B$ net-neighbouring $x_{r-1}$. Removing the netpath $B_{\rightsquigarrow x_{r-1}} \setminus B_{\rightsquigarrow x}$, we have that $x$ is a vertex of degree 2 in $B^* = B \setminus (B_{\rightsquigarrow x_{r-1}} \setminus B_{\rightsquigarrow x})$ and therefore $B \in \mathbf{B}\left((n_1, \dots, n_{r-2}), (x_1, \dots, x_{r-2})\right)$ contains $r-3$ branchpoints, each of degree 3, none of which is the origin. Thus $B^*$ is nondegenerate. By definition of a netpath and the fact that $x$ is not a branch point of $B^*$, we see that $B^*$ contains two fewer netpaths than $B$. The induction hypothesis gives that $B^*$ consists of $2(r-1)-3$ net paths joined together with the topology of $\alpha^*$ for some $\alpha^* \in \Sigma_{r-1}$. Therefore $B$ contained $2r-3$ netpaths joined together with the topology of $\alpha \in \Sigma_{r-1}$, where $\alpha$ is the shape obtained by adding a vertex to the edge of $\alpha^*$ corresponding to the unique net-path in $B^*$ containing $x = \mathbf{x}_j^*$ and adding an edge to that vertex.

2. Suppose now that $B \in \mathbf{B}_D(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$. If $B$ contains no branch point other than perhaps 0, then trivially for $r \geq 3$, $B$ contains fewer than $2r-3$ net paths. Otherwise we use the same decomposition as for part 1, and let the degree of the branch point $x \neq 0$ be $l$. If $l = 3$ then $B^*$ above is a degenerate bare tree and the result hold by induction. If $l > 3$ then $B^*$ contains one fewer netpath and the same number of branch points as $B$. By induction $B^* \in \mathbf{B}\left((n_1, \dots, n_{r-2}), (x_1, \dots, x_{r-2})\right)$ contains at most $2(r-1)-3$ netpaths and $(r-1)-2 = r-3$ branch points that are not the origin. Therefore $B$ contained at most $2r-4$ netpaths and $r-3$ branch points that are not the origin.

□

**Definition 4.1.6.** *For a fixed shape $\alpha \in \Sigma_r$ and $\vec{n} \in \mathbb{N}_+^{2r-3}$ we let $\mathcal{N}(\alpha, \vec{n})$ be the abstract* network shape *obtained by inserting $n_j - 1$ vertices onto edge $j$ of $\alpha$, $j = 1, \dots, 2r-3$. Each edge $j$ of $\alpha$ has two vertices $j_1, j_2$ in $\alpha$ incident to it. We define* branch $\mathcal{N}_j$ *of $\mathcal{N}$ to be the smallest connected subnetwork of $\mathcal{N}$ that contains the vertices $j_1, j_2$.*

*Let $B \in \mathbf{B}_D^c(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$. We say that $B$ has network shape $\mathcal{N}(\alpha, \vec{n})$ if $B$ and $\mathcal{N}(\alpha, \vec{n})$ are graph isomorphic and for each $i$ the graph isomorphism maps leaf $i$ of $\mathcal{N}(\alpha, \vec{n})$ to $x_i$. For $\vec{y} = (y_1, \dots, y_{2r-3}) \in \mathbb{Z}^{d(2r-3)}$, we define $\mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})$ to be the set of lattice trees $T \in \mathcal{T}_0$ such that there exists $\tilde{\mathbf{x}} \in \mathbb{Z}^{d(r-1)}$, $\tilde{\mathbf{n}} \in \mathbb{N}^{r-1}$ and $B \in \mathbf{B}_D^c(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$ such that*

1. *$T \in \mathcal{T}_B$,*

2. *$B$ has network shape $\mathcal{N}(\alpha, \vec{n})$, and*

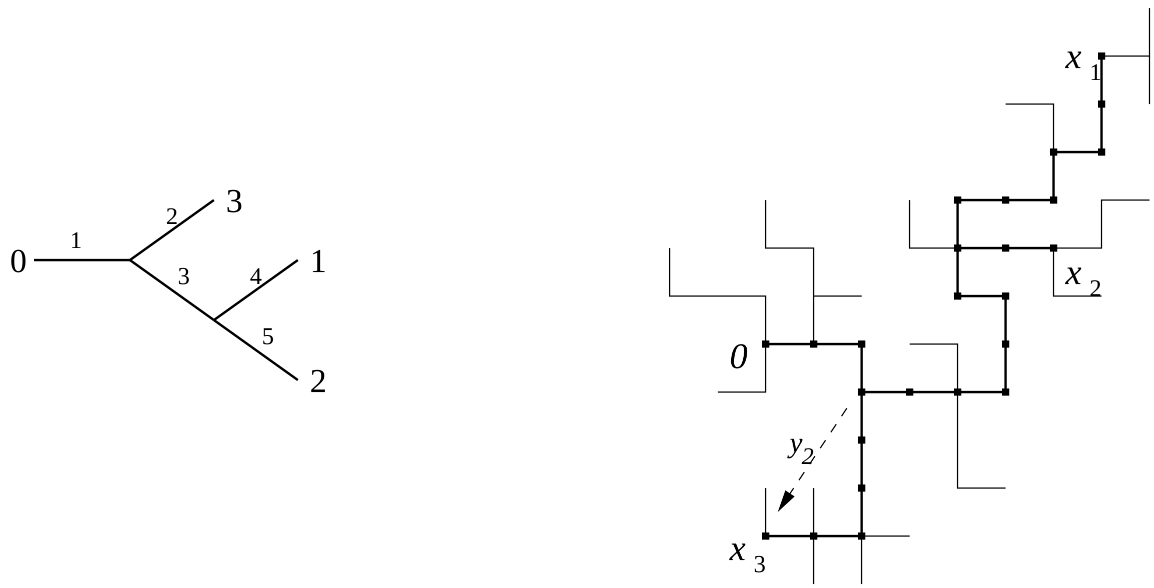


Figure 4.1: A shape $\alpha \in \Sigma_4$ with labelled edges, and a nearest neighbour lattice tree $T \in \mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})$ for $\vec{n} = (3,5,7,7,2)$, $\vec{y} = ((2,-1),(-2,-3),(2,3),(3,4),(2,0))$. Also $T \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})$ where $\tilde{\mathbf{n}} = (17,12,8)$ and $\tilde{\mathbf{x}} = ((7,6),(6,2),(0,-4))$. Note for example that $y_1 + y_3 + y_4 = x_1$.

*3. if the endvertices of netpath $B_j$ are $u_j, v_j \in \mathbb{R}^d$, where $B_{\rightsquigarrow u_j} \subset B_{\rightsquigarrow v_j}$ then $v_j - u_j = y_j$, for each $j = 1, \dots, 2r-3$.*

Suppose $T \in \mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})$, with corresponding $\tilde{\mathbf{x}}$, $\tilde{\mathbf{n}}$, $B$ as in Definition 4.1.6. Since $B$ has shape $\mathcal{N}(\alpha,\vec{n})$, we may label the netpaths $\{B_1, \dots, B_{2r-3}\}$ of $B$ according to the edge labels $\{1, \dots, 2r-3\}$ of $\alpha$. Let $E_i = \{j : B_j \subset B_{\rightsquigarrow x_i}\}$, and note that $E_i$ is equal to the set of edges in the unique path in $\alpha$ from the root to leaf $i$, defined in Section 1.4. By definition we have $\sum_{j \in E_i} y_j = \mathbf{x_i}$ and $\sum_{j \in E_i} n_j = \mathbf{n_i}$. See Figure 4.1 for an illustration of this.

Lemma 4.1.5 implies that if $T \in \mathcal{T}_B$ for some non-degenerate $B \in \mathbf{B}^c_D(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$, then $T \in \mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})$ for some $\alpha \in \Sigma_r$, $\vec{n} \in \mathbb{N}^{2r-3}$, $\vec{y} \in \mathbb{Z}^{d(2r-3)}$ satisfying $\sum_{j \in E_i} n_j = \mathbf{n}_i$, $\sum_{j \in E_i} y_j = \mathbf{x}_i$, $i \in \{1, \dots, r-1\}$. On the other hand suppose $T \in \mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})$. Let $x_i$ be the vertex in $T$ corresponding to leaf $i$ of $\alpha$, $i = 1, \dots, r-1$, and let $\mathbf{n_i} = |T_{\rightsquigarrow x_i}|$. Then $T \in \mathcal{T}_{\tilde{\mathbf{n}}}(\tilde{\mathbf{x}})$ by definition. Choosing $B = \cup_{i=1}^{r-1} T_{\rightsquigarrow x_i}$, it is easy to see that $B \in \mathbf{B}(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$ and $T \in \mathcal{T}_B$. Finally since $\mathcal{N}(\alpha,\vec{n})$ contains $r-2$ distinct branch points, each of degree 3 (of which none are the origin), $B$ must also have this property and thus $B \in \mathbf{B}^c_D(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$.

For fixed $\alpha \in \Sigma_r$, $\tilde{\mathbf{n}} \in \mathbb{N}^{r-1}$ and $\tilde{\mathbf{x}} \in \mathbb{Z}^{d(r-1)}$ we write $\sum_{\vec{n} \overset{\alpha}{\rightsquigarrow} \tilde{\mathbf{n}}}$ to mean the sum over $\{\vec{n} \in \mathbb{N}^{2r-3} : \sum_{j\in E_i} n_j = \mathbf{n_i},\ i = 1,\dots,r-1\}$, and $\sum_{\vec{y}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{x}}}$ to mean the sum over $\{\vec{y} \in \mathbb{Z}^{d(2r-3)} : \sum_{j\in E_i} y_j = \mathbf{x_i},\ i = 1,\dots,r-1\}$. Then

$$\sum_{B\in\mathbf{B}^c_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})}\sum_{T\in\mathcal{T}_B} W(T) = \sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{n}}}\sum_{\vec{y}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{x}}}\sum_{T\in\mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})} W(T). \tag{4.5}$$

See Figure 4.1 for a concrete example of this idea.

**Definition 4.1.7.** *For fixed $r \geq 2$, $\alpha \in \Sigma_r$, network shape $\mathcal{N} = \mathcal{N}(\alpha,\vec{n})$, and netpath displacements $\vec{y} = (y_1,\dots y_{2r-3}) \in \mathbb{Z}^{d(2r-3)}$ we define*

$$t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) = \sum_{T\in\mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})} W(T). \tag{4.6}$$

Recall the definition of $\vec{\kappa}$ from (1.29). We are now able to state the main result of this chapter, Theorem 4.1.8.

**Theorem 4.1.8.** *Fix $d > 8$, $\gamma \in (0, 1\wedge\frac{d-8}{2})$ and $\delta \in (0,(1\wedge\frac{d-8}{2})-\gamma)$. There exists $L_0 = L_0(d) \gg 1$ such that: for each $L \geq L_0$ there exists $V = V(d,L) > 0$ such that for every $r \geq 2$, $\alpha \in \Sigma_r$, $\vec{n} \in \mathbb{N}^{2r-3}$, $R > 0$, and $\vec{\kappa} \in [-R,R]^{(2r-3)d}$,*

$$\widehat{t}_{\mathcal{N}(\alpha,\vec{n})}\left(\frac{\vec{\kappa}}{\sqrt{\sigma^2 vn}}\right) = V^{r-2}A^{2r-3}\prod_{j=1}^{2r-3} e^{\frac{-\kappa_j^2}{2d}\left(\frac{n_j}{n}\right)} + \mathcal{O}\left(\sum_{j=1}^{2r-3}\frac{1}{n_j^{\frac{d-8}{2}}}\right) + \mathcal{O}\left(\sum_{j=1}^{2r-3}\frac{|\kappa|^2 n_j^{1-\delta}}{n}\right), \tag{4.7}$$

*where $A$ and $v$ are the constants appearing in Theorem 1.4.3 and the constants in the error terms may depend on $r$ and $R$.*

The constant $V$ is defined in Definition 4.3.1 and reflects the presence of non-trivial interaction near branch points of our binary tree networks $\mathcal{N}$ where three trees must meet at a single point but are otherwise mutually avoiding.

In view of (4.4) and (4.5) we have that

$$\begin{aligned}\widehat{t}^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{k}}) &= \sum_{\tilde{\mathbf{x}}} e^{i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}}\sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{n}}}\sum_{\vec{y}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{x}}} t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) + \sum_{\tilde{\mathbf{x}}} e^{i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}}\sum_{B\in\mathbf{B}_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})}\sum_{T\in\mathcal{T}_B} W(T)\\ &\equiv \sum_{\tilde{\mathbf{x}}} e^{i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}}\sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{n}}}\sum_{\vec{y}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{x}}} t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) + \widehat{\phi}^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{k}}).\end{aligned} \tag{4.8}$$

We will show that $\widehat{\phi}^r_{\tilde{\mathbf{n}}}(\bullet)$ gives rise to an error term.

Recall the definition of the set of edges $E_j$ of the unique path in $\alpha$ from 0 to leaf $j$. Then $\mathbf{x}_j = \sum_{l=1}^{2r-3} y_l I_{\{l\in E_j\}}$ and in (4.8) we use

$$\sum_{j=1}^{r-1}\mathbf{k}_j\cdot\mathbf{x}_j = \sum_{j=1}^{r-1}\mathbf{k}_j\cdot\sum_{l=1}^{2r-3} y_l I_{\{l\in E_j\}} = \sum_{l=1}^{2r-3} y_l\cdot\sum_{j=1}^{r-1}\mathbf{k}_j I_{\{l\in E_j\}} = \sum_{l=1}^{2r-3} y_l\cdot\kappa_l = \vec{\kappa}\cdot\vec{y}, \tag{4.9}$$

where $\kappa_l$ was defined in (1.29). Thus the first term on the right of (4.8) is equal to

$$\begin{aligned}\sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{n}}}\sum_{\tilde{\mathbf{x}}} e^{i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}}\sum_{\vec{y}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{x}}} t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) &= \sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{n}}}\sum_{\vec{y}} e^{i\vec{\kappa}\cdot\vec{y}} t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) \\ &= \sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\tilde{\mathbf{n}}} \widehat{t}_{\mathcal{N}(\alpha,\vec{n})}(\vec{\kappa}).\end{aligned} \tag{4.10}$$

This is more clear if we consider the case $r = 3$, for which there is a unique shape $\alpha$ (which we suppress in the notation for $\mathcal{N}$), and a single branch point. If we denote the spatial location of the branch point by $y$ then

$$\begin{aligned}t^3_{(\mathbf{n}_1,\mathbf{n}_2)}(\mathbf{k}_1,\mathbf{k}_2) = \sum_{\mathbf{x}_1,\mathbf{x}_2}\sum_{n=1}^{(\mathbf{n}_1\wedge\mathbf{n}_2)-1} & e^{i\mathbf{k}_1\cdot\mathbf{x}_1}e^{i\mathbf{k}_2\cdot\mathbf{x}_2}\sum_y t_{\mathcal{N}(n,\mathbf{n}_1-n,\mathbf{n}_2-n)}(y,\mathbf{x}_1-y,\mathbf{x}_2-y) \\ & + \widehat{\phi}^3_{\tilde{\mathbf{n}}}(\tilde{\mathbf{k}}),\end{aligned} \tag{4.11}$$

where informally one may think of $\widehat{\phi}^3$ as consisting of the $n = 0$ and $n = \mathbf{n_1} \wedge \mathbf{n_2}$ terms of the sum. The first term on the right of (4.11) is equal to

$$\begin{aligned}&\sum_{n=1}^{(\mathbf{n}_1\wedge\mathbf{n}_2)-1}\sum_{\mathbf{x}_1,\mathbf{x}_2}\sum_y e^{i(\mathbf{k}_1\cdot(\mathbf{x}_1-y)+\mathbf{k}_2\cdot(\mathbf{x}_2-y)+(\mathbf{k}_1+\mathbf{k}_2)\cdot y)} t_{\mathcal{N}(n,\mathbf{n}_1-n,\mathbf{n}_2-n)}(y,\mathbf{x}_1-y,\mathbf{x}_2-y) \\ &= \sum_{n=1}^{(\mathbf{n}_1\wedge\mathbf{n}_2)-1} \widehat{t}_{\mathcal{N}(n,\mathbf{n}_1-n,\mathbf{n}_2-n)}(\kappa_1,\kappa_2,\kappa_3).\end{aligned} \tag{4.12}$$

Recall from (3.4)–(3.5), and the fact that $\zeta_c = 1$ that we were able to express the critical 2-point function as

$$t_n(x) = \sum_{\substack{\omega:0\to x,\\ |\omega|=n}} W(\omega)\prod_{i=0}^{n}\sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i)\prod_{0\le s<t\le n}[1+U_{st}], \tag{4.13}$$

using the notation $\prod_{i=0}^n \sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i)\prod_{0\le s<t\le n}[1+U_{st}]$ to represent

$$\sum_{R_0\in\mathcal{T}_{\omega(0)}} W(R_0)\cdots\sum_{R_n\in\mathcal{T}_{\omega(n)}} W(R_n)\prod_{0\le s<t\le n}[1+U_{st}]. \tag{4.14}$$

The product $\prod[1+U_{st}]$ incorporats the mutual avoidance of the branches $R_i$ emanating from the backbone $\omega$ (which is a random walk), and we analysed this product using the lace expansion. For higher-point functions, the backbone structure in question may be interpreted as a branching random walk, with the temporal (resp. spatial) location and ancestry of the branching given by $\mathcal{N}(\vec{n},\alpha)$ (resp. $\vec{y}$).

**Definition 4.1.9.** *Fix $\mathcal{N}(\vec{n},\alpha)$. We say that $\omega$ is an* embedding *of $\mathcal{N}$ into $\mathbb{Z}^d$ if $\omega$ is a map from the vertex set of $\mathcal{N}$ into $\mathbb{Z}^d$ that maps the root to $0$ and adjacent vertices in $\mathcal{N}$ to $D(\cdot)$ neighbours in $\mathbb{Z}^d$. Let $\Omega_{\mathcal{N}}(\vec{y})$ be the set of embeddings $\omega$ of $\mathcal{N}$ into $\mathbb{Z}^d$ such that the embedding $\omega_i$ of branch $i$ has displacement $y_i$.*

We now express the $r$-point function (4.6) in a form similar to that previously obtained for the two point function (3.5). For a collection of sets of vertices $\{R_s\}_{s\in\mathcal{N}}$, define as in (3.3),

$$U_{st} = U(R_s, R_t) = \begin{cases} -1, & \text{if } R_s \cap R_t \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{4.15}$$

Recall from Definition 2.1.1 that $\mathbf{E}_{\mathcal{N}} = \{st : s,t \in \mathcal{N}, s \neq t\}$. Also note that a vertex $s \in \mathcal{N}$ is uniquely described by a pair $(i, m_i)$, where $i$ is an edge in $\alpha$ and $m_i \leq n_i$. We write $\prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}_{\omega(s)}}$ as shorthand notation for

$$\sum_{R_0\in\mathcal{T}_{\omega(0)}} \sum_{R_{(1,1)}\in\mathcal{T}_{\omega(1,1)}} \sum_{R_{(1,2)}\in\mathcal{T}_{\omega(1,2)}} \cdots \sum_{R_{(2r-3,n_{2r-3})}\in\mathcal{T}_{\omega(2r-3,n_{2r-3})}} . \tag{4.16}$$

Then,

$$t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) = \sum_{\omega\in\Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}_{\omega(s)}} W(R_s) \prod_{\flat\in\mathbf{E}_{\mathcal{N}}} [1+U_\flat], \tag{4.17}$$

where

1. the sum over $\omega$ is a sum over all embeddings of the network shape i.e. over all bare trees with the required network shape and displacements,
2. the sums over $R_s$ are sums over all branches at vertices $s$ of the embedding $\omega$,
3. the factor $\prod_\flat [1+U_\flat]$ ensures that the branches are mutually avoiding so that only combinations of branches that form lattice trees are counted.

Equation (4.17) follows from (4.6) since any combination $(\omega \in \Omega_{\mathcal{N}}(\vec{y}),\ \{R_s\}_{s\in\omega})$ such that the $R_s$ are all mutually avoiding lattice trees, uniquely defines a lattice tree $T \in \mathcal{T}_{\mathcal{N}(\alpha,\vec{n})}$ and vice versa.

## 4.2 Application of the Lace Expansion

We now apply the expansion described in Section 2.2. Let

$$\phi^{\mathcal{R}}_{\mathcal{N}}(\vec{y}) = \sum_{\omega\in\Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}(\omega(s))} W(R_s) \left(\prod_{\flat\in\mathcal{R}^c} [1+U_\flat]\right) \left(1 - \prod_{\flat\in\mathcal{R}} [1+U_\flat]\right). \tag{4.18}$$

Then by expressions (2.4) and (4.17) we can write

$$t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) = \sum_{\omega\in\Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}(\omega(s))} W(R_s)K(\mathcal{N}) - \phi^{\mathcal{R}}_{\mathcal{N}}(\vec{y}), \tag{4.19}$$

where $K(\mathcal{N}) = \prod_{\flat\in\mathcal{R}^c}[1+U_\flat]$. We will see shortly that $\widehat{\phi}^{\mathcal{R}}_{\mathcal{N}}(\vec{\kappa})$ is an error term. Another such error term comes from

$$\phi^{b}_{\mathcal{N}}(\vec{y}) = \sum_{\omega\in\Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}(\omega(s))} W(R_s) \sum_{\Gamma\in\mathcal{E}^b_{\mathcal{N}}} \prod_{\flat\in\Gamma} U_\flat, \tag{4.20}$$

where $b$ is the branch point neighbouring the origin and $\mathcal{E}^b_{\mathcal{N}}$ is defined in part 8 of Definition 2.1.1.

Recall the definition of a branch from the second paragraph of 2.1. Let $\vec{n}_b = (n_1, n_2, n_3)$ be the vector of branch lengths for branches incident to $b$ and let $G = G(\mathcal{N}) \subset \{2,3\}$ be the set of branch labels for branches incident to $b$ and another branch point of $\mathcal{N}$. Define $\mathcal{H}_{\vec{n}_b}(\mathcal{N}) \subset \mathbb{Z}^3_+$ and $\overline{\mathcal{H}}_{\vec{n}_b}(\mathcal{N}) \subset \mathbb{Z}^3_+$ by

$$\begin{aligned} \mathcal{H}_{\vec{n}_b} &= \{\vec{m} : 0 \le m_i \le \frac{n_i}{3},\ \ i = 1,2,3\} \cap \{\vec{m} : m_i \le n_i - 2, i \in G\} \\ \overline{\mathcal{H}}_{\vec{n}_b} &= (\{\vec{m} : 0 \le m_i \le n_i,\ \ i = 1,2,3\} \cap \{\vec{m} : m_i \le n_i - 2, i \in G\}) \setminus \mathcal{H}_{\vec{n}_b}. \end{aligned} \tag{4.21}$$

Note from (2.11) that $\mathcal{H}_{\vec{n}_b} \cup \overline{\mathcal{H}}_{\vec{n}_b} = \{\vec{m} : m_1 \le n_1, m_2 \le n_2 - 2I_2, m_3 \le n_3 - 2I_3\}$ and that this is empty if $n_i = 1$ for some $i \in G$. Equations (2.8)–(2.11) give an expansion for $K(\mathcal{N})$ which yields

$$\begin{aligned} t_{\mathcal{N}}(\vec{y}) = &\sum_{\omega\in\Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}(\omega(s))} W(R_s) \sum_{\vec{m}\in\mathcal{H}_{\vec{n}_b}} J(\mathcal{S}^\Delta(\vec{m})) \prod_{i=1}^{3} K\left((\mathcal{N}\setminus\mathcal{S}^\Delta(\vec{m}))_i\right) \\ &+ \phi^{\pi}_{\mathcal{N}}(\vec{y}) + \phi^{F}_{\mathcal{N}}(\vec{y}) - \phi^{\mathcal{R}}_{\mathcal{N}}(\vec{y}), \end{aligned} \tag{4.22}$$

where

$$\phi^{\pi}_{\mathcal{N}}(\vec{y}) \equiv \sum_{\omega\in\Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}(\omega(s))} W(R_s) \sum_{\vec{m}\in\overline{\mathcal{H}}_{\vec{n}_b}} J(\mathcal{S}^\Delta(\vec{m})) \prod_{i=1}^{3} K\left((\mathcal{N}\setminus\mathcal{S}^\Delta(\vec{m}))_i\right). \tag{4.23}$$

See Figure 4.2 for an illustration of these definitions. In accordance with Definition 2.1.1, the first term on the right side of (4.22) does not contribute in cases where $b$ is adjacent to another branch point of $\mathcal{N}$ (which implies that $r \ge 4$ and $n_2 \wedge n_3 = 1$).

For $r = 3$ there is only one branch point, $b$, hence $\phi^b_{\mathcal{N}}(\vec{y}) = \phi^{\mathcal{R}}_{\mathcal{N}}(\vec{y}) = 0$. Lemma 4.2.1 states that in fact for large $\vec{n}_{-\infty} \equiv \inf_{1\le j\le 2r-3} n_j$, all the terms $\widehat{\phi}^{\mathcal{R}}_{\mathcal{N}}$, $\widehat{\phi}^{b}_{\mathcal{N}}$ and $\widehat{\phi}^{\pi}_{\mathcal{N}}$ are error terms, so the main term in (4.22) is

$$Q_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) = t_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) - \phi^{b}_{\mathcal{N}}(\vec{y}) - \phi^{\pi}_{\mathcal{N}}(\vec{y}) + \phi^{\mathcal{R}}_{\mathcal{N}}(\vec{y}), \tag{4.24}$$

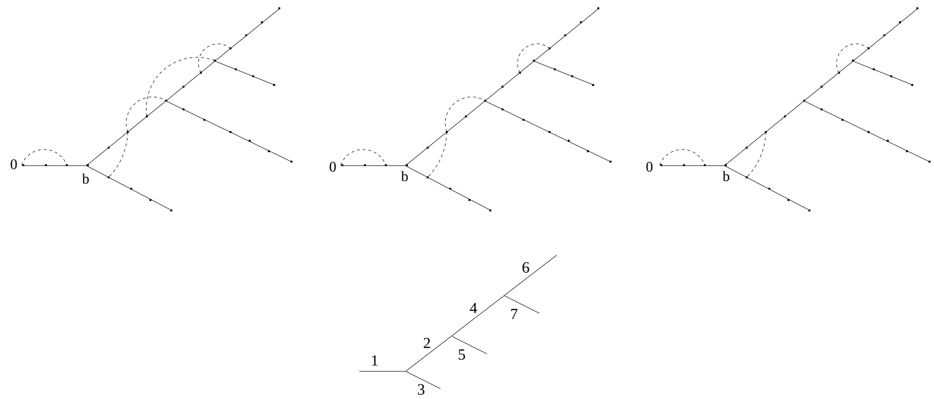


Figure 4.2: An example of graphs on $\mathcal{N}(\alpha, \vec{n})$ with $\alpha \in \Sigma_5$ a shape with edge labels shown at the bottom and $\vec{n} = (3, 4, 4, 3, 6, 4, 3)$. The first graph contains an edge in $\mathcal{R}$ so contributes to $\phi^{\mathcal{R}}$. The second graph does not contain such an edge but branch 2 is covered so this graph contributes to $\phi^{F}$. In the third graph, branches 2 and 3 are not covered, but $n_2 - 2 \geq m_2 = 2 > \frac{n_2}{3} = \frac{4}{3}$ and this graph contributes to $\phi^{\pi}$.

which is the first term on the right of (4.22). Taking Fourier transforms of (4.22) or (4.24) we obtain

$$\widehat{t}_{\mathcal{N}}(\vec{\kappa}) = \widehat{Q}_{\mathcal{N}}(\vec{\kappa}) + \widehat{\phi}^{b}_{\mathcal{N}}(\vec{\kappa}) + \widehat{\phi}^{\pi}_{\mathcal{N}}(\vec{\kappa}) - \widehat{\phi}^{\mathcal{R}}_{\mathcal{N}}(\vec{\kappa}). \tag{4.25}$$

**Lemma 4.2.1.** *The error terms defined in (4.18)–(4.23) satisfy*

$$\begin{aligned} \sum_{\vec{y}} |\phi^{\mathcal{R}}_{\mathcal{N}}(\vec{y})| &= \mathcal{O}\left(\sum_{i=1}^{2r-3} \frac{1}{n_i^{\frac{d-8}{2}}}\right), \\ \sum_{\vec{y}} |\phi^{b}_{\mathcal{N}}(\vec{y})| &= \mathcal{O}\left(\sum_{i=2}^{3} \frac{1}{n_i^{\frac{d-8}{2}}}\right), \\ \sum_{\vec{y}} |\phi^{\pi}_{\mathcal{N}}(\vec{y})| &= \mathcal{O}\left(\sum_{i=1}^{3} \frac{1}{n_i^{\frac{d-8}{2}}}\right), \end{aligned} \tag{4.26}$$

*where the constants implied by the $\mathcal{O}$ notation depend on $r$.*

The proof of Lemma 4.2.1 involves estimating diagrams and is postponed until Chapter 6.

## 4.3 Decomposition of $Q_{\mathcal{N}}$

In this section we show that $Q_{\mathcal{N}}$ can be expressed as a convolution of a function $\pi_{\vec{M}}$ and functions $t_{\mathcal{N}_j}$, for $j = 1, 2, 3$, and ultimately that $\widehat{Q}_{\mathcal{N}}$ can be expressed as a Gaussian term plus some error terms. The $\mathcal{N}_j$ are network shapes with $\alpha_j \in \Sigma_{r_j}$ and $r_j < r$. This permits analysis by induction on $r$.

We first define the quantity $\pi_{\vec{M}}(\vec{u})$ and then the constant $V$ appearing in Theorem 1.4.5 in terms of this function. We then state some bounds on the function $\pi_{\vec{M}}(\vec{u})$ in Proposition 4.3.2 and Lemma 4.3.3 that are the main ingredient for the proof of Theorem 1.4.5. The proofs of Proposition 4.3.2 and Lemma 4.3.3 are postponed until Chapter 5. The convolution expression for $Q_{\mathcal{N}}(\vec{y})$ involving $\pi_{\vec{M}}$ appears in Lemma 4.3.4, and for the Fourier transform in (4.42). Finally we express $\widehat{Q}_{\mathcal{N}}$ as a Gaussian term plus some error terms in (4.43). These error terms are bounded in Section 4.4.

**Definition 4.3.1.** *Suppose $\mathcal{S}^{\Delta}_{\vec{M}}$ is a star-shaped network of degree $\Delta \in \{1, 2, 3\}$ defined by branch lengths $\vec{M}$ as in (2.12). Let $\vec{u} \in \mathbb{Z}^{3d}$. We define*

$$\pi_{\vec{M}}(\vec{u}) = \sum_{\omega \in \Omega_{\mathcal{S}^{\Delta}_{\vec{M}}(\vec{u})}} W(\omega) \prod_{i \in \mathcal{S}^{\Delta}_{\vec{M}}} \sum_{R_i \in \mathcal{T}_{\omega(i)}} W(R_i) J(\mathcal{S}^{\Delta}_{\vec{M}}). \tag{4.27}$$

*Note that if $M_j = 0$ then $\Omega_{\mathcal{S}^{\Delta}_{\vec{M}}(\vec{u})}$ is empty unless $u_j = 0$. In particular if $\mathcal{S}^{\Delta}_{\vec{0}} = \{\vec{0}\}$ is a single vertex (star-shaped network of degree 0) then we define $\pi_{\vec{0}}(\vec{u}) = \rho(0) I_{\{\vec{u}=\vec{0}\}}$.*

Now by (2.7) we can write

$$\begin{aligned} J(\mathcal{S}^{\Delta}_{\vec{M}}) &= \sum_{N=1}^{\infty} \sum_{L \in \mathcal{L}^N(\mathcal{S}^{\Delta}_{\vec{M}})} \prod_{\flat \in L} U_\flat \prod_{\flat' \in \mathcal{C}(L)} [1 + U_{\flat'}] \\ &= \sum_{N=1}^{\infty} (-1)^N \sum_{L \in \mathcal{L}^N(\mathcal{S}^{\Delta}_{\vec{M}})} \prod_{\flat \in L} (-U_\flat) \prod_{\flat' \in \mathcal{C}(L)} [1 + U_{\flat'}], \end{aligned} \tag{4.28}$$

so that for $\vec{M} \neq \vec{0}$, $\pi_{\vec{M}}(\vec{u}) = \sum_{N=1}^{\infty} (-1)^N \pi^N_{\vec{M}}(\vec{u})$ where

$$\pi^N_{\vec{M}}(\vec{u}) \equiv \sum_{L \in \mathcal{L}^N(\mathcal{S}^{\Delta}_{\vec{M}})} \sum_{\omega \in \Omega_{\mathcal{S}^{\Delta}_{\vec{M}}(\vec{u})}} W(\omega) \prod_{i \in \mathcal{S}^{\Delta}_{\vec{M}}} \sum_{R_i \in \mathcal{T}(\omega(i))} W(R_i) \prod_{\flat \in L} (-U_\flat) \prod_{\flat' \in \mathcal{C}(L)} [1 + U_{\flat'}]. \tag{4.29}$$

Note that $\pi^N_{\vec{M}}(\vec{u}) \geq 0$ since $-U_\flat \geq 0$. We also define

$$V \equiv \sum_{\vec{M} \in \mathbb{Z}^3_+} \sum_{\vec{u} \in \mathbb{Z}^{3d}} \sum_{\vec{v} \in \mathbb{Z}^{3d}} \pi_{\vec{M}}(\vec{v}) \prod_{i=1}^{3} p_c D(u_i - v_i) = p_c^3 \sum_{\vec{M}} \sum_{\vec{v}} \pi_{\vec{M}}(\vec{v}). \tag{4.30}$$

The following Proposition is proved in Chapter 6 and is the main ingredient for the proof of Theorem 4.1.8. In order to state the proposition in a tidy manner, we introduce the notation

$$[M] \equiv M \vee 1. \tag{4.31}$$

**Proposition 4.3.2.** *There exists a constant $C$ independent of $L$ such that for $N \geq 1$ and $q \in \{0,1\}$,*

$$\sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^{2q} \pi^N_{\vec{M}}(\vec{u}) \leq N^3 (N^2\sigma^2 \|\vec{M}\|_\infty)^q B_N(\vec{M}), \tag{4.32}$$

*where $\vec{u} = (u_1, u_2, u_3)$, and*

$$B_N(\vec{M}) \equiv \left(C\beta^{2-\frac{8\nu}{d}}\right)^N \times \left[\prod_{i=1}^3 \frac{1}{[M_i]^{\frac{d-6}{2}}} + \sum_{i=1}^3 \frac{1}{[M_i]^{\frac{d-6}{2}}} \sum_{j\neq i} \sum_{m_j \leq M_j} \frac{1}{[M_j - m_j]^{\frac{d-6}{2}}} \frac{1}{[M_k + m_j]^{\frac{d-4}{2}}}\right]. \tag{4.33}$$

**Lemma 4.3.3.** *Let $B_N(\vec{M})$ be defined by (4.33) there is a constant $C$ independent of $L$ such that*

$$\begin{aligned} &\sum_{N=1} N^3 \sum_{\vec{M}: M_j \geq n_j} B_N(\vec{M}) \leq \frac{C\beta^{2-\frac{8\nu}{d}}}{[n_j]^{\frac{d-8}{2}}}, \quad j = 1,2,3, \quad \textit{and} \\ &\sum_{N=1} N^5 \sum_{\vec{M} \leq \vec{n}} \|\vec{M}\|_\infty B_N(\vec{M}) \leq \begin{cases} \|\vec{n}\|_\infty^{\frac{10-d}{2}\vee 0}, & \textit{if } d \neq 10 \\ \log \|n\|_\infty, & \textit{if } d = 10. \end{cases} \end{aligned} \tag{4.34}$$

Given $\vec{M} \in \mathcal{H}_{\vec{n}_b}$ we define $\mathcal{N}_i^- = (\mathcal{N} \setminus \mathcal{S}^{\Delta}_{\vec{M}})_i$, where the notation $(\mathcal{N} \setminus \mathcal{M})_i$ was defined in Definition 2.2.1. Note that the dependence of $\mathcal{N}_i^-$ on $\vec{M}$ is suppressed in the notation. Let vectors $\vec{y} \in \mathbb{Z}^{(2r-3)d}$ and $\vec{v} \in \mathbb{Z}^{3d}$ and $\mathcal{S}^{\Delta}_{\vec{M}} \subset \mathcal{N}$ with $\vec{M} \in \mathcal{H}_{\vec{n}_b}$ be given. We write $\mathcal{B}_{\mathcal{N}_i^-}$ for the set of branch labels of $\mathcal{N}$ that are branches in $\mathcal{N}_i^-$ but not $\mathcal{S}^{\Delta}_{\vec{M}}$ and we write $\vec{y}_i$ for the vector of $y_j$ such that $j \in \mathcal{B}_{\mathcal{N}_i^-}$. Then we define

$$\vec{y}_{v_i} = (y_i - v_i, \vec{y}_i). \tag{4.35}$$

**Lemma 4.3.4.** *Let $\vec{y}_{v_i}$ denote the vector of displacements associated to the branches of $\mathcal{N}_i$ (determined by $\vec{v}$, $\vec{y}$, and the labelling of the branches of $\mathcal{N}$ as in (4.35)). Then*

$$Q_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) = \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \sum_{\vec{u}} \pi_{\vec{M}}(\vec{u}) \prod_{i=1}^3 p_c \sum_{v_i} D(v_i - u_i) t_{\mathcal{N}_i^-}(\vec{y}_{v_i}). \tag{4.36}$$

*Proof.* First from (4.22) and (4.24) we have

$$Q_{\mathcal{N}(\alpha,\vec{n})}(\vec{y}) = \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \sum_{\omega\in\Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}_{\omega(s)}} W(R_s) J(\mathcal{S}^{\Delta}(\vec{M})) \prod_{i=1}^{3} K(\mathcal{N}_i^-). \tag{4.37}$$

However, as in the proof of (3.9) for the two point function, we may split up the branching random walk $\omega \in \Omega_{\mathcal{N}}(\vec{y})$ into 4 branching random walks (some of which may be empty) to obtain

$$\sum_{\omega\in\Omega_{\mathcal{N}(\vec{y})}} W(\omega) = \sum_{\vec{u}} \sum_{\omega\in\Omega_{\mathcal{S}^{\Delta}_{\vec{M}}}(\vec{u})} W(\omega) \prod_{i=1}^{3} \sum_{v_i} p_c D(v_i - u_i) \sum_{\omega_i\in\Omega_{\mathcal{N}_i^-}(\vec{y}_{v_i})} W(\omega_i). \tag{4.38}$$

Trivially,

$$\prod_{s\in\mathcal{N}} \sum_{R_s\in\mathcal{T}_{\omega(s)}} W(R_s) = \prod_{s\in\mathcal{S}^{\Delta}_{\vec{M}}} \sum_{R_s\in\mathcal{T}_{\omega(s)}} W(R_s) \prod_{i=1}^{3} \prod_{s_i\in\mathcal{N}_i^-} \sum_{R_{s_i}\in\mathcal{T}_{\omega_i(s_i)}} W(R_{s_i}), \tag{4.39}$$

where the products of the form $s \in \mathcal{N}^-$ are products over vertices in the network shape $\mathcal{N}^-$.

Since by definition, $\mathcal{N}_i^-$ and $\mathcal{S}^{\Delta}_{\vec{M}}$ are vertex disjoint (i.e. have no vertex in common), equations (4.37)–(4.39) show that $Q_{\mathcal{N}(\alpha,\vec{n})}(\vec{y})$ is equal to

$$\left( \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \sum_{\vec{u}} \sum_{\omega\in\Omega_{\mathcal{S}^{\Delta}_{\vec{M}}}(\vec{u})} W(\omega) \prod_{s\in\mathcal{S}^{\Delta}(\vec{M})} \sum_{R_s\in\mathcal{T}_{\omega(s)}} W(R_s) J(\mathcal{S}^{\Delta}_{\vec{M}}) \right) \times$$

$$\left( \prod_{i=1}^{3} \sum_{v_i} p_c D(v_i - u_i) \sum_{\omega_i\in\Omega_{\mathcal{N}_i^-}(\vec{y}_{v_i})} W(\omega_i) \prod_{s_i\in\mathcal{N}_i} \sum_{R_{s_i}\in\mathcal{T}_{\omega_i(s_i)}} W(R_{s_i}) K(\mathcal{N}_i^-) \right) \tag{4.40}$$

$$= \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \sum_{\vec{u}} \pi_{\vec{M}}(\vec{u}) \prod_{i=1}^{3} p_c \sum_{v_i} D(v_i - u_i) t_{\mathcal{N}_i^-}(\vec{y}_{v_i}),$$

as required. □

Given $\vec{\kappa} \in [-\pi, \pi]^{2r-3}$ we let $\vec{\kappa}^b = (\kappa_1, \kappa_2, \kappa_3)$, and $\vec{\kappa}_j^*$ denote the vector of $\kappa_i$, $i = 1, 2, \ldots, 2r-3$ such that $i$ is the label (inherited from $\mathcal{N}$) of a branch of $\mathcal{N}_j^-$. Then

$$e^{i\vec{\kappa}\cdot\vec{y}} = e^{i\vec{\kappa}^b\cdot\vec{u}} \prod_{j=1}^{3} e^{i\kappa_j\cdot(v_j-u_j)} e^{i\vec{\kappa}_j^*\cdot\vec{y}_{v_j}}. \tag{4.41}$$

From Lemma 4.3.4 we have

$$\widehat{Q}_{\mathcal{N}}(\vec{\kappa}) = \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \widehat{\pi}_{\vec{M}}\left(\vec{\kappa}^b\right) \prod_{j=1}^{3} p_c \widehat{D}(\kappa_j)\widehat{t}_{\mathcal{N}_j}(\vec{\kappa}_j^*). \tag{4.42}$$

Finally we write

$$\widehat{Q}_{\mathcal{N}}(\vec{\kappa}) = V^{r-2}\prod_{i=1}^{2r-3} Ae^{-\frac{\kappa_i^2}{2d}n_i\sigma^2 v} + \mathcal{E}_{\vec{n}}^{D}(\vec{\kappa}) + \mathcal{E}_{\vec{n}}^{\vec{0}}(\vec{\kappa}) + \mathcal{E}_{\vec{n}}^{\mathrm{ind}}(\vec{\kappa}) + \mathcal{E}_{\vec{n}}^{V}(\vec{\kappa}), \tag{4.43}$$

where the $\mathcal{E}_{\vec{n}}^{\cdot}$ are defined by

$$\begin{aligned}
\mathcal{E}_{\vec{n}}^{D}(\vec{\kappa}) &\equiv \sum_{\substack{E\subset\{1,2,3\}\\ E\neq\emptyset}} \left(\prod_{l\in E}(\widehat{D}(\kappa_l)-1)\right) p_c^3 \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \widehat{\pi}_{\vec{M}}(\vec{\kappa}^b)\prod_{j=1}^{3}\widehat{t}_{\mathcal{N}_j}(\vec{\kappa}_j^*),\\
\mathcal{E}_{\vec{n}}^{\vec{0}}(\vec{\kappa}) &= p_c^3 \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \left(\widehat{\pi}_{\vec{M}}(\vec{\kappa}^b) - \widehat{\pi}_{\vec{M}}(\vec{0})\right)\prod_{j=1}^{3}\widehat{t}_{\mathcal{N}_j}(\vec{\kappa}_j^*),\\
\mathcal{E}_{\vec{n}}^{\mathrm{ind}}(\vec{\kappa}) &= p_c^3 \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \widehat{\pi}_{\vec{M}}(\vec{0})\left(\prod_{j=1}^{3}\widehat{t}_{\mathcal{N}_j^-}(\vec{\kappa}_j^*) - V^{r-3}\prod_{l=1}^{2r-3} Ae^{-\frac{\kappa_l^2}{2d}n_l\sigma^2 v}\right),\\
\mathcal{E}_{\vec{n}}^{V}(\vec{\kappa}) &= V^{r-3}\prod_{l=1}^{2r-3} Ae^{-\frac{\kappa_l^2}{2d}n_l\sigma^2 v} p_c^3 \sum_{\vec{M}\in\overline{\mathcal{H}}_{\vec{n}_b}} \widehat{\pi}_{\vec{M}}(\vec{0}).
\end{aligned} \tag{4.44}$$

The first term is obtained by writing $\widehat{D}(\kappa_j) = \left(1 + (\widehat{D}(\kappa_j) - 1)\right)$, the second by writing $\widehat{\pi}_{\vec{M}}(\vec{\kappa}^b) = \left(\widehat{\pi}_{\vec{M}}(\vec{0}) + (\widehat{\pi}_{\vec{M}}(\vec{\kappa}^b) - \widehat{\pi}_{\vec{M}}(\vec{0}))\right)$ and so on.

## 4.4 Bounds on the $\mathcal{E}_\bullet$

In this section we prove bounds on the quantities (4.44), as stated in Lemma 4.4.2. All of these terms will turn out to be error terms in our analysis and in general rely on estimates for $\widehat{\pi}_{\vec{M}}(\vec{\kappa})$ such as those appearing in Proposition 4.3.2. Each term except $\mathcal{E}^{\mathrm{ind}}$ will also use naive bounds of the form appearing in Lemma 4.4.1, in which $\#_{\mathcal{M}}$ denotes the number of branches in $\mathcal{M}$ (recall the definition of a branch from the second paragraph of Section 2.1).

**Lemma 4.4.1.** *There exists a constant $K$, independent of $L$, $\mathcal{M}$ and $\vec{\kappa}$ such that for any network $\mathcal{M}$*

$$\widehat{t}_{\mathcal{M}}(\vec{\kappa}) \leq K^{\#_{\mathcal{M}}}. \tag{4.45}$$

The proof of Lemma 4.4.1 is elementary, but we also postpone this proof until Chapter 6.

Using Lemma 4.3.3 with $n_j = 1$,

$$\sum_{\vec{M}\neq\vec{0}} |\widehat{\pi}_{\vec{M}}(\vec{\kappa}^b)| = \sum_{N=1}^{\infty} \sum_{\vec{M}\neq\vec{0}} \sum_{\vec{u}} \pi^N_{\vec{M}}(\vec{u}) \leq \sum_N \sum_{\vec{M}} B_N(\vec{M}) \leq C\beta^{2-\frac{8\nu}{d}}, \tag{4.46}$$

where the constant is independent of $L$. In particular since $\widehat{\pi}_{\vec{0}}(\vec{0}) = 1$, this proves that $V = 1 + \mathcal{O}(\beta^{2-\frac{8\nu}{d}})$.

**Lemma 4.4.2 ($\mathcal{E}^{\bullet}_{\vec{n}}$ bounds).** *For all $\vec{\kappa}$,*

$$\mathcal{E}^D_{\vec{n}}(\vec{\kappa}) = \mathcal{O}\left(L^2 \sum_{j=1}^{3} \kappa_j^2\right), \tag{4.47}$$

$$\mathcal{E}^{\vec{0}}_{\vec{n}}(\vec{\kappa}) = \begin{cases} \mathcal{O}\left(|\vec{\kappa}^b|^2\sigma^2\|\vec{n}^b\|_\infty^{(\frac{10-d}{2}\vee 0)}\right), & \textit{if } d \neq 10 \\ \mathcal{O}\left(|\vec{\kappa}^b|^2\sigma^2 \log\|n\|_\infty\right), & \textit{if } d = 10, \end{cases} \tag{4.48}$$

$$\mathcal{E}^V_{\vec{n}}(\vec{\kappa}) = \mathcal{O}\left(\sum_{j=1}^{3} \frac{1}{n_i^{\frac{d-8}{2}}}\right). \tag{4.49}$$

*Proof of (4.47).* For $l \notin E$ we bound $\prod_{j=1}^3 \widehat{t}_{\mathcal{N}_j}(\vec{\kappa}^*_j)$ and $\sum_{\vec{M}} \widehat{\pi}_{\vec{M}}(\vec{\kappa}^b)$ by constants using Lemma 4.4.1 and (4.46). This leaves us with

$$|\mathcal{E}^D_{\vec{n}}(\vec{\kappa})| \leq C \sum_{\substack{E \subset \{1,2,3\} \\ E \neq \emptyset}} \prod_{j\in E} a(\kappa_j). \tag{4.50}$$

For each nonempty $E$ we may bound all but one of the $a(k_i)$ by 2. This gives

$$|\mathcal{E}^D_{\vec{n}}(\vec{\kappa})| \leq C \sum_{j=1}^{3} a(\kappa_j). \tag{4.51}$$

In particular since $a(\kappa_j) \leq 2$ this quantity is also bounded by a constant $C'$. If $\|\vec{\kappa}_j\|_\infty \geq L^{-1}$, then $C' \leq c\|\kappa^b\|^2 L^2$ and we have obtained (4.47) for $\|\vec{\kappa}_j\|_\infty \geq L^{-1}$. By (3.29) we have for $\|\vec{\kappa}_j\|_\infty \leq L^{-1}$ that

$$a(\kappa_j) \leq CL^2\kappa_j^2 \tag{4.52}$$

which proves the first bound for $\|\vec{\kappa}_j\|_\infty \leq L^{-1}$. ☐

*Proof of (4.48).* We bound the $\widehat{t}_{\mathcal{N}}$ by a constant and apply (3.48) with $3d$ instead of $d$ with $\vec{\kappa^b} = (\kappa_{1,1}, \dots, \kappa_{1,d}, \kappa_{2,1}, \dots, \kappa_{2,d}, \kappa_{3,1}, \dots, \kappa_{3,d})$ to bound the difference $\widehat{\pi}_{\vec{M}}(\vec{0}) - \widehat{\pi}_{\vec{M}}(\vec{\kappa}^b)$. In doing so we obtain

$$\left|\widehat{\pi}_{\vec{M}}(\vec{0}) - \widehat{\pi}_{\vec{M}}(\vec{\kappa}^b)\right| \leq C|\vec{\kappa}^b|^2 \sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^2 \left|\widehat{\pi}_{\vec{M}}(\vec{u})\right|. \tag{4.53}$$

This gives us

$$|\mathcal{E}^{\vec{0}}_{\vec{n}}(\vec{\kappa})| \leq C \sum_{\vec{M}\leq\vec{n}^b} |\vec{\kappa}^b|^2 \sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^2 \left|\widehat{\pi}_{\vec{M}}(\vec{u})\right|. \tag{4.54}$$

Applying Proposition 4.3.2 and Lemma 4.3.3 we obtain

$$\left|\mathcal{E}^{\vec{0}}_{\vec{n}}(\vec{\kappa})\right| \leq C \sum_{\vec{M}\leq\vec{n}^b} |\vec{\kappa}|^2\sigma^2\|\vec{M}\|_\infty N^5 B_N(\vec{M}) \leq C\beta^{2-\frac{8\nu}{d}} \begin{cases} |\vec{\kappa}^b|^2\sigma^2\|\vec{n}^b\|_\infty^{(\frac{10-d}{2}\vee 0)} & \text{if } d\neq 10 \\ |\vec{\kappa}^b|^2\sigma^2\log\|\vec{n}^b\|_\infty, & \text{if } d=10, \end{cases} \tag{4.55}$$

as required. □

*Proof of (4.49).* We bound each exponential by a constant, leaving

$$|\mathcal{E}^{V}_{\vec{n}}(\vec{\kappa})| \leq C \sum_{\vec{M}\in\overline{\mathcal{H}}_{\vec{n}^b}} |\widehat{\pi}_{\vec{M}}(\vec{0})|. \tag{4.56}$$

Next we observe that $\vec{M} \in \overline{\mathcal{H}}_{\vec{n}^b}$ only if $M_j \geq \frac{n_j}{3}$ for some $j \in \{1,2,3\}$. The required bound then follows from Proposition 4.3.2 and Lemma 4.3.3. □

It follows immediately from (4.47) that

$$\mathcal{E}^{D}_{\vec{n}}(\vec{\kappa})\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right) = \mathcal{O}\left(\frac{L^2\sum_{j=1}^3 \kappa_j^2}{\sigma^2 n}\right) = \mathcal{O}\left(\frac{\sum_{j=1}^3 \kappa_j^2}{n}\right), \tag{4.57}$$

and from (4.48) that

$$\mathcal{E}^{\vec{0}}_{\vec{n}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right) = \begin{cases} \mathcal{O}\left(\frac{|\vec{\kappa}^b|^2\sigma^2\|\vec{n}^b\|_\infty^{(\frac{10-d}{2}\vee 0)}}{v\sigma^2 n}\right), & \text{if } d\neq 10, \\ \mathcal{O}\left(\frac{|\vec{\kappa}^b|^2\sigma^2\log\|\vec{n}^b\|_\infty}{v\sigma^2 n}\right), & \text{if } d=10. \end{cases} \tag{4.58}$$

## 4.5 Proof of Theorem 4.1.8.

We prove Theorem 4.1.8 by induction on $r$ (or equivalently on the number of branches $2r-3$ in $\mathcal{N}$). For $r=2$ recall that $A = A'\rho(0)$, so (as in the proof of Theorem 1.4.3) we have by Theorem 3.4.3 and Lemma 3.5.2 that

$$\widehat{t}_{n_1}\left(\frac{\kappa}{\sqrt{v\sigma^2 n}}\right) = Ae^{-\frac{\kappa^2}{2d}\frac{n_1}{n}} + \mathcal{O}\left(\frac{\kappa^2\frac{n_1}{n}}{n_1^{\delta}}\right) + \mathcal{O}\left(\frac{1}{n_1^{\frac{d-8}{2}}}\right), \tag{4.59}$$

with the error terms uniform in $\{\kappa \in \mathbb{R}^d : |\kappa|^2 \leq C_0 \log n_1\}$. This yields the required result for $r = 2$.

Now fix $r$ and $\mathcal{N} = \mathcal{N}(\alpha, \vec{n})$ with $\alpha \in \Sigma_r$ and $\vec{n} \in \mathbb{N}^{2r-3}$, and assume the theorem holds for all $r_i < r$. By (4.25) and Lemma 4.2.1, we have that

$$\widehat{t}_{\mathcal{N}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right) = \widehat{Q}_{\mathcal{N}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right) + \mathcal{O}\left(\sum_{l=1}^{2r-3}\frac{1}{n_l^{\frac{d-8}{2}}}\right). \tag{4.60}$$

Next from (4.43), (4.57)–(4.58) and (4.49), we have that $\widehat{Q}_{\mathcal{N}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right)$ is equal to $V^{r-2}\prod_{l=1}^{2r-3} Ae^{-\frac{\kappa_l^2}{2d}\frac{n_l}{n}}$ plus

$$\mathcal{E}_{\vec{n}}^{\text{ind}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right) + \mathcal{O}\left(\frac{\sum_{j=1}^{3}\kappa_j^2}{n}\right) + \mathcal{O}\left(\frac{|\vec{\kappa}^b|^2\sigma^2\|\vec{n}^b\|_\infty^{(\frac{10-d}{2}\vee 0)}}{v\sigma^2 n}\right) + \mathcal{O}\left(\sum_{j=1}^{3}\frac{1}{n_j^{\frac{d-8}{2}}}\right), \tag{4.61}$$

plus the error term (4.58) Since $\delta < \frac{d-8}{2} \wedge 1$ in the statement of Theorem 4.1.8 we have $\frac{10-d}{2} \vee 0 < 1 - \delta$ and these error terms satisfy the error bounds of the Theorem.

It remains to show that $\mathcal{E}_{\vec{n}}^{\text{ind}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right)$ is an error term of the required type. From (4.44) we have

$$\mathcal{E}_{\vec{n}}^{\text{ind}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right) = p_c^3 \sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}} \widehat{\pi}_{\vec{M}}(\vec{0})\left(\prod_{j=1}^{3}\widehat{t}_{\mathcal{N}_j^-}\left(\frac{\vec{\kappa}_j^*}{\sqrt{v\sigma^2 n}}\right) - V^{r-3}\prod_{l=1}^{2r-3} Ae^{-\frac{\kappa_l^2 n_l}{2dn}}\right). \tag{4.62}$$

By the induction hypothesis applied to $r_j < r$, we have

$$\begin{aligned}\widehat{t}_{\mathcal{N}_j^-}\left(\frac{\vec{\kappa}_j^*}{\sqrt{v\sigma^2 n}}\right) =& V^{r_j-2}A^{2r_j-3}\prod_{l\in\mathcal{B}_{\mathcal{N}_j^-}} e^{-\frac{\kappa_{jl}^{*2} n_{jl}^*}{2dn}} \\ &+ \mathcal{O}\left(\sum_{l\in\mathcal{B}_{\mathcal{N}_j^-}}\frac{1}{n_{jl}^{*\frac{d-8}{2}}}\right) + \mathcal{O}\left(\sum_{l\in\mathcal{B}_{\mathcal{N}_j^-}}\frac{|\vec{\kappa}_j^*|^2 n_{jl}^{*(1-\delta)}}{n}\right),\end{aligned} \tag{4.63}$$

where the sums and products are over branch labels of branches in $\mathcal{N}_j^-$. If $\mathcal{H}_{\vec{n}_b} = \emptyset$ then $\mathcal{E}_{\vec{n}}^{\text{ind}} = 0$. For $\vec{M} \in \mathcal{H}_{\vec{n}_b}$, for every $j \in \{1, 2, 3\}$ we have $\frac{2n_l}{3} \leq n_{jl} \leq n_l$. This

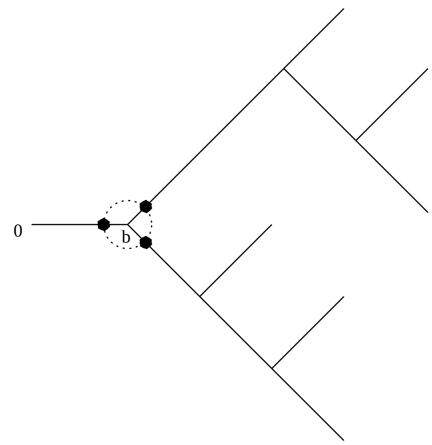


Figure 4.3: An illustration of the relation $\sum_{i=1}^3 r_i = r+3$ resulting from the decomposition of a network $\mathcal{N}$ into $\mathcal{N}_i$, when $\vec{M} \in \mathcal{H}_{\vec{n}_b}$. The 3 extra vertices generated by this decomposition are indicated.

enables us to replace $n_{jl}$ by $n_l$ if necessary in the error terms of (4.63). Additionally since $\vec{M} \in \mathcal{H}_{\vec{n}_b}$ we have $r = \sum_{i=1}^3 (r_i - 1)$, or equivalently $\sum_{i=1}^3 r_i = r+3$ (see Figure 4.3) and

$$\begin{aligned} \prod_{j=1}^{3} \widehat{t}_{\mathcal{N}_j^-}\left(\frac{\vec{\kappa}_j^*}{\sqrt{v\sigma^2 n}}\right) =& V^{r-3} A^{2r-3} \prod_{l=4}^{2r-3} e^{-\frac{\kappa_l^2 n_l}{2dn}} \prod_{j=1}^{3} e^{-\frac{\kappa_j^2 (n_j - M_j)}{2dn}} \\ &+ \mathcal{O}\left(\sum_{l=1}^{2r-3} \frac{1}{n_l^{\frac{d-8}{2}}}\right) + \mathcal{O}\left(\sum_{l=1}^{2r-3} \frac{|\vec{\kappa}|^2 n_l^{(1-\delta)}}{n}\right). \end{aligned} \tag{4.64}$$

Thus, $\prod_{j=1}^3 \widehat{t}_{\mathcal{N}_j}\left(\frac{\vec{\kappa}_j^*}{\sqrt{v\sigma^2 n}}\right) - V^{r-3} \prod_{l=1}^{2r-3} A e^{-\frac{\kappa_l^2 n_l}{2dn}}$ is equal to

$$\begin{aligned} & V^{r-3} A^{2r-3} \prod_{l=4}^{2r-3} e^{-\frac{\kappa_l^2 n_l}{2dn}} \left[\prod_{j=1}^{3} e^{-\frac{\kappa_j^2 (n_j - M_j)}{2dn}} - \prod_{j=1}^{3} e^{-\frac{\kappa_j^2 n_j}{2dn}}\right] \\ &+ \mathcal{O}\left(\sum_{l=1}^{2r-3} \frac{1}{n_l^{\frac{d-8}{2}}}\right) + \mathcal{O}\left(\sum_{l=1}^{2r-3} \frac{|\vec{\kappa}|^2 n_l^{(1-\delta)}}{n}\right). \end{aligned} \tag{4.65}$$

Next using a telescoping sum and the inequality $e^{-a} - e^{-b} \le C(b-a)$ for $b \ge a \ge 0$

we see that $\left[\prod_{j=1}^3 e^{-\frac{\kappa_j^2(n_j-M_j)}{2dn}} - \prod_{j=1}^3 e^{-\frac{\kappa_j^2 n_j}{2dn}}\right]$ is equal to

$$\begin{aligned}\sum_{l=1}^{3}&\left(\prod_{j<l} e^{-\frac{\kappa_j^2 n_j}{2dn}}\right)\left[e^{-\frac{\kappa_l^2(n_l-M_l)}{2dn}} - e^{-\frac{\kappa_l^2 n_l}{2dn}}\right]\left(\prod_{j>l} e^{-\frac{\kappa_j^2(n_j-M_j)}{2dn}}\right)\\ &\le C\sum_{l=1}^{3}\frac{\kappa_l^2}{2dn}\left[n_l-(n_l-M_l)\right] = C\sum_{l=1}^{3}\frac{\kappa_l^2 M_l}{2dn}.\end{aligned} \tag{4.66}$$

Collecting terms and applying Proposition 4.3.2 and Lemma 4.3.3 we have

$$\begin{aligned}\left|\mathcal{E}_{\vec{n}}^{\text{ind}}\left(\frac{\vec{\kappa}}{\sqrt{v\sigma^2 n}}\right)\right| \le& p_c^3\sum_{l=1}^{3}\frac{\kappa_l^2}{2dn}\sum_{N}\sum_{\vec{M}\in\mathcal{H}_{\vec{n}_b}}\widehat{\pi}_{\vec{M}}^{N}(\vec{0})M_l\\ &+\mathcal{O}\left(\sum_{l=1}^{2r-3}\frac{1}{n_l^{\frac{d-8}{2}}}\right)+\mathcal{O}\left(\sum_{l=1}^{2r-3}\frac{|\vec{\kappa}|^2 n_l^{(1-\delta)}}{n}\right)\\ =&\mathcal{O}\left(\sum_{l=1}^{3}\frac{|\vec{\kappa}|^2 n_l^{\frac{10-d}{2}\vee 0}}{n}\right)+\mathcal{O}\left(\sum_{l=1}^{2r-3}\frac{1}{n_l^{\frac{d-8}{2}}}\right)+\mathcal{O}\left(\sum_{l=1}^{2r-3}\frac{|\vec{\kappa}|^2 n_l^{(1-\delta)}}{n}\right).\end{aligned} \tag{4.67}$$

Since $1-\delta > \frac{10-d}{2}\vee 0$ these are all error terms, and the proof is complete. □

## 4.6 Proof of Theorem 1.4.5.

In this section we prove Theorem 1.4.5. By (4.8) there are two terms to consider. From (4.9), the first term of (4.8) involves a quantity that is treated in Theorem 4.1.8, summed over shapes and temporal locations of the branch points. We shall see in the proof of Theorem 1.4.5 that with the appropriate scaling this first term approximates the sum over shapes of the integral in Theorem 1.4.5.

The second term of (4.8) is the contribution from degenerate trees and Lemma 4.6.2 shows that this is an error term. In proving this Lemma we will make use of an expression of the form (4.5) for degenerate trees. As such we introduce the notion of a degenerate shape.

**Definition 4.6.1 (Degenerate Shape).** *For $r \ge 3$, let $\overline{\Sigma}_r$ be the set of rooted trees $\bar{\alpha}$ such that*

1. *$\bar{\alpha}$ contains fewer than $2r-3$ edges, and fewer than $r-2$ branch points (vertices of degree $\ge 3$) that are not the root, and*

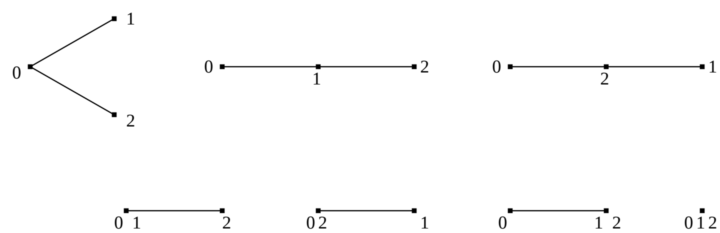


Figure 4.4: The seven possible degenerate shapes for $r = 3$. The second (resp. third) shape is only a possible candidate for the shape of $B \in \mathbf{B}_D(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$ if $\mathbf{n}_2 > \mathbf{n}_1$ (resp. $\mathbf{n_2} < \mathbf{n_1}$).

2. *for each* $i \in \{0, \dots, r-1\}$ *there exists a vertex in* $\bar{\alpha}$ *with label* $i$ *(each vertex may have more than one label), and each leaf (vertex of degree* 1*) of* $\bar{\alpha}$ *has at least one label.*

*We call* $\bar{\alpha} \in \overline{\Sigma}_r$ *a* degenerate shape. *Clearly there are only finitely many degenerate shapes for each fixed* $r$. *See Figure 4.4 for the set* $\overline{\Sigma_3}$.

By Definition (4.1.3) and Lemma 4.1.5, if $B \in \mathbf{B}_D(\tilde{\mathbf{n}}, \tilde{\mathbf{x}})$ for some $\tilde{\mathbf{n}} \in \mathbb{N}^{r-1}$, $\tilde{\mathbf{x}} \in \mathbb{Z}^{d(r-1)}$ then $B$ has the topology of some $\bar{\alpha} \in \overline{\Sigma}_r$. For $\bar{\alpha} \in \overline{\Sigma}_r$ consisting of $l < 2r-3$ edges and $\vec{n} \in \mathbb{N}^l$ we define $\mathcal{D}(\bar{\alpha}, \vec{n})$ to be the abstract *network shape* obtained by inserting $n_j - 1$ vertices onto edge $j$ of $\bar{\alpha}$, $j = 1, \dots, l$. Furthermore for $\vec{y} \in \mathbb{Z}^{dl}$ we define $\mathcal{T}_{\mathcal{D}(\bar{\alpha},\vec{n})}(\vec{y})$ to be the set of lattice trees $T \in \mathcal{T}_0$ with network shape $\mathcal{D}(\bar{\alpha}, \vec{n})$ such that for each edge $i$ in $\bar{\alpha}$ with endvertices $i_1, i_2$ ($i_1$ is closer to the root), the corresponding vertices $u$, $v$ in $T$ satisfy $v - u = y_i$. Furthermore we define

$$t_{\mathcal{D}(\bar{\alpha},\vec{n})}(\vec{y}) = \sum_{T \in \mathcal{T}_{\mathcal{D}(\bar{\alpha},\vec{n})}(\vec{y})} W(T). \tag{4.68}$$

Then as in the nondegenerate case (4.5),

$$\begin{aligned} \sum_{\tilde{\mathbf{x}}} \sum_{B \in \mathbf{B}_D(\tilde{\mathbf{n}},\tilde{\mathbf{x}})} \sum_{T \in \mathcal{T}_B} W(T) &\leq \sum_{\bar{\alpha} \in \overline{\Sigma}_r} \sum_{\vec{n} \overset{\bar{\alpha}}{\rightsquigarrow} \tilde{\mathbf{n}}} \sum_{\tilde{\mathbf{x}}} \sum_{\vec{y} \overset{\bar{\alpha}}{\rightsquigarrow} \tilde{\mathbf{x}}} \sum_{T \in \mathcal{T}_{\mathcal{D}(\bar{\alpha},\vec{n})}(\vec{y})} W(T) \\ &= \sum_{\bar{\alpha} \in \overline{\Sigma}_r} \sum_{\vec{n} \overset{\bar{\alpha}}{\rightsquigarrow} \tilde{\mathbf{n}}} \widehat{t}_{\mathcal{D}(\bar{\alpha},\vec{n})}(\vec{0}). \end{aligned} \tag{4.69}$$

Note that for any given $\tilde{\mathbf{n}} \in \mathbb{N}^{r-1}$ we may have many $\bar{\alpha}$ for which the set $\{\vec{n} : \vec{n} \overset{\bar{\alpha}}{\rightsquigarrow} \tilde{\mathbf{n}}\}$ is empty.

We are now able to prove the following Lemma.

**Lemma 4.6.2.** *For all* $\tilde{\mathbf{k}} \in [-\pi, \pi]^{(r-1)d}$,

$$|\widehat{\phi}^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{k}})| \leq C_r \|\tilde{\mathbf{n}}\|_\infty^{r-3}. \tag{4.70}$$

*Proof.* Let $l = l(\bar{\alpha})$ be the number of edges in $\bar{\alpha}$. Applying Lemma 4.4.1 to $\mathcal{D}$ we obtain

$$\widehat{t}_{\mathcal{D}(\bar{\alpha},\vec{n})}(\vec{0}) \leq K^l. \tag{4.71}$$

Therefore, (4.69) implies that

$$|\widehat{\phi}^r_{\tilde{\mathbf{n}}}(\tilde{\mathbf{k}})| \leq \sum_{\bar{\alpha}\in\overline{\Sigma}_r} \sum_{\vec{n}\overset{\bar{\alpha}}{\rightsquigarrow}\tilde{\mathbf{n}}} K^l \leq \sum_{\bar{\alpha}\in\overline{\Sigma}_r} \|\tilde{\mathbf{n}}\|_\infty^{r-3} K^{2r-4} \leq C_r \|\tilde{\mathbf{n}}\|_\infty^{r-3}. \tag{4.72}$$

The second inequality holds since $\sum_{\vec{n}\overset{\bar{\alpha}}{\rightsquigarrow}\tilde{\mathbf{n}}}$ is a sum over at most $r-3$ temporal locations of branch points which are not the origin, each of which must be smaller than $\|\tilde{\mathbf{n}}\|_\infty$ by definition. □

We are now ready to prove Theorem 1.4.5 which we restate below.

**Theorem (1.4.5).** *Fix $d > 8$, $\gamma \in (0, 1 \wedge \frac{d-8}{2})$ and $\delta \in (0, (1 \wedge \frac{d-8}{2}) - \gamma)$. There exists $L_0 = L_0(d) \gg 1$ such that: for each $L \geq L_0$, $\tilde{\mathbf{t}} \in (0,\infty)^{(r-1)}$, $r \geq 3$, $R > 0$, and $\|\vec{k}\|_\infty \leq R$,*

$$\widehat{t}^{(r)}_{\lfloor n\tilde{\mathbf{t}}\rfloor}\left(\frac{\tilde{\mathbf{k}}}{\sqrt{v\sigma^2 n}}\right) = n^{r-2}V^{r-2}A^{2r-3}\left[\sum_{\alpha\in\Sigma_r}\int_{R_{\tilde{\mathbf{t}}}(\alpha)} \prod_{l=1}^{2r-3} e^{-\frac{\kappa_l(\alpha)^2 s_l}{2d}} d\vec{s} + \mathcal{O}\left(\frac{1}{n^\delta}\right)\right], \tag{4.73}$$

*where the constant in the error term depends on $\tilde{\mathbf{t}}, R$ and $L$, and where $V$ is the constant of Theorem 4.1.8.*

*Proof.* From (4.8) and Lemma 4.6.2 we have

$$\begin{aligned}\widehat{t}^{(r)}_{\lfloor n\tilde{\mathbf{t}}\rfloor}\left(\frac{\tilde{\mathbf{k}}}{\sqrt{\sigma^2 vn}}\right) &= \sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor} \widehat{t}_{\mathcal{N}(\alpha,\vec{n})}\left(\frac{\vec{\kappa}}{\sqrt{\sigma^2 vn}}\right) + \widehat{\phi}^r_{\lfloor n\tilde{\mathbf{t}}\rfloor}\left(\frac{\tilde{\mathbf{k}}}{\sqrt{\sigma^2 vn}}\right) \\ &= \sum_{\alpha\in\Sigma_r}\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor} \widehat{t}_{\mathcal{N}(\alpha,\vec{n})}\left(\frac{\vec{\kappa}}{\sqrt{\sigma^2 vn}}\right) + n^{r-2}\mathcal{O}\left(\frac{\|\tilde{\mathbf{t}}\|_\infty^{r-3}}{n}\right),\end{aligned} \tag{4.74}$$

where $\vec{\kappa} = \vec{\kappa}(\alpha, \tilde{\mathbf{k}})$ as described in (1.29). Theorem 4.1.8 may be applied to the first term, giving

$$\begin{aligned}\widehat{t}^{(r)}_{\lfloor n\tilde{\mathbf{t}}\rfloor}\left(\frac{\tilde{\mathbf{k}}}{\sqrt{\sigma^2 vn}}\right) = \sum_{\alpha\in\Sigma_r} \sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor :\\ \vec{n}\in\mathbb{N}^{2r-3}}} &\left[V^{r-2}A^{2r-3}\prod_{j=1}^{2r-3} e^{\frac{-\kappa_j^2}{2d}\left(\frac{n_j}{n}\right)} + \right. \\ &\left.\mathcal{O}\left(\sum_{j=1}^{2r-3}\frac{1}{n_j^{\frac{d-8}{2}}}\right) + \mathcal{O}\left(\sum_{j=1}^{2r-3}\frac{|\vec{\kappa}|^2 n_j^{1-\delta}}{n}\right)\right] + n^{r-2}\mathcal{O}\left(\frac{\|\tilde{\mathbf{t}}\|_\infty^{r-3}}{n}\right).\end{aligned} \tag{4.75}$$

Considering the first error term, note that

$$
\begin{aligned}
\sum_{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor} \frac{1}{n_j^{\frac{d-8}{2}}} &= \sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor :\\ n_j\le \frac{\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty}{2}}} \frac{1}{n_j^{\frac{d-8}{2}}} + \sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor :\\ n_j> \frac{\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty}{2}}} \frac{1}{n_j^{\frac{d-8}{2}}} \\
&\le \sum_{m\le \frac{\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty}{2}} \sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor :\\ n_j=m}} \frac{1}{n_j^{\frac{d-8}{2}}} + \frac{C}{n^{\frac{d-8}{2}}} \sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor :\\ n_j> \frac{\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty}{2}}} 1 \\
&\le \sum_{m\le \frac{\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty}{2}} \frac{1}{m^{\frac{d-8}{2}}} \sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor :\\ n_j=m}} 1 + \frac{C}{n^{\frac{d-8}{2}}}\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty^{r-2} \\
&\le C\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty^{(\frac{10-d}{2}\vee 0)}\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty^{r-3} + \frac{C}{n^{\frac{d-8}{2}}}\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty^{r-2},
\end{aligned}
\tag{4.76}
$$

where in the last step we used the fact that since $n_j$ is fixed, the sum over $\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor : n_j=m$ is a sum over the locations of $r-3$ branch points. Note that if $d=10$ we interpret the quantity $\|\lfloor n\tilde{\mathbf{t}}\rfloor\|_\infty^{(\frac{10-d}{2}\vee 0)}$ as $\log(\lfloor n\tilde{\mathbf{t}}\rfloor)$. Thus, since $|\Sigma_r|$ is a finite quantity depending only on $r$, the first error term in (4.75) is

$$
n^{r-2}\mathcal{O}\left(\frac{1}{n^\delta}\right) \tag{4.77}
$$

where the constant in the error term depends on $r$ and $\vec{t}$ (and goes to 0 as $\vec{t}\searrow\vec{0}$).

The second error term in (4.75) is

$$
\mathcal{O}\left(\frac{|\vec{\kappa}|^2 n_j^{1-\delta}}{n}\right) = n^{r-2}\mathcal{O}\left(\frac{|\tilde{\mathbf{k}}|^2\|\tilde{\mathbf{t}}\|_\infty^{r-1-\delta}}{n^\delta}\right), \tag{4.78}
$$

where we have used (1.29) with $\kappa_j^2 \le (r-1)\sum_{j=1}^{r-1}\left(k_j I_{l\in E_j}\right)^2$.

The third error term is already of the form $n^{r-2}\mathcal{O}\left(\frac{1}{n^\delta}\right)$ where the constant depends on $\tilde{\mathbf{t}}$. Thus it remains to show that

$$
\left| \sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor :\\ \vec{n}\in\mathbb{N}^{2r-3}}} \prod_{j=1}^{2r-3} e^{\frac{-\kappa_j^2}{2d}\left(\frac{n_j}{n}\right)} - n^{r-2}\int_{R_{\tilde{\mathbf{t}}}(\alpha)} \prod_{j=1}^{2r-3} e^{-\frac{\kappa_j^2 s_j}{2d}}\, d\vec{s} \right| = \mathcal{O}\left(\frac{1}{n^\delta}\right), \tag{4.79}
$$

for each $\alpha\in\Sigma_r$, where the constant depends on $\tilde{\mathbf{t}}$, $r$ and $R$. We rewrite the left hand side as

$$n^{r-2}\left|\frac{1}{n^{r-2}}\sum_{\substack{\vec{n}\overset{\alpha}{\rightsquigarrow}\lfloor n\tilde{\mathbf{t}}\rfloor\,:\\ \vec{n}\in\mathbb{N}^{2r-3}}}\prod_{j=1}^{2r-3}e^{\frac{-\kappa_j^2}{2d}\left(\frac{n_j}{n}\right)}-\int_{R_{\tilde{\mathbf{t}}}(\alpha)}\prod_{j=1}^{2r-3}e^{-\frac{\kappa_j(\alpha)^2 s_j}{2d}}d\vec{s}\right|. \tag{4.80}$$

Observe that the left hand term inside the absolutely value is the Riemann sum approximation to the integral on the right, with the approximation breaking $R_{\tilde{\mathbf{t}}}(\alpha)$ into cubes of side $\frac{1}{n}$, with some overcounting or undercounting at the boundary. The set $R_{\tilde{\mathbf{t}}}(\alpha)$ is a convex $r-2$ dimensional subset of $\mathbb{R}^{2r-3}$. As such there are at most $C_1 n^{r-3}$ boundary cubes in the discrete approximation, each of volume $\frac{1}{n^{r-2}}$, where $C_1$ is a constant depending on $\vec{t}$ and $r$. Since the integrand (and summand) is uniformly bounded by 1, the contribution to the left hand side of (4.80) is $\mathcal{O}\left(\frac{1}{n}\right)$ where the constant depends on $\vec{t}$ and $r$. Within each cube of side $\frac{1}{n}$ we have, for all $\vec{s}$ in that cube,

$$\left|e^{\frac{-\kappa_j^2}{2d}\frac{n_j}{n}}-e^{-\frac{\kappa_j^2 s_j}{2d}}\right|\le\frac{\kappa_j^2}{2d}\left|s_j-\frac{n_j}{n}\right|=\mathcal{O}\left(\frac{\kappa_j^2}{n}\right). \tag{4.81}$$

By a telescoping sum representation (as in (4.66)) this gives us that for all $\vec{s}$ in that cube,

$$\prod_{j=1}^{2r-3}e^{\frac{-\kappa_j^2}{2d}\left(\frac{n_j}{n}\right)}-\prod_{j=1}^{2r-3}e^{-\frac{\kappa_j^2 s_j}{2d}}=\mathcal{O}\left(\frac{|\vec{\kappa}|^2}{n}\right). \tag{4.82}$$

Using $\kappa_j^2\le(r-1)\sum_{j=1}^{r-1}\left(k_j I_{l\in E_j}\right)^2$, this verifies (4.79) and hence proves the Theorem. □

# Chapter 5

# Diagrams for the 2-point function

Proposition 3.4.1 was needed to advance the induction argument for the 2-point function in Chapter 3. In this chapter we estimate various diagrams arising from the lace expansion on an interval (star-shaped network of degree 1) and prove Proposition 3.4.1. In Section 5.1 we introduce some definitions and notation that will be used throughout this chapter, and state Propositions 5.1.1, 5.1.7 and 5.1.4, and Lemma 5.1.6. Proposition 3.4.1 follows immediately from Proposition 5.1.1 by summing over $N$. In Section 5.2 we prove Proposition 5.1.1 assuming Propositions 5.1.7 and 5.1.4 and Lemma 5.1.6. Proposition 5.1.7 and Lemma 5.1.6 are proved in Section 5.3 and Proposition 5.1.4 is proved in Section 5.4.

## 5.1 Definitions and Notation

In this section we introduce some notation and results that we need to prove Proposition 3.4.1.

Let $\pi_m(x;\zeta)$ be defined by (3.7), with $U_{st}$ given by (3.3). Then from (2.7) and writing $U_{st} = (-1)(-U_{st})$ we have that for $m \geq 1$,

$$\begin{aligned}\pi_m(x;\zeta) =&\zeta^m \sum_{N=1}^{\infty}(-1)^N \sum_{L\in\mathcal{L}^N([0,m])} \sum_{\substack{\omega\,:\,0\to x\\ |\omega|=m}} W(\omega)\times \\ &\prod_{i=0}^{m} \sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i) \prod_{st\in L}[-U_{st}] \prod_{s't'\in\mathcal{C}(L)}[1+U_{s't'}].\end{aligned} \tag{5.1}$$

The sum over $N$ is actually finite, since a lace on $[0,m]$ can contain at most $m$

bonds. We define

$$\pi_m^N(x;\zeta) = \zeta^m \sum_{L\in\mathcal{L}^N([0,m])} \sum_{\substack{\omega:0\to x\\ |\omega|=m}} W(\omega)\times \prod_{i=0}^{m} \sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i) \prod_{st\in L}[-U_{st}] \prod_{s't'\in\mathcal{C}(L)}[1+U_{s't'}], \tag{5.2}$$

and from (5.1) we have for $m \geq 1$ that $\pi_m(x;\zeta) = \sum_{N=1}^{\infty}(-1)^N \pi_m^N(x;\zeta)$ and hence $|\pi_m(x;\zeta)| \leq \sum_N \pi_m^N(x;\zeta)$. Therefore Proposition 3.4.1 follows immediately (by summing over $N$) from the following Proposition.

**Proposition 5.1.1.** *Suppose the bounds (3.33) hold for some $z^* \in (0,2)$, $K > 1$, $L \geq L_0$ and every $m \leq n$. Then for that $K, L$, and for all $z \in [0,z^*]$, $m \leq n+1$ and $q \in \{0,1,2\}$,*

$$\sum_x |x|^{2q}\pi_m^N(x;\zeta) \leq \frac{\sigma^{2q}\left(C\beta^{2-\frac{6\nu}{d}}\right)^N}{m^{\frac{d-4}{2}-q}}, \tag{5.3}$$

*where $\zeta = \frac{z}{\rho(0)p_c}$, the constant $C = C(K,d)$ does not depend on $L$, $m$, $z$, $N$, and where $\nu > 0$ is the constant appearing in Theorem 1.2.9.*

Throughout this chapter, unless otherwise specified, $C$ denotes a constant that depends on $d$ and $K$ but not on $L$, $m$, $z$, or $N$. It may change from line to line without explicit comment.

Define $h_{m_i}(u) = h_{m_i}(u,\zeta)$ by

$$h_{m_i}(u) = \begin{cases} \zeta^2 p_c^2 (D * t_{m_i-2} * D)(u), & \text{if } m_i \geq 2 \\ \zeta p_c D(u), & \text{if } m_i = 1 \\ I_{\{u=0\}}, & \text{if } m_i = 0 \end{cases} \tag{5.4}$$

where $t_0(u) = \rho(0)I_{\{u=0\}}$.

**Definition 5.1.2.** *For $q_i \in \{0,1\}$, $m_i \in \mathbb{Z}_+$ we define $s_{m_i,q_i}(x) = |x|^{2q_i}h_{m_i}(x)$. For $l \leq 4$ we define $s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}(x)$ to be the $l$-fold spatial convolution of the $s_{m_i,q_i}$.*

**Definition 5.1.3.** *For $r_i \in \{0,1\}$, let $\phi_{r_i}(x) = |x|^{2r_i}\rho(x)$. For $l \in \{1,2,3,4\}$, let $\phi^{(l)}_{\vec{r}^{(l)}}(x)$ denote the $l$-fold spatial convolution of the $\phi_{r_i}$, and define $\phi^{(0)}(x) = I_{\{x=0\}}$.*

**Proposition 5.1.4.** *Let $l \in \{1,2,3,4\}$, and $k \in \{0,1,2,3,4\}$. Let $\vec{m}^{(l)} \in \mathbb{Z}_+^l$ and $m = \sum_{i=1}^l m_i$. If the bounds (3.33) hold for $1 \leq m \leq n$ and $z \in [0,2]$ then for all $m \leq n+1$, and $z \in [0,2]$,*

$$\|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}} * \phi^{(k)}_{\vec{r}^{(k)}}\|_\infty \leq m^{\sum q_i + \sum r_j}\sigma^{2(\sum q_i + \sum r_j)}\frac{c\beta^{2-\frac{2k\nu}{d}}}{m^{\frac{d-2k}{2}}}, \tag{5.5}$$

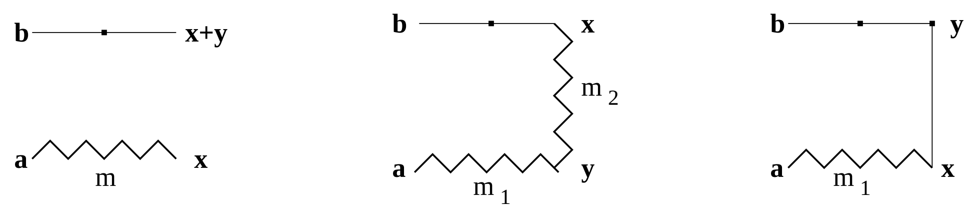


Figure 5.1: Feynman diagrams for $M_m^{(1)}(a,b,x,y)$, $A_{m_1,m_2}(a,b,x,y)$ and $A_{m_1,0}(a,b,x,y)$. A jagged line between two vertices $u$ and $v$ represents a quantity $h_{m_i}(v-u)$. A straight line between two vertices $u$ and $v$ represents the quantity $\rho(v-u)$.

*and*

$$\|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}\|_1 \le C m^{\sum q_i}\sigma^{2\sum q_i}. \tag{5.6}$$

**Definition 5.1.5.** *Let*

$$M_m^{(1)}(a,b,x,y) \equiv h_m(x-a)\rho^{(2)}(x+y-b), \tag{5.7}$$

*and*

$$A_{m_1,m_2}(a,b,x,y) \equiv \begin{cases} h_{m_1}(y-a)h_{m_2}(x-y)\rho^{(2)}(b-x), & m_2 \neq 0, \\ h_{m_1}(x-a)\rho(y-x)\rho^{(2)}(b-y), & m_2 = 0. \end{cases} \tag{5.8}$$

*We recursively define*

$$M_{\vec{m}}^{(N)}(a,b,x,y) \equiv \sum_{u,v} A_{m_1,m_2}(a,b,u,v) M^{(N-1)}_{(m_3,\ldots,m_{2N-1})}(u,v,x,y). \tag{5.9}$$

The diagrammatic representation of these quantities appears in Figures 5.1 and 5.2.

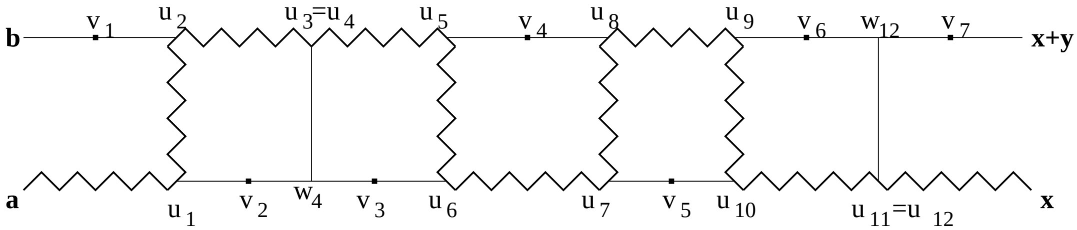


Figure 5.2: An example of an "opened" Feynman diagram, $M^{(7)}_{\vec{m}}(a,b,x,y)$ arising from the lace expansion. A jagged line from $u_{i-1}$ to $u_i$ represents the quantity $h_{m_i}(u_i - u_{i-1})$ (derived from the backbone from $a$ to $x$). A straight line between two vertices $u$ and $v$ represents the quantity $\rho(v-u)$ (derived from intersections of branches emanating from the backbone).

**Lemma 5.1.6.** *Setting $u_0 = a$ and $u_{2N-1} = x$, for every $N \geq @$,*

$$\begin{aligned} M^{(N)}_{\vec{m}}(a,b,x,y) &= \sum_{u_1} \cdots \sum_{u_{2N-2}} \left[ \prod_{i=1}^{2N-1} h_{m_i}(u_i - u_{i-1}) \right] \times \\ &\sum_{v_1,\dots,v_N} \rho(v_1 - b)\rho(v_N - (x+y)) \times \\ &\left[ \prod_{l \geq 2 : m_l = 0} \sum_{w_l} \rho(w_l - u_{l-1})\rho(v_{\frac{l+2}{2}} - w_l)\rho(v_{\frac{l}{2}} - w_l) \right] \times \\ &\prod_{\substack{1 \leq l \leq 2N-2 : \\ m_l, m_{l+1} \neq 0}} \left( \rho(v_{\frac{l}{2}} - u_l) I_{\{l \ even\}} + \rho(v_{\frac{l+3}{2}} - u_l) I_{\{l \ odd\}} \right) \\ &= \sum_{u,v} M^{(N-1)}_{(m_1,\dots,m_{2N-3})}(a,b,u,v) A_{m_{2N-1},m_{2N-2}}(x,y,u,v). \end{aligned} \tag{5.10}$$

We also make use of the following notation. Let

$$\mathcal{H}_{m,N} = \left\{ \vec{m} \in \mathbb{Z}_+^{2N-1} : \sum_{i=1}^{2N-1} m_i = m, \ m_{2j} \geq 0, m_{2j-1} > 0 \right\}. \tag{5.11}$$

For general $N \geq 2$ we let

$$\begin{aligned} E^N_m &= \left\{ \vec{m} \in \mathcal{H}_{m,N} : m_2 + m_1 \leq \frac{2m}{3} \right\} \\ F^N_m &= \left\{ \vec{m} \in \mathcal{H}_{m,N} : m_{2N-2} + m_{2N-1} \leq \frac{2m}{3} \right\}, \end{aligned} \tag{5.12}$$

and for $N = 2$ we also define

$$G_m^2 = \{\vec{m} \in \mathcal{H}_{m,2} : (m_1 \vee m_3) \le m_2\} . \tag{5.13}$$

Note that for $N \ge 3$, $E_m^N \cup F_m^N = \mathcal{H}_{m,N}$ since $m_1 + m_2 + m_{2N-2} + m_{2N-1} \le m$. Similarly for $N = 2$, $E_m^2 \cup F_m^2 \cup G_m^2 = \mathcal{H}_{m,2}$.

**Proposition 5.1.7.** *For $q \in \{0, 1, 2\}$ and $N \ge 1$,*

$$\sum_x |x|^{2q} \pi_m^N(x;\zeta) \le \sum_{\vec{m} \in \mathcal{H}_{m,N}} \sum_x |x|^{2q} M_{\vec{m}}^{(N)}(0,0,x,0). \tag{5.14}$$

Observe that there are two disjoint paths in the diagram $M_{\vec{m}}^{(N)}(a,a,x,0)$ from $a$ to $x$, corresponding to taking the uppermost path and the lowest path, each have displacement $x - a$. In the opened diagram $M_{\vec{m}}^{(N)}(a,b,x,y)$, the corresponding uppermost path may be from $b$ to $x$ or from $b$ to $x + y$ depending on $\vec{m}$. Similarly the right endpoint of the lowest path depends on $\vec{m}$. We define $\overline{z} = \overline{z}(\vec{m}, x, b, y)$ and $\underline{z} = \underline{z}(\vec{m}, x, a, y)$ by

$$\begin{aligned} \overline{z} &= \begin{cases} x - b & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is odd} \\ x + y - b & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is even} \end{cases} \\ \underline{z} &= \begin{cases} x + y - a & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is odd} \\ x - a & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is even .} \end{cases} \end{aligned} \tag{5.15}$$

## 5.2 Proof of Proposition 5.1.1

In this section we prove Proposition 5.1.1, assuming Propositions 5.1.4 and 5.1.7. We prove the three cases $q = 0, 1, 2$ separately.

**Case 1:** $q = 0$. Our induction hypothesis is that

$$\sum_{\vec{m} \in \mathcal{H}_{m,N}} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(N)}(a,b,x,y) \le \frac{(C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-4}{2}}}. \tag{5.16}$$

In view of Proposition 5.1.7 with $q = 0$, this clearly implies Proposition 5.1.1 with $q = 0$.

For $N = 1$ note that

$$\begin{aligned} \sup_{a,b,y} \sum_x M_m^{(1)}(a,b,x,y) &= \sup_{a,b,y} \sum_x h_m(x-a)\rho^{(2)}(x+y-b) \\ &= \sup_{a,b,y} \sum_x h_m(x)\rho^{(2)}(x+y-b+a) \\ &= \sup_z \sum_x h_m(x)\rho^{(2)}(x+z). \end{aligned} \tag{5.17}$$

Applying (5.5) with $l = 1$, $k = 2$ and all $q_i = r_j = 0$, this is bounded by $\frac{C\beta^{2-\frac{4\nu}{d}}}{m^{\frac{d-4}{2}}}$ as required.

We consider separately the contributions to (5.16) from $E_m^N$, $F_m^N$ and in the case $N = 2$ also the contribution from $G_m^2$.

Now by (5.9) we have

$$\begin{aligned}
&\sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(N)}(a,b,x,y) \\
&= \sum_{m_1\le\frac{2m}{3}} \sum_{m_2\le\frac{2m}{3}-m_1} \sum_{\vec{m}'\in\mathcal{H}_{m-(m_1+m_2),N-1}} \sup_{a,b} \sum_{u,v} A_{m_1,m_2}(a,b,u,v) \\
&\quad \times \sup_y \sum_x M_{\vec{m}'}^{(N-1)}(u,v,x,y) \\
&= \sum_{m_1\le\frac{2m}{3}} \sum_{m_2\le\frac{2m}{3}-m_1} \sup_{a,b} \sum_{u,v} A_{m_1,m_2}(a,b,u,v) \\
&\quad \times \sum_{\vec{m}'\in\mathcal{H}_{m-(m_1+m_2),N-1}} \sup_{u',v',y} \sum_x M_{\vec{m}'}^{(N-1)}(u',v',x,y) \\
&\le \sum_{m_1\le\frac{2m}{3}} \sum_{m_2\le\frac{2m}{3}-m_1} \sup_{a,b} \sum_{u,v} A_{m_1,m_2}(a,b,u,v) \frac{(C\beta^{2-\frac{6\nu}{d}})^{N-1}}{(m-(m_1+m_2))^{\frac{d-4}{2}}},
\end{aligned} \tag{5.18}$$

where we have applied the induction hypothesis in the last step. Since $m_1+m_2 \le \frac{2m}{3}$ in the range we are summing over, the last line of (5.18) is bounded by

$$C' \frac{(C\beta^{2-\frac{6\nu}{d}})^{N-1}}{m^{\frac{d-4}{2}}} \sum_{m_1\le\frac{2m}{3}} \sum_{m_2\le\frac{2m}{3}-m_1} \sup_{a,b} \sum_{u,v} A_{m_1,m_2}(a,b,u,v), \tag{5.19}$$

where the constant $C' = 3^{\frac{d-4}{2}}$ is independent of $N$. Finally we split the sum over $m_2$ into the two cases $m_2 = 0$, $m_2 > 0$ to get

$$\begin{aligned}
&\sum_{m_1\le\frac{2m}{3}} \sum_{m_2\le\frac{2m}{3}-m_1} \sup_{a,b} \sum_{u,v} A_{m_1,m_2}(a,b,u,v) \\
&= \sum_{m_1\le\frac{2m}{3}} \sup_{a,b} \sum_{u,v} h_{m_1}(u-a)\rho(v-u)\rho^{(2)}(b-v) \\
&\quad + \sum_{m_1\le\frac{2m}{3}} \sum_{0<m_2\le\frac{2m}{3}-m_1} \sup_{a,b} \sum_{u,v} h_{m_1}(v-a)h_{m_2}(u-v)\rho^{(2)}(b-u) \\
&\le \sum_{m_1\le\frac{2m}{3}} \frac{C\beta^{2-\frac{6\nu}{d}}}{m_1^{\frac{d-6}{2}}} + \sum_{m_1\le\frac{2m}{3}} \sum_{0<m_2\le\frac{2m}{3}-m_1} \frac{C\beta^{2-\frac{4\nu}{d}}}{[m_1+m_2]^{\frac{d-4}{2}}} \le C\beta^{2-\frac{6\nu}{d}},
\end{aligned} \tag{5.20}$$

where we have applied (5.5) with all $q_i = r_j = 0$ in the penultimate step and the fact that $d > 8$ in the last step . Combining (5.18)–(5.20), we get that

$$\sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(N)}(a,b,x,y) \leq \frac{(C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-4}{2}}}, \tag{5.21}$$

as required.

Similarly using the symmetry of $M_{\vec{m}}^{(N)}$ (in the form of the second equality of (5.10)) and writing $n_1$ for $m_{2N-1}$ and $n_2$ for $m_{2N-2}$ we get

$$\begin{aligned} &\sum_{\vec{m}\in F_m^N} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(N)}(a,b,x,y) \\ &= \sum_{n_1\leq\frac{2m}{3}} \sum_{n_2\leq\frac{2m}{3}-n_1} \sum_{\vec{m}'\in\mathcal{H}_{m-(n_1+n_2),N-1}} \sup_{u',y} \sum_{x,v} A_{n_1,n_2}(x,y,u',v) \\ &\times \sup_{a,b,v'} \sum_u M_{\vec{m}'}^{(N-1)}(a,b,u,v'). \end{aligned} \tag{5.22}$$

Using translation invariance of $A_{n_1,n_2}(x,y,u',v)$ we proceed as in (5.18)–(5.20) to get

$$\sum_{\vec{m}\in F_m^N} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(N)}(a,b,x,y) \leq \frac{(C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-4}{2}}}, \tag{5.23}$$

as required.

It remains to prove the bound (5.16) for the sum over $\vec{m} \in G_m^2$. Note that in this case $m_2 \neq 0$ and so $M_{\vec{m}}^{(2)}(a,b,x,y)$ is equal to

$$\sum_{u,v} \rho^{(2)}(b-v)h_{m_1}(u-a)h_{m-(m_1+m_3)}(v-u)h_{m_3}(x-v)\rho^{(2)}(x+y-u). \tag{5.24}$$

We break the sum over $\vec{m} \in G_m^2$ according to which of $m_1$ and $m_3$ is larger and note that $m_2 = m-(m_1+m_3)$. By symmetry of $M_{\vec{m}}^{(2)}(a,b,x,y)$ and translation invariance we have

$$\begin{aligned}
&\sum_{\vec{m}\in G_m^2} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(2)}(a,b,x,y) \\
\le& 2 \sum_{m_1<\frac{m}{2}} \sum_{\substack{m_3\le m_1:\\ m_2\ge m_1}} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(2)}(a,b,x,y) \\
=& 2 \sum_{m_1<\frac{m}{2}} \sum_{\substack{m_3\le m_1:\\ m_2\ge m_1}} \sup_{a,b,y} \sum_x \sum_{u,v} \rho^{(2)}(b-v) h_{m_1}(u-a) \\
&\times h_{m-(m_1+m_3)}(v-u) h_{m_3}(x-v)\rho^{(2)}(x+y-u) \\
=& 2 \sum_{m_1<\frac{m}{2}} \sum_{\substack{m_3\le m_1:\\ m_2\ge m_1}} \sup_{b,y} \sum_x \sum_{u,v} \rho^{(2)}(b-v) h_{m_1}(u) \\
&\times h_{m-(m_1+m_3)}(v-u) h_{m_3}(x-v)\rho^{(2)}(x+y-u),
\end{aligned} \tag{5.25}$$

where in the last step we have subtracted $a$ from each vertex and correspondingly changed variables (i.e. we have used tranlations invariance). This is bounded by

$$\begin{aligned}
&2 \sum_{m_1<\frac{m}{2}} \sum_{\substack{m_3\le m_1:\\ m_2\ge m_1}} \left(\sup_{b,u'} \sum_v \rho^{(2)}(b-v) h_{m-(m_1+m_3)}(v-u')\right)\left(\sup_{v'} \sum_x h_{m_3}(x-v')\right) \\
&\times \left(\sup_z \sum_u h_{m_1}(u)\rho^{(2)}(z-u)\right).
\end{aligned} \tag{5.26}$$

Applying (5.5) with all $q_i, r_j = 0$ for the term inside the first and third braces and (5.6) with $l=1$ and $q_i=0$ for all $i$ for the term inside the second braces, (5.26) is bounded by

$$\begin{aligned}
&2 \sum_{m_1<\frac{m}{2}} \sum_{\substack{m_3\le m_1:\\ m_2\ge m_1}} \frac{C\beta^{2-\frac{4\nu}{d}}}{(m-(m_1+m_3))^{\frac{d-4}{2}}} K \frac{C\beta^{2-\frac{4\nu}{d}}}{m_1^{\frac{d-4}{2}}} \\
\le& C' \frac{(C\beta^{2-\frac{4\nu}{d}})^2}{m^{\frac{d-4}{2}}} \sum_{m_1<\frac{m}{2}} \frac{1}{m_1^{\frac{d-4}{2}}} \sum_{\substack{m_3\le m_1:\\ m-(m_1+m_3)\ge m_1}} 1 \\
\le& C' \frac{(C\beta^{2-\frac{4\nu}{d}})^2}{m^{\frac{d-4}{2}}} \sum_{m_1<\frac{m}{2}} \frac{1}{m_1^{\frac{d-6}{2}}},
\end{aligned} \tag{5.27}$$

and we have the desired bound since $d > 8$. This completes the proof of Proposition 5.1.1 for $q = 0$. $\square$

**Case 2:** $q = 1$. Our induction hypotheses are that

$$\begin{aligned} \sum_{\vec{m}\in\mathcal{H}_{m,N}} \sup_{a,b,y} \sum_x |\overline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y) &\leq \frac{\sigma^2 (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-6}{2}}}, \quad \text{and} \\ \sum_{\vec{m}\in\mathcal{H}_{m,N}} \sup_{a,b,y} \sum_x |\underline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y) &\leq \frac{\sigma^2 (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-6}{2}}}. \end{aligned} \tag{5.28}$$

In view of Proposition 5.1.7 with $q = 1$, these clearly imply Proposition 5.1.1 with $q = 1$.

For $N = 1$, the first statement of (5.28) is

$$\sup_{a,b,y} \sum_x |x+y-b|^2 h_m(x-a)\rho^{(2)}(x+y-b) \leq \frac{\sigma^2 (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-6}{2}}}. \tag{5.29}$$

Writing $\rho^{(2)}(x+y-b) = \sum_u \rho(u-b)\rho(x+y-u)$ and using $|x+y-b|^2 \leq 2(|u-b|^2 + |x+y-u|^2)$, (5.29) is bounded by

$$2\sup_{a,b,y}\sum_x \phi_1(u-b)\rho(x+y-u)h_m(x-a) + 2\sup_{a,b,y}\sum_x \rho(u-b)\phi_1(x+y-u)h_m(x-a). \tag{5.30}$$

Applying (5.5) to each term with $l = 1$, $k = 2$, $q_1 = 0$ and exactly one $r_j = 1$, (5.30) is bounded by $\frac{\sigma^2(C\beta^{2-\frac{4\nu}{d}})}{m^{\frac{d-6}{2}}}$ as required. The second statement for $N = 1$ is

$$\sup_{a,b,y} \sum_x |x-a|^2 h_m(x-a)\rho^{(2)}(x+y-b) \leq \frac{\sigma^2 (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-6}{2}}}, \tag{5.31}$$

which follows immediately by applying (5.5) with $l = 1$, $k = 2$, $q_1 = 1$ and all $r_j = 0$.

For the inductive step, for each statement of (5.28) we break up the sum over $\vec{m} \in \mathcal{H}_{m,N}$ into sums over $\vec{m} \in E_m^N$, $\vec{m} \in F_m^N$, and when $N = 2$, also $\vec{m} \in G_m^2$. For the contribution from $\vec{m} \in E_m^N$ we write $|\overline{z}|^2 \leq 2(|\overline{z}_A|^2 + |z_M|^2)$ where

$$(\overline{z}_A, z_M) = \begin{cases} (x-u, u-b) & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is odd and } m_2 > 0 \\ (x+y-u, u-b) & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is even and } m_2 > 0 \\ (x-v, v-b) & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is odd and } m_2 = 0 \\ (x+y-v, v-b) & , \text{ if } \#\{m_{2j} : m_{2j} \neq 0\} \text{ is even and } m_2 = 0. \end{cases} \tag{5.32}$$

Thus

$$\begin{aligned}
&\sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_x |\overline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y) \\
\le& 2 \sum_{\vec{m}\in E_{m,N}} \sup_{a,b,y} \sum_{x,u,v} |\overline{z}_A|^2 A_{m_1,m_2}(a,b,u,v) M_{\vec{m}'}^{(N-1)}(u,v,x,y) \\
&+ 2 \sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_{x,u,v} A_{m_1,m_2}(a,b,u,v)|z_M|^2 M_{\vec{m}'}^{(N-1)}(u,v,x,y).
\end{aligned} \tag{5.33}$$

As in (5.18) the first term on the right of (5.33) is equal to

$$\begin{aligned}
&2 \sum_{m_1\le \frac{2m}{3}} \sum_{m_2\le \frac{2m}{3}-m_1} \sup_{a,b} \sum_{u,v} |\overline{z}_A|^2 A_{m_1,m_2}(a,b,u,v) \frac{(C\beta^{2-\frac{6\nu}{d}})^{N-1}}{(m-(m_1+m_2))^{\frac{d-4}{2}}} \\
\le& 2\frac{(C\beta^{2-\frac{6\nu}{d}})^{N-1}}{m^{\frac{d-4}{2}}} \sum_{m_1\le \frac{2m}{3}} \sum_{m_2\le \frac{2m}{3}-m_1} \sup_{a,b} \sum_{u,v} |\overline{z}_A|^2 A_{m_1,m_2}(a,b,u,v).
\end{aligned} \tag{5.34}$$

We now proceed exactly as in (5.19)–(5.21) except that we use (5.5) with exactly one $r_j = 1$ (instead of all $r_j = 0$ as we did in (5.20). This yields an upper bound on (5.34) of

$$\frac{\sigma^2 m (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-4}{2}}}. \tag{5.35}$$

For the second term on the right of (5.33) note that by definition, $z_M$ is either $\overline{z}'$ or $\underline{z}'$, the displacement of the upper or lower path of $M^{(N-1)}(u,v,x,y)$. We proceed exactly as in (5.18)–(5.20) except that the induction hypotheses give a bound

$$\begin{aligned}
\sum_{\vec{m}'\in \mathcal{H}_{m-(m_1+m_2),N-1}} \sup_{u',v',y} \sum_x |z_M|^2 M_{\vec{m}'}^{(N-1)}(u',v',x,y) &\le \sigma^2 \frac{(C\beta^{2-\frac{6\nu}{d}})^{N-1}}{(m-(m_1+m_2))^{\frac{d-6}{2}}} \\
&\le \sigma^2 m \frac{(C\beta^{2-\frac{6\nu}{d}})^{N-1}}{(m-(m_1+m_2))^{\frac{d-4}{2}}},
\end{aligned} \tag{5.36}$$

which contains an extra factor of $\sigma^2 m$ compared to that appearing in (5.18). We now proceed exactly as in (5.19)–(5.21) to get a bound on the first term of (5.33) of

$$\sigma^2 m \frac{(C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-4}{2}}}. \tag{5.37}$$

This proves that

$$\sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_x |\overline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y) \le \frac{\sigma^2 (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-6}{2}}}. \tag{5.38}$$

As in the $q = 0$ case of Proposition 5.1.1, the bound

$$\sum_{\vec{m}\in F_m^N} \sup_{a,b,y} \sum_x |\overline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y) \le \frac{\sigma^2 (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-6}{2}}}, \tag{5.39}$$

follows by symmetry.

When $N = 2$, the contribution to (5.28) from $\vec{m} \in G_m^2$ is easily bounded as in (5.25) by applying (5.5) and (5.6) with exactly one of these having one $q_i$ or $r_j = 0$. This gives the desired bound of $\frac{\sigma^2(C\beta^{2-\frac{4\nu}{d}})^2}{m^{\frac{d-6}{2}}}$ as required. This completes the proof of Proposition 5.1.1 for $q = 1$ □

**Case 3:** $q = 2$. Our induction hypothesis is that

$$\sum_{\vec{m}\in \mathcal{H}_{m,N}} \sup_{a,b,y} \sum_x |\overline{z}|^2 |\underline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y) \le \frac{\sigma^4 (C\beta^{2-\frac{6\nu}{d}})^N}{m^{\frac{d-8}{2}}}. \tag{5.40}$$

the induction hypothesis In view of Proposition 5.1.7 with $q = 2$, this clearly implies Proposition 5.1.1 with $q = 2$. The proof of (5.40) is very similar to the proof of (5.28) so we just present the main ideas.

The $N = 1$ case follows from (5.5) with $l = 1$, $k = 2$, $q_1 = 1$ and exactly one $r_i = 1$. To bound

$$\sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_x |\overline{z}|^2 |\underline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y), \tag{5.41}$$

we use the expansions $|\overline{z}|^2 \le 2(|\bullet|^2 + |\bullet|^2)$ and $|\underline{z}|^2 \le 2(|\bullet|^2 + |\bullet|^2)$ yielding 4 terms instead of the two in (5.33). One such term is

$$4 \sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_{x,u,v} |\overline{z}_A|^2 |\underline{z}_A|^2 A_{m_1,m_2}(a,b,u,v) M_{\vec{m}'}^{(N-1)}(u,v,x,y), \tag{5.42}$$

on which we use the $q = 0$ case of Proposition 5.1.1, and (5.5) with $q_1 = 1$ and exactly one of the $r_j = 1$. For two of the remaining three terms arising from (5.41) we use the $q = 1$ case of Proposition 5.1.1 and (5.5) with exactly one of $q_1 = 1$ or some $r_j = 1$. The remaining term arising from (5.41) is

$$4 \sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_{x,u,v} A_{m_1,m_2}(a,b,u,v) |\overline{z}'|^2 |\underline{z}'|^2 M_{\vec{m}'}^{(N-1)}(u,v,x,y), \tag{5.43}$$

which we bound using the induction hypothesis and (5.5) with all $q_i, r_j = 0$ and . Collecting the 4 terms we obtain the bound

$$\sum_{\vec{m}\in E_m^N} \sup_{a,b,y} \sum_x |\overline{z}|^2 |\underline{z}|^2 M_{\vec{m}}^{(N)}(a,b,x,y) \le \frac{\sigma^4 (C\beta^{2-\frac{4\nu}{d}})^2}{m^{\frac{d-8}{2}}}. \tag{5.44}$$

The contribution from $\vec{m} \in F_m^N$ also obeys the bound (5.44) by symmetry, while the contribution from $\vec{m} \in G_m^2$ when $N = 2$ is handled as for the $q = 1$ case of Proposition 5.1.1 except that we have exactly two of the $q_i, r_j$ equal to 1 when we apply (5.5) and (5.6).

This completes the proof of Proposition 5.1.1 for $q = 2$, and hence completes the proof of Proposition 5.1.1. □

**Remark 5.2.1.** *Observe that apart from the recursive representations of the diagrams $M^{(N)}$ in (5.9) and (5.10), the only information we used to bound the diagrams was Proposition 5.1.4. This will become important when we estimate more complicated diagrams in Chapter 6.*

### 5.2.1 Diagrams with an extra vertex.

We say that a diagram *has an extra vertex on some* $\rho$ if it is the same as a diagram corresponding to some $M_{\vec{m}}^N$ except one $\rho(z)$ in that diagram is replaced with $\rho^{(2)}(z)$. We say that a diagram *has an extra vertex on some* $h_m$ if it is the same as a diagram corresponding to some $M_{\vec{m}}^{(N)}$ except one $h_{m_j}(z)$ in that diagram is replaced with $h_{m'} * h_{m_j - m'}(z)$. When we consider the diagrams arising from the lace expansion on a star-shape of degree 3 we will encounter diagrams with an extra vertex on some $\rho$ or $h_m$. We bound the contribution from all such diagrams by repeating the inductive analysis used in the proof of Proposition 5.1.1. We do not show all the details but the main ideas are as follows.

We let $n$ denote the location along the branch point where the extra vertex is located. If $n = \sum_{i=1}^{j} m_i$ for some $1 \leq j \leq 2N - 2$ then the vertex is on the $\rho$ emanating from the backbone at $n$, or a $\rho$ incident to that $\rho$ (of which there are at most two). If $n = 0$ (resp. $n = m$) then the vertex is on the first $\rho$ (resp. last $\rho$) in the diagram, or the $\rho$ incident to it. Otherwise the vertex is at position $n$ on the backbone (i.e. on some $h_{m_i}$). Let $M_{\vec{m}}^{(N),n}(a, b, x, y)$ denote the corresponding diagram with an extra vertex at $n$.

We prove by induction on $N$ that

$$\sum_{\vec{m} \in \mathcal{H}_{m,N}} \sum_{n \leq m} \sup_{a,b,y} \sum_{x} M_{\vec{m}}^{(N),n}(a, b, x, y) \leq \frac{(C\beta^{2-\frac{8\nu}{d}})^N}{m^{\frac{d-6}{2}}}. \tag{5.45}$$

For $N = 1$ the left hand side of (5.45) is

$$\sum_{0<n<m} \sup_{a,b,y} \sum_{x} (h_n * h_{m-n})(x - a)\rho^{(2)}(x + y - b) + 2 \sup_{a,b,y} \sum_{x} h_m(x - a)\rho^{(3)}(x + y - b). \tag{5.46}$$

Using (5.5) with $l = 2$, $k = 2$ and all $q_i, r_j = 0$, the first term in (5.46) is bounded by

$$\sum_{0<n<m} \frac{C\beta^{2-\frac{4\nu}{d}}}{m^{\frac{d-4}{2}}} \le \frac{C\beta^{2-\frac{4\nu}{d}}}{m^{\frac{d-6}{2}}}. \tag{5.47}$$

Similarly using (5.5) with $l = 2$, $k = 3$ and all $q_i, r_j = 0$, the second term is bounded by $\frac{C\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}}$. Adding these together we get a bound of $\frac{C\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}}$ which satisfies the induction hypothesis with $N = 1$.

For general $N \ge 2$ we bound

$$\sum_{\vec{m}\in E_m^N} \sum_{n\le m} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(N),n}(a, b, x, y), \tag{5.48}$$

by using (5.9), and splitting the sum over $n \le m$ into sums over $n \le m_1 + m_2 : n \ne m_1$, and $n > m_1 + m_2$, and the final case $n = m_1$. In each case the extra vertex is either on $A_{m_1,m_2}$ or $M_{\vec{m}'}^{(N-1)}$. In the former case we use the $q = 0$ result in the proof of 5.1.1 on the $M_{\vec{m}'}^{(N-1)}$ part and (5.5) (increasing $k$ or $l$ by one due to the extra vertex) on the $A_{m_1,m_2}^n$ part. In the latter case we use the induction hypothesis on the $M_{\vec{m}'}^{(N-1),n}$ part and (5.5) on the $A_{m_1,m_2}^n$ part. The contributions from $\vec{m} \in F_m^N$ and $\vec{m} \in G_m^2$ are dealt with as usual.

Similarly we prove

$$\sum_{\vec{m}\in\mathcal{H}_{m,N}} \sum_{n\le m} \sup_{a,b,y} \sum_x |x-a|^2 M_{\vec{m}}^{(N),n}(a, b, x, y) \le \frac{\sigma^2(C\beta^{2-\frac{8\nu}{d}})^N}{m^{\frac{d-8}{2}}}. \tag{5.49}$$

Note the factor $|x-a|^2$ in (5.49) rather than $|\overline{z}|$ or $|\underline{z}|$. This is to avoid the situation that could arise of having a convolution of four $\rho$'s with one of them having an extra factor $|u|^2$ on the same diagram piece. This would violate the condition $k + \sum_{i=1}^k r_i \le 4$ in Proposition 5.1.4. Using $|x - a|^2$ instead, we will use path along the backbone from $a$ to $x$ rather than the top path or bottom path, and the induction argument goes through as before.

## 5.3 General Diagrams

In this section we prove Proposition 5.1.7 and Lemma 5.1.6. We begin with the proof of Lemma 5.1.6.

**Lemma (5.1.6).** *Setting $u_0 = a$ and $u_{2N-1} = x$, for every $N \geq 2$,*

$$\begin{aligned} M_{\vec{m}}^{(N)}(a,b,x,y) = &\sum_{u_1} \cdots \sum_{u_{2N-2}} \left[ \prod_{i=1}^{2N-1} h_{m_i}(u_i - u_{i-1}) \right] \times \\ &\sum_{v_1,\ldots,v_N} \rho(v_1 - b)\rho(v_N - (x+y)) \times \\ &\left[ \prod_{l \geq 2 : m_l = 0} \sum_{w_l} \rho(w_l - u_{l-1})\rho(v_{\frac{l+2}{2}} - w_l)\rho(v_{\frac{l}{2}} - w_l) \right] \times \\ &\prod_{\substack{1 \leq l \leq 2N-2 : \\ m_l, m_{l+1} \neq 0}} \Big( \rho(v_{\frac{l}{2}} - u_l) I_{\{l \text{ even}\}} + \rho(v_{\frac{l+3}{2}} - u_l) I_{\{l \text{ odd}\}} \Big) \\ = &\sum_{u,v} M_{(m_1,\ldots,m_{2N-3})}^{(N-1)}(a,b,u,v) A_{m_{2N-1},m_{2N-2}}(x,y,u,v). \end{aligned} \tag{5.50}$$

*Proof.* For the first equality of (5.50), we prove the result by induction on $N$ and leave the reader to verify the easiest case, $N = 2$ (consider the two cases $m_2 > 0$, $m_2 = 0$).

For $N \geq 3$, if $m_2 > 0$ then by separating the terms $l = 1, 2$ from the initial and final products in the right side of (5.9) we have that $M_{\vec{m}}^{(N)}(a,b,x,y)$ is equal to

$$\begin{aligned} &\sum_{u_1,u_2} \Bigg( h_{m_1}(u_1 - a) h_{m_2}(u_2 - u_1) \sum_{v_1} \rho(v_1 - b)\rho(v_{\frac{2}{2}} - u_2) \Bigg) \\ &\times \Bigg( \sum_{u_3} \cdots \sum_{u_{2N-2}} \left[ \prod_{i=3}^{2N-1} h_{m_i}(u_i - u_{i-1}) \right] \times \\ &\sum_{v_2,\ldots,v_N} \rho(v_{\frac{1+3}{2}} - u_1)\rho(v_N - (x+y)) \left[ \prod_{l \geq 4 : m_l = 0} \sum_{w_l} \rho(w_l - u_{l-1})\rho(v_{\frac{l+2}{2}} - w_l)\rho(v_{\frac{l}{2}} - w_l) \right] \\ &\times \prod_{\substack{3 \leq l \leq 2N-2 : \\ m_l, m_{l+1} \neq 0}} \Big( \rho(v_{\frac{l}{2}} - u_l) I_{\{l \text{ even}\}} + \rho(v_{\frac{l+3}{2}} - u_l) I_{\{l \text{ odd}\}} \Big) \Bigg) \\ &= \sum_{u_1,u_2} A_{m_1,m_2}(a,b,u_1,u_2) M_{(m_3,\ldots,m_{2N-1})}^{(N-1)}(u_1,u_2,x,y), \end{aligned} \tag{5.51}$$

by definition of $A_{m_1,m_2}$ and the induction hypothesis. This proves the result when $m_2 \neq 0$

If $m_2=0$ then by separating the $l=1,2$ terms from the first product and $l=2$ term from the second product in (5.9) we have that $M^{(N)}_{\vec{m}}(a,b,x,y)$ is equal to

$$\begin{aligned}
&\sum_{u_1,w_2}\sum_{u_2}\Bigg(h_{m_1}(u_1-a)h_0(u_2-u_1)\sum_{v_1}\rho(v_1-b)\rho(w_2-u_1)\rho(v_{\frac{2}{2}}-w_2)\Bigg)\\
&\times\Bigg(\sum_{u_3}\cdots\sum_{u_{2N-2}}\left[\prod_{i=3}^{2N-1}h_{m_i}(u_i-u_{i-1})\right]\times\\
&\sum_{v_2,\dots,v_N}\rho(v_{\frac{2+2}{2}}-w_2)\rho(v_N-(x+y))\left[\prod_{l\geq 4:m_l=0}\sum_{w_l}\rho(w_l-u_{l-1})\rho(v_{\frac{l+2}{2}}-w_l)\rho(v_{\frac{l}{2}}-w_l)\right]\\
&\times\prod_{\substack{3\leq l\leq 2N-2:\\ m_l,m_{l+1}\neq 0}}\Big(\rho(v_{\frac{l}{2}}-u_l)I_{\{l\text{ even}\}}+\rho(v_{\frac{l+3}{2}}-u_l)I_{\{l\text{ odd}\}}\Big)\Bigg)\\
&=\sum_{u_1,w_2}A_{m_1,m_2}(a,b,u_1,w_2)M^{(N-1)}_{(m_3,\dots,m_{2N-1})}(u_1,w_2,x,y),
\end{aligned}\tag{5.52}$$

by definition of $A_{m_1,m_2}$ and the induction hypothesis. This proves the result when $m_2=0$, and thus completes the proof of the first equality of (5.50).

The proof of the second equality is the same by symmetry of the expression for $M^{(N)}_{\vec{m}}$ in the first equality, by considering the cases $m_{2N-2}>0$ and $m_{2N-2}=0$ and separating the terms $l=2N-1,2N-2$. □

We now prove Proposition 5.1.7.

**Proposition (5.1.7).** *For $q\in\{0,1,2\}$ and $N\geq 1$*

$$\sum_x |x|^{2q}\pi^N_m(x;\zeta)\leq\sum_{\vec{m}\in\mathcal{H}_{m,N}}\sum_x|x|^{2q}M^{(N)}_{\vec{m}}(0,0,x,0).\tag{5.53}$$

*Proof.* We prove the stronger result that

$$\pi^N_m(x;\zeta)\leq\sum_{\vec{m}\in\mathcal{H}_{m,N}}M^{(N)}_{\vec{m}}(0,0,x,0).\tag{5.54}$$

Recall the definition of $\pi^N_m(x;\zeta)$ from (5.2).

For $N=1$ there is only one lace $L=\{0m\}$ on $[0,m]$ and every other bond

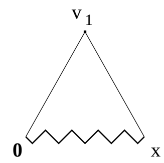


Figure 5.3: The Feynman diagram corresponding to the lace containing one bond. The jagged line represents the quantity $h_m(x)$, while straight line between 0 (resp. $x$) and $v_1$ represents the quantity $\rho(v_1)$ (resp. $\rho(x-v_1)$).

is compatible with $\{0m\}$, so by (5.2)

$$\begin{aligned}\pi^1_m(x;\zeta) =& \zeta^m \sum_{\substack{\omega:0\to x\\ |\omega|=m}} W(\omega)\prod_{i=0}^{m}\sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i)[-U_{0m}]\prod_{\flat\neq 0m}[1+U_\flat]\\ =& \sum_{R_0\in\mathcal{T}_0} W(R_0)\sum_{R_m\in\mathcal{T}_x} W(R_m)[-U_{0m}]\times \\ & \zeta^m \sum_{\substack{\omega:0\to x\\ |\omega|=m}} W(\omega)\prod_{i=1}^{m-1}\sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i)\prod_{\flat\neq 0m}[1+U_\flat].\end{aligned} \tag{5.55}$$

Note that everything in this expression is non-negative. Now $-U_{0m} = I_{\{R_0\cap R_m\neq\emptyset\}}$ so $\pi^1_m(x;\zeta)$ is nonzero if and only if there exists $v\in\mathbb{Z}^d$ such that $v\in R_0\cap R_m$ and therefore

$$\begin{aligned}\sum_{R_0\in\mathcal{T}_0} W(R_0)\sum_{R_m\in\mathcal{T}_x} W(R_m)[-U_{0m}] &\leq \sum_v \sum_{R_0\in\mathcal{T}_0(v)} W(R_0)\sum_{R_m\in\mathcal{T}_x(v)} W(R_m)\\ &= \sum_v \rho(v)\rho(v-x).\end{aligned} \tag{5.56}$$

If $m=1$ then the last line of (5.55) is

$$\zeta \sum_{\substack{\omega:0\to x\\ |\omega|=1}} W(\omega) = \zeta p_c D(x), \tag{5.57}$$

as required. For $m\geq 2$, $\prod_{\flat\neq 0m}[1+U_\flat] \leq \prod_{1\leq s<t\leq m-1}[1+U_{st}]$ and letting $y_1$ (resp.

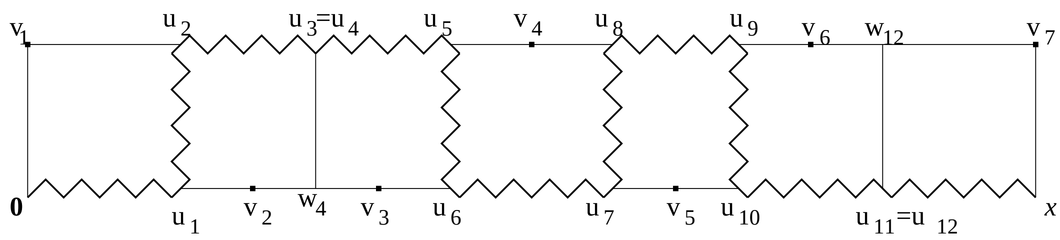


Figure 5.4: An example of the Feynman diagrams arising from the lace expansion. A jagged lines from $u_{i-1}$ to $u_i$ represents the quantity $h_{m_i}(u_i - u_{i-1})$ (derived from the backbone from 0 to $x$). A straight line between two vertices $u$ and $v$ represents the quantity $\rho(v-u)$ (derived from intersections of branches emanating from the backbone).

$y_2$) be the location of the walk $\omega$ after 1 step (resp. $m-1$ steps) we have

$$\begin{aligned}
&\zeta^m \sum_{\substack{\omega:0\to x\\|\omega|=m}} W(\omega)\prod_{i=1}^{m-1}\sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i)\prod_{\flat\neq 0m}[1+U_\flat]\\
\leq &\sum_{y_1}\sum_{y_2}\zeta p_c D(y_1)\zeta p_c D(x-y_2)\times \\
&\zeta^{m-2}\sum_{\substack{\omega':y_1\to y_2\\|\omega'|=m-2}} W(\omega')\prod_{j=0}^{m-2}\sum_{R_j\in\mathcal{T}_{\omega'(j)}} W(R_j)\prod_{\flat}[1+U_\flat]\\
=&h_m(x).
\end{aligned} \tag{5.58}$$

Combining (5.55)–(5.58) gives the desired result for $N=1$. See Figure 5.3 for the diagrammatic representation of this bound.

For $N \geq 2$ the reader should refer to Figure 5.4 to help understand the following derivation. Firstly $L \in \mathcal{L}^N([0,m])$ if and only if $L = \{s_1t_1,\ldots,s_Nt_N\}$ where $s_1 = 0, t_N = m$ and for each $i$, $s_{i+1} \leq t_i$ and $s_{i+1} - t_{i-1} > 0$. Hence from (5.2), $\pi_m^N(x;\zeta)$ is equal to

$$\zeta^m \sum_{\substack{\{s_1t_1,\ldots s_Nt_N\}\\ \in\mathcal{L}^N([0,m])}} \sum_{\substack{\omega:0\to x\\|\omega|=m}} W(\omega)\prod_{i=0}^{m}\sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i)\prod_{i=1}^{N}[-U_{s_it_i}]\prod_{\flat\in\mathcal{C}(L)}[1+U_\flat]. \tag{5.59}$$

Now everything in this expression is positive, and every bond $\flat = st$ such that $s_1 < s < t < s_2$, or $t_{N-1} < s < t < t_N$, or $s_{i+1} < s < t < t_i$, or $t_i < s < t < s_{i+2}$, is compatible with $L = \{s_1t_1, \ldots, s_Nt_N\}$. Therefore (5.59) is bounded above by

$$\zeta^m \sum_{\substack{\{s_1t_1,\ldots s_Nt_N\} \\ \in \mathcal{L}^N([0,m])}} \sum_{\substack{\omega:0\to x \\ |\omega|=m}} W(\omega) \prod_{i=0}^{m} \sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i) \prod_{i=1}^{N} [-U_{s_it_i}] \times$$

$$\prod_{\flat\in(s_1,s_2)} [1+U_\flat] \prod_{\flat\in(t_{N-1},t_N)} [1+U_\flat] \prod_{i=1}^{N-1} \prod_{\flat\in(s_{i+1},t_i)} [1+U_\flat] \prod_{j=1}^{N-2} \prod_{\flat\in(t_j,s_{j+2})} [1+U_\flat], \tag{5.60}$$

where for $\flat = st$ we are using the notation $\flat \in (a,b)$ to mean $a < s < t < b$.

For $L = \{s_1t_1, \ldots, s_Nt_N\} \in \mathcal{L}^N([0,m])$ we define $\vec{m}(L) \in \mathbb{Z}_+^{2N-1}$ by

$$m_1 = s_2 - 0, \quad m_{2N-1} = m - t_{N-1}, \quad m_{2i} = t_i - s_{i+1}, \quad m_{2i-1} = s_{i+1} - t_{i-1}. \tag{5.61}$$

Then $m_{2i} \geq 0$, $m_{2i-1} > 0$ and $\sum_{i=1}^{2N-1} m_i = m$, so $\vec{m} \in \mathcal{H}_{m,N}$. Similarly for any $\vec{m} \in \mathcal{H}_{m,N}$ we define $L(\vec{m}) = \{s_1t_1, \ldots, s_Nt_N\} \in \mathcal{G}([0,m])$ by

$$\begin{aligned} s_1 &= 0, \quad t_N = m, \\ t_i &= \sum_{j=1}^{2i} m_j, \quad i = 1, \ldots, N-1, \\ s_l &= \sum_{j=1}^{2l-1} m_j, \quad l = 2, \ldots N. \end{aligned} \tag{5.62}$$

Then for each $i$, $s_{i+1} \leq t_i$ and $s_{i+1} - t_{i-1} > 0$ so that $L(\vec{m}) \in \mathcal{L}^N([0,m])$. Thus (5.61)–(5.62) defines a bijection between $\mathcal{L}^N([0,m])$ and $\mathcal{H}_{m,N}$.

We now break up the sum over walks $\omega$ in (5.60) according to the intervals on the right of (5.60). Doing so we obtain

$$\begin{aligned} &\sum_{\substack{\omega:0\to x \\ |\omega|=m}} W(\omega) \\ &= \sum_{u_1,\ldots,u_{2N-1}} \sum_{\substack{\omega_1:0\to u_1 \\ |\omega|=s_2-s_1}} W(\omega_1) \sum_{\substack{\omega_{2N-1}:u_{2N-2}\to x \\ |\omega|=s_2-s_1}} W(\omega_{2N-1}) \times \\ &\prod_{i=1}^{N-1} \sum_{\substack{\omega_{2i}:u_{2i-1}\to u_{2i} \\ |\omega_{2i}|=t_i-s_{i+1}}} W(\omega_{2i}) \prod_{j=1}^{N-2} \sum_{\substack{\omega_{2j+1}:u_{2j}\to u_{2j+1} \\ |\omega_{2j+1}|=s_{j+2}-t_j}} W(\omega_{2j+1}). \end{aligned} \tag{5.63}$$

Then under this scheme, $\prod_{i=0}^{m}\sum_{R_i\in\mathcal{T}_{\omega(i)}} W(R_i)$ becomes

$$\sum_{R_0\in\mathcal{T}_0} W(R_0) \prod_{\substack{1\le i\le 2N-1:\\ m_i\neq 0}} \left(\sum_{R_{i,m_i}\in\mathcal{T}_{\omega_i(m_i)}} W(R_{i,m_i})\prod_{j=1}^{m_i-1}\sum_{R_{i,j}\in\mathcal{T}_{\omega_i(j)}} W(R_{i,j})\right), \tag{5.64}$$

where $\omega_i(m_i)=u_i$ ($\omega_{2N-1}(m_{2N-1})=x$) and the product over $i$ ensures that if some $s_l=t_{l-1}$ then we do not count the tree emanating from this vertex twice. Similarly the term $\prod_{i=1}^{N}[-U_{s_it_i}]=\prod_{i=1}^{N} I_{\{R_{s_i}\cap R_{t_i}\neq\emptyset\}}$ becomes

$$\begin{aligned}&\left(I_{\{m_i\neq 0\}}+I_{\{m_i=0\}}I_{\{R_{i,m_i}=R_{i-1,m_{i-1}}\}}\right)\times\\ &I_{\{R_0\cap R_{2,m_2}\neq\emptyset\}}I_{\{R_{2N-3,m_{2N-3}}\cap R_{2N-1,m_{2N-1}}\neq\emptyset\}}\prod_{l=1}^{N-2} I_{\{R_{2l-1,m_{2l-1}}\cap R_{2l+2,m_{2l+2}}\neq\emptyset\}}.\end{aligned} \tag{5.65}$$

Note that (5.65) contains no information about $R_{i,j}$ for $0<j<m_i$.

Lastly we have that the second line of (5.60) becomes

$$\prod_{i=1}^{2N-1}\left(\prod_{1\le s<t\le m_i-1} I_{\{R_{i,s}\cap R_{i,t}=\emptyset\}}\right). \tag{5.66}$$

Combining (5.60) with (5.63)–(5.66), and writing $u_0=0, u_{2N-1}=x$ we have that (5.60) is equal to

$$\begin{aligned}&\sum_{\vec{u}}\sum_{\vec{m}\in\mathcal{H}_{m,N}}\sum_{R_0\in\mathcal{T}_0} W(R_0)\prod_{\substack{1\le i\le 2N-1:\\ m_i\neq 0}}\left(\sum_{R_{i,m_i}\in\mathcal{T}_{u_i}} W(R_{i,m_i})\right)\times\\ &\left(I_{\{m_i\neq 0\}}+I_{\{m_i=0\}}I_{\{R_{i,m_i}=R_{i-1,m_{i-1}}\}}\right)\times\\ &I_{\{R_0\cap R_{2,m_2}\neq\emptyset\}}I_{\{R_{2N-3,m_{2N-3}}\cap R_{2N-1,m_{2N-1}}\neq\emptyset\}}\prod_{l=1}^{N-2} I_{\{R_{2l-1,m_{2l-1}}\cap R_{2l+2,m_{2l+2}}\neq\emptyset\}}\times\\ &\prod_{i=1}^{2N-1}\left(\zeta^{m_i}\sum_{\substack{\omega_i:u_{i-1}\to u_i\\ |\omega_i|=m_i}} W(\omega_i)\prod_{j=1}^{m_i-1}\sum_{R_{i,j}\in\mathcal{T}_{\omega_i(j)}} W(R_{i,j})\left(\prod_{1\le s<t\le m_i-1} I_{\{R_{i,s}\cap R_{i,t}=\emptyset\}}\right)\right).\end{aligned} \tag{5.67}$$

The last line of (5.67) is $\prod_{i=1}^{2N-1} h_{m_i}(u_i-u_{i-1})$ by definition.

For any collection of trees $\{R_{i,m_i}: 1\le i\le 2N-1\}$ for which (5.65) is equal to one (i.e. nonzero) we choose $v_i\in\mathbb{Z}^d,\ \ i=1,\dots,N$ as follows.

(a) $I_{\{R_0 \cap R_{2,m_2} \neq \emptyset\}} = 1$ if and only if there exists a $v_1 \in \mathbb{Z}^d$ such that $v_1 \in R_0 \cap R_{2,m_2}$. This means that $R_0 \in \mathcal{T}_0(v_1)$ and $R_{2,m_2} \in \mathcal{T}_{u_2}(v_1)$.

(b) Similarly $I_{\{R_{2N-3,m_{2N-3}} \cap R_{2N-1,m_{2N-1}} \neq \emptyset\}} = 1$ if and only if there exists a $v_N \in \mathbb{Z}^d$ such that $v_N \in R_{2N-3,m_{2N-3}} \cap R_{2N-1,m_{2N-1}}$. This means that $R_{2N-3,m_{2N-3}} \in \mathcal{T}_{u_{2N-2}}(v_N)$ and $R_{2N-1,m_{2N-1}} \in \mathcal{T}_x(v_1)$.

(c) For each $i \in \{3, \dots, 2N-5\}$ such that $i$ is odd, $I_{\{R_{i,m_i} \cap R_{i+3,m_{i+3}} \neq \emptyset\}} = 1$ if and only if there exists $v_{\frac{i+3}{2}} \in \mathbb{Z}^d$ such that $v_{\frac{i+3}{2}} \in R_{i,m_i} \cap R_{i+3,m_{i+3}}$. This means that $R_{i,m_i} \in \mathcal{T}_{u_i}(v_{\frac{i+3}{2}})$ and $R_{i+3,m_{i+3}} \in \mathcal{T}_{u_{i+3}}(v_{\frac{i+3}{2}})$ where $i+3$ is even.

Now if $m_l = 0$ (in particular this forces $i$ to be even) then $h_{m_l}(u_l - u_{l-1})$ in (5.67) is nonzero if and only if $u_l = u_{l-1}$. In addition $I_{\{R_{l,m_l} = R_{l-1,m_{l-1}}\}} = 1$ if and only if $R_{l,m_l} = R_{l-1,m_{l-1}}$. By the above construction we have that $v_{\frac{l}{2}} \in R_{l,m_l}$, and $v_{\frac{l+2}{2}} \in R_{l-1,m_{l-1}}$, i.e. $v_{\frac{l}{2}}, v_{\frac{l+2}{2}}, u_l \in R_{l,m_l}$. For $T = R_{l,m_l}$ let $T_{u_l \rightsquigarrow v_{\frac{l}{2}}}$ and $T_{u_l \rightsquigarrow v_{\frac{l+2}{2}}}$ denote the backbones in $T$ joining the specified verticies. Then there exists a unique $w_l \in T$ such that

$$T_{u_l \rightsquigarrow v_{\frac{l}{2}}} \cap T_{u_l \rightsquigarrow v_{\frac{l+2}{2}}} = T_{u_l \rightsquigarrow w_l}. \tag{5.68}$$

Collecting the above statements we have that

$$\begin{aligned}
&\sum_{R_0 \in \mathcal{T}_0} W(R_0) \prod_{\substack{1 \le i \le 2N-1: \\ m_i \neq 0}} \left( \sum_{R_{i,m_i} \in \mathcal{T}_{u_i}} W(R_{i,m_i}) \right) \times \\
&\left( I_{\{m_i \neq 0\}} + I_{\{m_i = 0\}} I_{\{R_{i,m_i} = R_{i-1,m_{i-1}}\}} \right) \times \\
&I_{\{R_0 \cap R_{2,m_2} \neq \emptyset\}} I_{\{R_{2N-3,m_{2N-3}} \cap R_{2N-1,m_{2N-1}} \neq \emptyset\}} \prod_{l=1}^{N-2} I_{\{R_{2l-1,m_{2l-1}} \cap R_{2l+2,m_{2l+2}} \neq \emptyset\}} \\
\le &\sum_{\vec{v}} \sum_{R_0 \in \mathcal{T}_0(v_1)} W(R_0) \sum_{R_{2N-1,m_{2N-1}} \in \mathcal{T}_x(v_N)} W(R_{2N-1,m_{2N-1}}) \times \\
&\prod_{l: m_l = 0} \sum_{R_{l,m_l} \in \mathcal{T}_{u_l}(v_{\frac{l}{2}}, v_{\frac{l+2}{2}})} W(R_{l,m_l}) \times \\
&\prod_{\substack{l: m_l \neq 0 \\ m_{l+1} \neq 0}} \left[ \left( \sum_{R_{l,m_l} \in \mathcal{T}_{u_l}(v_{\frac{l}{2}})} W(R_{l,m_l}) \right) I_{\{l \text{ even}\}} + \left( \sum_{R_{l,m_l} \in \mathcal{T}_{u_l}(v_{\frac{l+3}{2}})} W(R_{l,m_l}) \right) I_{\{l \text{ odd}\}} \right].
\end{aligned} \tag{5.69}$$

Now observe that $\sum_{R\in\mathcal{T}_{y_1}(y_2)} W(R) = \rho(y_2-y_1)$ and

$$\sum_{R_{l,m_l}\in\mathcal{T}_{u_l}(v_{\frac{l}{2}},v_{\frac{l+2}{2}})} W(R_{l,m_l}) \leq \sum_{w_l}\sum_{R_1\in\mathcal{T}_{u_l}(w_l)} W(R_1) \sum_{R_2\in\mathcal{T}_{w_l}(v_{\frac{l}{2}})} W(R_2) \sum_{R_3\in\mathcal{T}_{w_l}(v_{\frac{l+2}{2}})} W(R_3)$$
$$= \sum_{w_l}\rho(w_l-u_l)\rho(v_{\frac{l}{2}}-w_l)\rho(v_{\frac{l+3}{2}}-w_l). \tag{5.70}$$

This completes the proof of (5.54), and hence Proposition 5.1.7. □

## 5.4 Diagram pieces

In this section we first prove Proposition 5.1.4 assuming the following two lemmas, which we prove later in this section.

**Lemma 5.4.1.** *Let $k\in\{1,2,3,4\}$ and $\vec{r}^{(k)}\in\{0,1\}^k$ be such that $k+\sum_{i=1}^k r_i\leq 4$, then*

$$\sum_{0\leq|x|\leq\sqrt{m}L}\phi^{(k)}_{\vec{r}^{(k)}}(x)\leq Cm^{k+\sum r_j}\sigma^{k\nu+2\sum r_j}, \quad \textit{and} \quad \sup_{|x|>\sqrt{m}L}\phi^{(k)}_{\vec{r}^{(k)}}(x)\leq\frac{C\sigma^{2\sum r_j}\beta^{2-\frac{2k\nu}{d}}}{m^{\frac{d-2k-2\sum r_j}{2}}}. \tag{5.71}$$

**Lemma 5.4.2.** *If the bounds (3.33) hold for $1\leq m\leq n$ and $z\in[0,2]$, then for all $z\in[0,2]$, $l\in\{1,2,3,4\}$, $\vec{q}\in\{0,1\}^l$ and $\vec{m}^{(l)}\in\mathbb{Z}^l_+$ such that $\sum m_i=m\leq n+1$,*

$$\|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}\|_\infty\leq\frac{C\sigma^{2\sum q_i}\beta^2m^{\sum q_i}}{m^{\frac{d}{2}}}, \qquad \textit{and } \|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}\|_1\leq C\sigma^{2\sum q_i}m^{\sum q_i}. \tag{5.72}$$

**Proposition (5.1.4).** *Let $l\in\{1,2,3,4\}$, and $k\in\{0,1,2,3,4\}$, $\vec{q}\in\{0,1\}^l$ and $\vec{r}\in\{0,1\}^k$ be such that $k+\sum_{i=1}^k r_i\leq 4$. Let $\vec{m}^{(l)}\in\mathbb{Z}^l_+$ and $m=\sum_{i=1}^l m_i$. If the bounds (3.33) hold for $1\leq m\leq n$ and $z\in[0,2]$ then for all $m\leq n+1$ and $z\in[0,2]$,*

$$\|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}*\phi^{(k)}_{\vec{r}^{(k)}}\|_\infty\leq m^{\sum q_i+\sum r_j}\sigma^{2(\sum q_i+\sum r_j)}\frac{c\beta^{2-\frac{2k\nu}{d}}}{m^{\frac{d-2k}{2}}}, \tag{5.73}$$

*and*

$$\|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}\|_1\leq Cm^{\sum q_i}\sigma^{2\sum q_i}. \tag{5.74}$$

*Proof.* Firstly (5.73) with $k=0$ and (5.74) follow immediately from Lemma 5.4.2. We must therefore prove (5.73) with $k\geq 1$.

By definition $\|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}} * \phi^{(k)}_{\vec{r}^{(k)}}\|_\infty$ is equal to $\sup_x \sum_u s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}(x-u)\phi^{(k)}_{\vec{r}^{(k)}}(u)$ which is equal to

$$\begin{aligned}
&\sup_x \sum_{|u|>\sqrt{mL}} s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}(x-u)\phi^{(k)}_{\vec{r}^{(k)}}(u) + \sup_x \sum_{|u|\le\sqrt{mL}} s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}(x-u)\phi^{(k)}_{\vec{r}^{(k)}}(u) \\
\le &\sup_{|u'|>\sqrt{mL}} \phi^{(k)}_{\vec{r}^{(k)}}(u') \sum_{|u|>\sqrt{mL}} s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}(x-u) + \sup_{x'} s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}(x') \sum_{|u|\le\sqrt{mL}} \phi^{(k)}_{\vec{r}^{(k)}}(u) \\
\le &\frac{C\sigma^{2\sum r_j}\beta^{2-\frac{2k\nu}{d}}}{m^{\frac{d-2k-2\sum r_j}{2}}} C\sigma^{2\sum q_i} m^{\sum q_i} + \frac{C\sigma^{2\sum q_i}\beta^2 m^{\sum q_i}}{m^{\frac{d}{2}}} C m^{k+\sum r_j}\sigma^{k\nu+2\sum r_j},
\end{aligned} \tag{5.75}$$

where we have applied Lemma (5.4.1) and Lemma (5.4.2) in the last step. Collecting terms we get the result. □

Let $[x] = |x| \vee 1$. In order to prove Lemma 5.4.1, we need the following convolution proposition which is proved in [11].

**Proposition 5.4.3 ([11] Prop. 1.7(i)).** *If functions $f, g$ on $\mathbb{Z}^d$ satisfy $|f(x)| \le \frac{1}{[x]^a}$ and $|g(x)| \le \frac{1}{[x]^b}$ with $a \ge b > 0$, then there exists a constant $C$ depending on $a, b, d$ such that*

$$|(f*g)(x)| \le \begin{cases} \frac{C}{[x]^b}, & \text{if } a > d \\ \frac{C}{[x]^{a+b-d}}, & \text{if } a < d \text{ and } a+b > d. \end{cases} \tag{5.76}$$

### 5.4.1 Proof of Lemma 5.4.1

We prove the result in two stages. We first prove that

$$\phi^{(k)}_{\vec{r}^{(k)}}(x) \le \sum_{j=0}^{k} \frac{C}{L^{j(2-\nu)}[x]^{d-2j-2\sum r_i}}. \tag{5.77}$$

For $k = 1$ we have from (1.13) that

$$\phi^{(1)}_0(x) \le CI_{0=x} + \frac{CI_{0\neq x}}{L^{2-\nu}[x]^{d-2}} \le \sum_{j=0}^{1} \frac{C}{L^{j(2-\nu)}[x]^{d-2j}}, \tag{5.78}$$

and

$$\phi^{(1)}_1(x) \le \frac{C}{L^{2-\nu}[x]^{d-4}} \le \sum_{j=0}^{1} \frac{C}{L^{j(2-\nu)}[x]^{d-2j-2}}. \tag{5.79}$$

Which verifies (5.77) for $k = 1$. For $k > 1$ we have

$$
\begin{aligned}
\phi^{(k)}_{\vec{r}(l)}(x) &= \sum_u \phi^{(1)}_{r_1}(u)\phi^{(k-1)}_{(r_2,\ldots,r_k)}(x-u) \\
&\leq \sum_u \sum_{j=0}^{1} \frac{C}{L^{j(2-\nu)}[u]^{d-2j-2r_1}} \sum_{n=0}^{k-1} \frac{C}{L^{n(2-\nu)}[x-u]^{d-2n-2\sum_{i=2}^k r_i}} \\
&\leq \sum_{j=0}^{1}\sum_{n=0}^{k-1} \frac{C}{L^{j(2-\nu)}L^{n(2-\nu)}} \sum_u \frac{1}{[u]^{d-2j-2r_1}} \frac{1}{[x-u]^{d-2n-2\sum_{i=2}^k r_i}} \qquad (5.80)\\
&\leq \sum_{j=0}^{1}\sum_{n=0}^{k-1} \frac{C}{L^{(j+n)(2+\nu)}} \frac{C}{[x]^{d-2(j+n)-2\sum_{i=1}^k r_i}},
\end{aligned}
$$

where we have used Proposition 5.4.3 with the fact that $k + \sum r_i < 4$ in the last step. With a different constant, (5.80) is bounded by

$$
\sum_{j=0}^{k} \frac{C}{L^{j(2+\nu)}[x]^{d-2j-2\sum_{i=1}^k r_i}} \tag{5.81}
$$

as required.

Therefore we have

$$
\begin{aligned}
\sum_{0\leq|x|\leq\sqrt{m}L} \phi^{(k)}_{\vec{r}(k)}(x) &\leq \sum_{j=0}^{k} \sum_{0\leq|x|\leq\sqrt{m}L} \frac{C}{L^{j(2-\nu)}[x]^{d-2j-2\sum r_i}} \\
&\leq \sum_{j=0}^{k} \frac{C\,(\sqrt{m}L)^{2j+2\sum r_i}}{L^{j(2-\nu)}} = C\sum_{j=0}^{k} m^{j+\sum r_i} L^{j\nu+2\sum r_i} \qquad (5.82)\\
&\leq Cm^{k+\sum r_i}L^{k\nu+2\sum r_i} \leq Cm^{k+\sum r_i}\sigma^{k\nu+2\sum r_i}
\end{aligned}
$$

which proves the first bound of Lemma 5.4.1. Similarly,

$$
\begin{aligned}
\sup_{|x|>\sqrt{m}L} \phi^{(k)}_{\vec{r}(k)}(x) &\leq \sum_{j=0}^{k} \sup_{|x|>\sqrt{m}L} \frac{C}{L^{j(2-\nu)}[x]^{d-2j-2\sum r_i}} \leq \sum_{j=0}^{k} \frac{C}{L^{j(2-\nu)}\left(\sqrt{m}L\right)^{d-2j-2\sum r_i}} \\
&= \sum_{j=0}^{k} \frac{C}{L^{d-j\nu-2\sum r_i} m^{\frac{d-2j-2\sum r_i}{2}}} \\
&\leq \frac{C\sigma^{2\sum r_j}\beta^{2-\frac{2k\nu}{d}}}{m^{\frac{d-2k-2\sum r_j}{2}}},
\end{aligned} \tag{5.83}
$$

which proves the second bound of Lemma 5.4.1. □

**Remark 5.4.4.** *Observe that the only information about $\rho(x)$ that we used to prove Lemma 5.4.1 (and hence Proposition 5.1.4) was (1.13). This will become important when we estimate more complicated diagrams in Chapter 6.*

### 5.4.2 Proof of Lemma 5.4.2.

In this section we prove Lemma 5.4.2 by induction on $l$.

For $l = 1$ we use induction on $m$. For $l = 1$ and $m = 1$ we have $h_1(x) = \zeta p_c D(x)$ and hence

$$\|h_1\|_\infty \leq \frac{C}{L^d} = C\beta^2, \quad \|h_1\|_1 \leq C. \tag{5.84}$$

Using the fact that $D(x) = 0$ for $|x|^2 > dL^2$,

$$\sup_x |x|^2 h_1(x) \leq CL^2 \frac{1}{L^d} \leq C\sigma^2\beta^2, \tag{5.85}$$

and by (3.28)

$$\sum_x |x|^2 h_1(x) \leq C \sum_x |x|^2 D(x) \leq C\sigma^2. \tag{5.86}$$

This proves the result for the case $l = 1$, $m = 1$. The case $l = 1$ and $m = 2$ is dealt with similarly using

$$|x|^2 h_2(x) \leq C \sum_u |u|^2 D(u) D(x-u) + C \sum_u |x-u|^2 D(u) D(x-u). \tag{5.87}$$

For $l = 1$ and $m \geq 2$ we use the inequality $h_m(x) \leq \rho(0)\zeta p_c \sum_u D(u) h_{m-1}(x-u)$ (which holds trivially by replacing the factor $\prod_{0 \leq s < t \leq m-2}[1+U_{st}]$ by $\prod_{1 \leq s < t \leq m-2}[1+U_{st}]$ in the definition of $t_{m-2}$) so that

$$\|h_m\|_\infty \leq \|h_{m-1}\|_\infty \leq \frac{C\beta^2}{m^{\frac{d}{2}}}, \quad \|h_m\|_1 \leq \|h_{m-1}\|_\infty \leq K. \tag{5.88}$$

Using $|x|^2 \leq 2(|u|^2 + |x-u|^2)$, we have

$$\begin{aligned} &\sup_x |x|^2 h_m(x) \leq \sigma^2 \|h_{m-1}\|_\infty + \sup_{x-u'} |x-u'|^2 h_{m-1}(x-u') \\ \leq &\frac{C\sigma^2\beta^2}{m^{\frac{d}{2}}} + \frac{C\sigma^2\beta^2}{m^{\frac{d-2}{2}}}. \end{aligned} \tag{5.89}$$

This proves the result for $l = 1$ and all $m \leq n+1$.

For $l \geq 2$ we have $s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}} = \sum_u s^{(1)}_{m_1,q_1}(u) s^{(l-1)}_{(m_2,\ldots,m_l),(q_2,\ldots q_l)}(x-u)$. If $m_1 \geq \frac{m}{2}$,

$$\begin{aligned} \|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}\|_\infty &\leq \|s^{(1)}_{m_1,q_1}\|_\infty \|s^{(l-1)}_{(m_2,\ldots,m_l),(q_2,\ldots q_l)}\|_1 \\ &\leq \frac{C\sigma^{2q_1}\beta^2 m^{q_1}}{m^{\frac{d}{2}}} C\sigma^{2\sum_{i=2}^l q_i} m^{\sum_{i=2}^l q_i} \\ &\leq \frac{C\sigma^{2\sum q_i} m^{\sum q_i}\beta^2}{m^{\frac{d}{2}}}, \end{aligned} \tag{5.90}$$

as required. Similarly if $m_1 < \frac{m}{2}$,

$$\begin{aligned} \|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}\|_\infty &\leq \|s^{(1)}_{m_1,q_1}\|_1 \|s^{(l-1)}_{(m_2,\ldots,m_l),(q_2,\ldots q_l)}\|_\infty \\ &\leq C\sigma^{2q_1} m^{q_1} \frac{C\sigma^{2\sum_{i=2}^l q_i}\beta^2 m^{\sum_{i=2}^l q_i}}{m^{\frac{d}{2}}} \\ &\leq \frac{C\sigma^{2\sum q_i} m^{\sum q_i}\beta^2}{m^{\frac{d}{2}}}, \end{aligned} \tag{5.91}$$

as required. This completes the proof of the first bound of Lemma 5.4.2 for all $l$.

For the second bound of Lemma 5.4.2, we have

$$\begin{aligned} \|s^{(l)}_{\vec{m}^{(l)},\vec{q}^{(l)}}\|_1 &\leq \|s^{(1)}_{m_1,q_1}\|_1 \|s^{(l-1)}_{(m_2,\ldots,m_l),(q_2,\ldots q_l)}\|_1 \\ &\leq C\sigma^{2q_1} m^{q_1} C\sigma^{2\sum_{i=2}^l q_i} m^{\sum_{i=2}^l q_i} \\ &\leq C\sigma^{2\sum_{i=1}^l q_i} m^{\sum_{i=1}^l q_i}, \end{aligned} \tag{5.92}$$

as required. This completes the proof of the second bound of Lemma 5.4.2, and thus completes the proof of Lemma 5.4.2. □

**Remark 5.4.5.** *Observe that the only information about $h_m$ that we used to prove Lemma 5.4.2 (and hence Proposition 5.1.4) was*

$$\|h_m\|_\infty \leq \frac{C\beta^2}{m^{\frac{d}{2}}}, \quad \|h_m\|_1 \leq C, \tag{5.93}$$

*and when some $q_i \neq 0$ we also used*

$$\sup_x |x|^2 h_m(x) \leq \frac{C\sigma^2 m\beta^2}{m^{\frac{d}{2}}}, \quad \sum_x |x|^2 h_m(x) \leq C\sigma^2 m. \tag{5.94}$$

*This will become important when we estimate more complicated diagrams in Chapter 6.*

# Chapter 6

# Diagrams for the $r$-point functions

In this chapter we prove Proposition 4.3.2, and Lemmas 4.3.3, 4.4.1 and 4.2.1. Note that since we have proved Proposition 3.4.1 in Chapter 5, one output of the inductive approach of Appendix A is that the bounds of equation (3.33) hold for all $n$. As a result, the conclusions of all the Lemmas and Propositions of Chapter 5 hold for all $n$. Another result of the inductive approach is that $\zeta_c = 1$ (see Lemma 3.5.2). In this chapter $\zeta = 1$ and hence it does not appear.

Proposition 4.3.2 is proved in Sections 6.1 to 6.4 using the lace expansion on a star-shaped network and the results of Chapter 5. Lemma 4.3.3 is proved in Section 6.5. The other results are proved in Section 6.6, also assuming the results of Chapter 5.

## 6.1 Proof of Proposition 4.3.2.

For $N \geq 1$, recall the definition of $\pi^N_{\vec{M}}(\vec{u})$ from (4.27) where $\mathcal{S}^{\Delta}_{\vec{M}}$ has at least one of $M_1, M_2, M_3$ nonzero (we defined $\pi_{\vec{0}}(\vec{u}) = \rho(0)I_{\{\vec{u}=\vec{0}\}}$).

**Proposition (4.3.2).** *There exists a constant $C$ independent of $L$ such that for $N \geq 1$ and $q \in \{0,1\}$,*

$$\sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^{2q}\pi^N_{\vec{M}}(\vec{u}) \leq N^3(N^2\sigma^2\|\vec{M}\|_\infty)^q B_N(\vec{M}), \tag{6.1}$$

*where* $\vec{u} = (u_1, u_2, u_3) \in \mathbb{Z}^{3d}$ *and*

$$B_N(\vec{M}) \equiv \left(C\beta^{2-\frac{8\nu}{d}}\right)^N \times \left[\prod_{i=1}^{3} \frac{1}{[M_i]^{\frac{d-6}{2}}} + \sum_{i=1}^{3} \frac{1}{[M_i]^{\frac{d-6}{2}}} \sum_{j\neq i} \sum_{m_j \le M_j} \frac{1}{[M_j - m_j]^{\frac{d-6}{2}}} \frac{1}{[M_k + m_j]^{\frac{d-4}{2}}}\right]. \tag{6.2}$$

We prove Proposition 4.3.2 assuming Lemmas 6.1.1, 6.1.2, and 6.1.3.

**Lemma 6.1.1.** *For* $q \in \{0, 1\}$*, when* $M_l = 0$ *but* $\vec{M} \neq \vec{0}$*,*

$$\sum_{\vec{u}} |u_j|^2 \pi^N_{\vec{M}}(\vec{u}) \le \frac{N^3 (N^2\sigma^2 \|\vec{M}\|_\infty)^q (C\beta^{2-\frac{6\nu}{d}})^N}{(\sum_{i=1}^3 M_i)^{\frac{d-4}{2}}}. \tag{6.3}$$

As stated in Chapter 2, laces on a star-shaped network of degree 3 can be classified as *cyclic* or *acyclic*. Let $\mathcal{L}^N_c$ (resp. $\mathcal{L}^N_a$) denote the set of cyclic (resp. acyclic) laces and define $\overset{a}{\pi}{}^N_{\vec{M}}(\vec{x})$ and $\overset{c}{\pi}{}^N_{\vec{M}}(\vec{x})$ to be the contributions to $\pi^N_{\vec{M}}(\vec{x})$ from acyclic and cyclic laces respectively so that when none of the $M_j = 0$, $\pi^N_{\vec{M}}(\vec{x}) = \overset{a}{\pi}{}^N_{\vec{M}}(\vec{x}) + \overset{c}{\pi}{}^N_{\vec{M}}(\vec{x})$. Figure 2.6 shows a basic cyclic lace and a basic acyclic lace with 3 bonds covering the branch point.

**Lemma 6.1.2.** *For* $q \in \{0, 1\}$*,*

$$\sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^{2q} \overset{c}{\pi}{}^N_{\vec{M}}(\vec{u}) \le N^3 (N^2\sigma^2 \|\vec{M}\|_\infty)^q \left(C\beta^{2-\frac{8\nu}{d}}\right)^N \prod_{i=1}^{3} \frac{1}{[M_i]^{\frac{d-6}{2}}}. \tag{6.4}$$

**Lemma 6.1.3.** *For* $q \in \{0, 1\}$*,*

$$\sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^{2q} \overset{a}{\pi}{}^N_{\vec{M}}(\vec{u}) \le N^3 (N^2\sigma^2 \|\vec{M}\|_\infty)^q \left(C\beta^{2-\frac{8\nu}{d}}\right)^N \times \sum_{i=1}^{3} \frac{1}{[M_i]^{\frac{d-6}{2}}} \sum_{j\neq i} \sum_{m_j \le M_j} \frac{1}{[M_j - m_j]^{\frac{d-6}{2}}} \frac{1}{[M_k + m_j]^{\frac{d-4}{2}}}. \tag{6.5}$$

Proof of Proposition 4.3.2. If $M_l = 0$ for some $l$ (but $\vec{M} \neq \vec{0}$) then from Lemma 6.1.1 we have

$$\begin{aligned}\sum_{\vec{u}} |u_j|^2 \pi^N_{\vec{M}}(\vec{u}) &\le \frac{N^3 (N^2\sigma^2 \|\vec{M}\|_\infty)^q (C\beta^{2-\frac{6\nu}{d}})^N}{(\sum_{i=1}^3 M_i)^{\frac{d-4}{2}}} \\ &\le \frac{N^3 (N^2\sigma^2 \|\vec{M}\|_\infty)^q (C\beta^{2-\frac{6\nu}{d}})^N}{[M_l - M_l]^{\frac{d-6}{2}} [M_{l_2} - M_{l_2}]^{\frac{d-6}{2}} [M_{l_1} + M_{l_2}]^{\frac{d-4}{2}}},\end{aligned} \tag{6.6}$$

where $l_1 \neq l$ and $l_2 \neq l$. Summing over $l$ this is trivially less than

$$N^3(N^2\sigma^2\|\vec{M}\|_\infty)^q(C\beta^{2-\frac{6\nu}{d}})^N B_N(\vec{M}), \tag{6.7}$$

as required.

Otherwise $M_l \neq 0$ for all $l$, and the result follows from Lemmas 6.1.2 and 6.1.3 using

$$\sum_{\vec{u}} |u_j|^2 \pi^N_{\vec{M}}(\vec{u}) \leq \sum_{\vec{u}} |u_j|^2 \overset{c}{\pi}{}^N_{\vec{M}}(\vec{u}) + \sum_{\vec{u}} |u_j|^2 \overset{a}{\pi}{}^N_{\vec{M}}(\vec{u}). \tag{6.8}$$

□

Before proving Lemma 6.1.1 in the next section, we introduce a Lemma which allows us to replace one or more lines (correpsonding to $\rho$'s and $h_m$'s) in a diagram with different quantities, in such a way that we can estimate the resulting diagrams without resorting to more inductive proofs such as in Section 5.2.

**Lemma 6.1.4.** *Given homogeneous functionals $F : E^m \to \mathbb{R}_+$ and $f_i : E \to \mathbb{R}_+$, suppose that whenever $f_i(a_i) \leq b_i$, we have $F(\vec{a}) \leq K$. Then for scalars $\alpha_i > 0$, the bounds $f_i(a_i^*) \leq \alpha_i b_i$ imply $F(\vec{a}^*) \leq K \prod_{i=1}^m \alpha_i$.*

*Proof.* By homogeneity,

$$F(\vec{a}^*) = F\left(\frac{a_1^*}{\alpha_1}, \dots, \frac{a_m^*}{\alpha_m}\right) \prod_{i=1}^m \alpha_i. \tag{6.9}$$

Also by homogeneity,

$$f_i\left(\frac{a_i^*}{\alpha_i}\right) = \frac{f_i(a_i^*)}{\alpha_i} \leq b_i, \quad \text{for each} i \tag{6.10}$$

which implies that $F\left(\frac{a_1^*}{\alpha_1}, \dots, \frac{a_m^*}{\alpha_m}\right) \leq K$ by hypothesis. This completes the proof. □

In most cases we will use Lemma 6.1.4 with each $a_i$ being either $h_{m_i}(u_i)$ or $\rho(u_i)$ and $F$ being a diagram (i.e. a large convolution of $h_m$'s and $\rho$'s). In fact Lemma 6.1.4 provides an alternative method of bounding the $q = 1, 2$ cases of Proposition 5.1.1.

## 6.2 Proof of Lemma 6.1.1.

Without loss of generality $M_3 = 0$. By (2.12), $\mathcal{S}^{\Delta}_{\vec{M}}$ is the interval (i.e. a star-shaped network of degree 1) $[-M_2, M_1]$ of length $M_1 + M_2$. Consider the lace $L = \{-M_2 M_1\}$ illustrated in Figure 6.1. Breaking up the walk corresponding to the backbone into

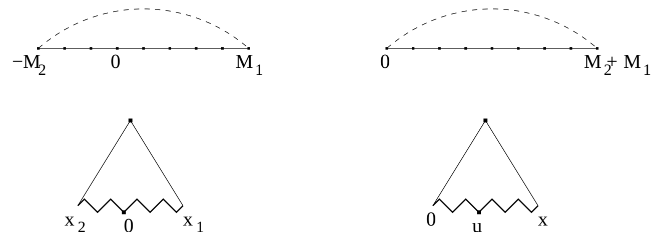


Figure 6.1: The single bond lace on $[-M_2, M_1]$ and the corresponding $[0, M_1+M_2]$.

two subwalks we can show that the contribution to $\sum_{x_1,x_2} \pi_{(M_1,M_2,0)}(x_1,x_2)$ from this lace is less than or equal to

$$\sum_{x_1,x_2} h_{M_2}(x_2)h_{M_1}(x_1)\rho^{(2)}(x_1-x_2). \tag{6.11}$$

Using translation invariance we can rewrite this as

$$\sum_{x,u} h_{M_2}(u)h_{M_1}(x-u)\rho^{(2)}(x). \tag{6.12}$$

Comparing this to the contribution to (4.29) from the lace on the right of Figure 6.1,

$$\sum_{x} h_{M_1+M_2}(x)\rho^{(2)}(x), \tag{6.13}$$

we see that the only difference is the replacement of $h_{M_2+m_1}(u_1)$ in (6.16) by $\sum_u h_{M_2}(u)h_{m_1}(u_1-u)$.

Now consider the lace $L = \{s_1t_1, s_2t_2\}$ where $s_1 = M_2$, $t_1 = m_1+m_2$, $s_2 = m_1$, $t_2 = M_1$ on the left of Figure 6.2. This lace divides the interval $[-M_2, M_1]$ into subintervals, one of which contains 0. Using the same method as in the proof of Proposition 5.1.7, but breaking up the walk corresponding to the subinterval containing the root into two subwalks, we can show that the contribution to $\sum_{x_1,x_2} \pi_{(M_1,M_2,0)}(x_1,x_2)$ from this lace is less than or equal to

$$\sum_{x_1,x_2}\sum_{u_1,u_2} h_{M_2}(x_2)h_{m_1}(u_1)h_{m_2}(u_2-u_1)h_{M_1-(m_1+m_2)}(x_1-u_2)\rho^{(2)}(u_2-x_2)\rho^{(1)}(u_1-x_1). \tag{6.14}$$

Using translation invariance we can rewrite this as

$$\sum_{x_1}\sum_{u,u_1,u_2} h_{M_2}(u)h_{m_1}(u_1-u)h_{m_2}(u_2-u_1)h_{M_1-(m_1+m_2)}(x_1-u_2)\rho^{(2)}(u_2)\rho^{(1)}(u_1-x_1). \tag{6.15}$$

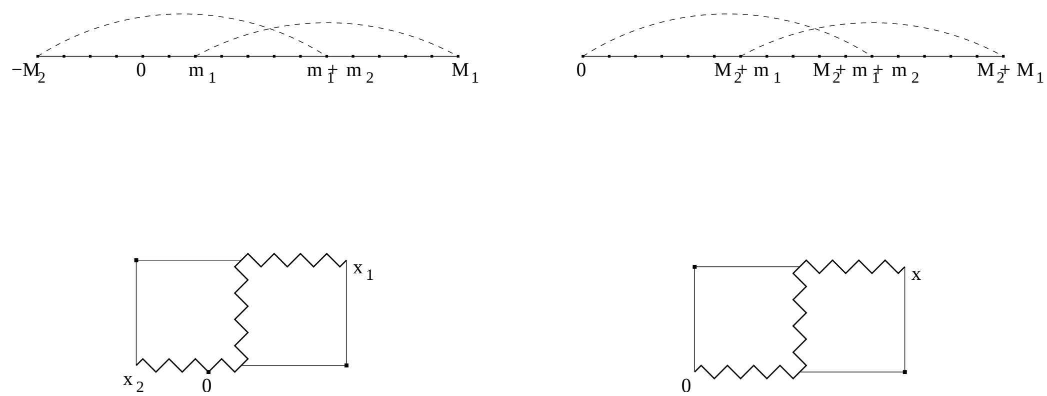


Figure 6.2: A lace on $[-M_2, M_1]$ and the corresponding $[0, M_1 + M_2]$.

Comparing this to the contribution to (4.29) from the lace on the right of Figure 6.2,

$$\sum_{x_1}\sum_{u_1,u_2} h_{M_2+m_1}(u_1)h_{m_2}(u_2-u_1)h_{M_1-(m_1+m_2)}(x_1-u_2)\rho^{(2)}(u_2)\rho^{(1)}(u_1-x_1), \quad (6.16)$$

we see that the only difference is the replacement of $h_{M_2+m_1}(u_1)$ in (6.16) by $\sum_u h_{M_2}(u)h_{m_1}(u_1-u)$.

In general, assuming $M_3 = 0$, the diagram arising from any lace on $[-M_2, M_1]$ is bounded by the opened diagram $M_{\vec{m}}^{(N)}$ of (5.9) that arises from the equivalent lace on $[0, M_1 + M_2]$, except for the replacement of at most one term $h_{m_i}(u_i)$ by a term of the form $(h_m * h_{m_i-m})(u)$ in $M_{\vec{m}}^{(N)}$. Note that this $m$ is fixed by $M_1$ and $M_2$ (i.e. it is not summed over). Proposition 5.1.4 states that the bound on a diagram piece does not depend on the degree of the convolution of $h_{m_i}$ (provided that degree is less than 4). Thus by Proposition 5.1.7 we have the same diagrammatic bounds for $\sum_{x_1,x_2}\pi^N_{M_1,M_2,0}(x_1,x_2)$ as for $\sum_{\vec{m}\in\mathcal{H}_{M_1+M_2,N}}\sup_{a,b,y}\sum_x M_{\vec{m}}^{(N)}(a,b,x,y)$. But from (5.16),

$$\sum_{x_1,x_2}\pi^N_{M_1,M_2,0}(x_1,x_2) \le \sum_{\vec{m}\in\mathcal{H}_{M_1+M_2,N}}\sup_{a,b,y}\sum_x M_{\vec{m}}^{(N)}(a,b,x,y) \le \frac{\left(C\beta^{2-\frac{6\nu}{d}}\right)^N}{[M_1+M_2]^{\frac{d-4}{2}}}, \quad (6.17)$$

which satisfies the claim for $q = 0$.

For $q = 1$ observe that $|x_j|^2 \le 2N\sum_{i=1}^{2N-1}|u_{j,i}|^2$, where the $u_{j,i}$ are the displacements of the $h_{m_i}$ along the backbone from 0 to $x_j$ (there are at most $2N-1$ of these). The resulting diagrams are the same as for the $q = 0$ case except that one

$h_{m_i}(u_{j,i})$ on the backbone from 0 to $x_j$ has been replaced with $|u_{j,i}|^2 h_{m_i}(u_{j,i})$. In view of Remarks 5.2.1 and 5.4.5, the only bounds on $h_{m_i}$ that we used to bound the diagrams $M_{\vec{m}}^{(N)}$ without the factor $|u_{j,i}|^2$ were $\|h_{m_i}\|_\infty \le \frac{C\beta^2}{m^{\frac{d}{2}}}$ and $\|h_{m_i}\|_1 \le C$. Since $m_i \le \|\vec{M}\|_\infty$, (5.5) with $l = 1, k = 0$ and (5.6) with $l = 1$ imply that $\sup_u |u|^2 h(u) \le \frac{C\sigma^2 \|\vec{M}\|_\infty \beta^2}{m^{\frac{d}{2}}}$ and $\sum_u |u|^2 h(u) \le C\sigma^2 \|\vec{M}\|_\infty$. We now apply Lemma 6.1.4 to get that the diagrams $M_{\vec{m}}^{(N)}$ with the extra factor of $|u_{j,i}|^2$ are bounded by $\sigma^2 \|\vec{M}\|_\infty$ times the bound for the diagrams $M_{\vec{m}}^{(N)}$ without the extra factor.

Therefore when $M_3 = 0$,

$$\begin{aligned}\sum_{\vec{u}} |u_j|^2 \pi_{\vec{M}}^N(\vec{u}) &\le \sigma^2 \|\vec{M}\|_\infty (2N)^2 \sum_{\vec{m} \in \mathcal{H}_{M_1+M_2,N}} \sup_{a,b,y} \sum_x M_{\vec{m}}^{(N)}(a,b,x,y) \\ &\le \sigma^2 \|\vec{M}\|_\infty (2N)^2 \frac{\left(C\beta^{2-\frac{6\nu}{d}}\right)^N}{[M_1+M_2]^{\frac{d-4}{2}}}.\end{aligned} \tag{6.18}$$

Similarly for $M_1 = 0$, and $M_2 = 0$. Since (6.18) is smaller than (6.3) this completes the proof. □

## 6.3 Proof of Lemma 6.1.2

In this section we prove Lemma 6.1.2, which gives a bound on the contribution to $\sum_x |x|^{2q} \pi_{\vec{M}}^N(x)$ from cyclic laces. We first consider the cyclic laces $L \in \mathcal{L}_c^3(\mathcal{S}_{\vec{M}}^{(3)})$ containing only 3 bonds. There are multiple cases to consider, depending on how many bonds have common endvertices. For example, one needs to consider the number of those bonds that have the branch point as one of its endpoints (see the second row of Figure 6.3).

- Consider the case where none of the three bonds in the lace $L$ have the branch point as an endpoint. Without loss of generality the branch point associated to branch 1 has its other endvertex on branch 3 as in the first lace in Figure 6.3. Then for each $i \in \{1,2,3\}$ there exists $1 \le n_i \le M_i$ that is the endpoint of $e_{i+1}$ (the bond associated to branch $i+1$) on branch $i$.

  If $n_i < M_i$ for all $i$ then by first breaking the sum over $\omega \in \Omega_{\mathcal{S}_{\vec{M}}^{(3)}}(\vec{x})$ into the sum over three walks $\omega_j \in \Omega_{\mathcal{S}_{M_i}^{(1)}}(x_i)$, and then each of these into two further subwalks (using the same methods as in the proof of Proposition 5.1.7), it is easy to show that the contribution to $\sum_{\vec{x}} \pi_{\vec{M}}^{(3)}(\vec{x})$ from this lace $L$ is bounded by

$$\sum_{\vec{x}} \sum_{\vec{u}} \prod_{j=1}^{3} h_{m_j}(u_j) h_{M_j - m_j}(x_j - u_j) \rho^{(2)}(x_{j+1} - u_j), \tag{6.19}$$

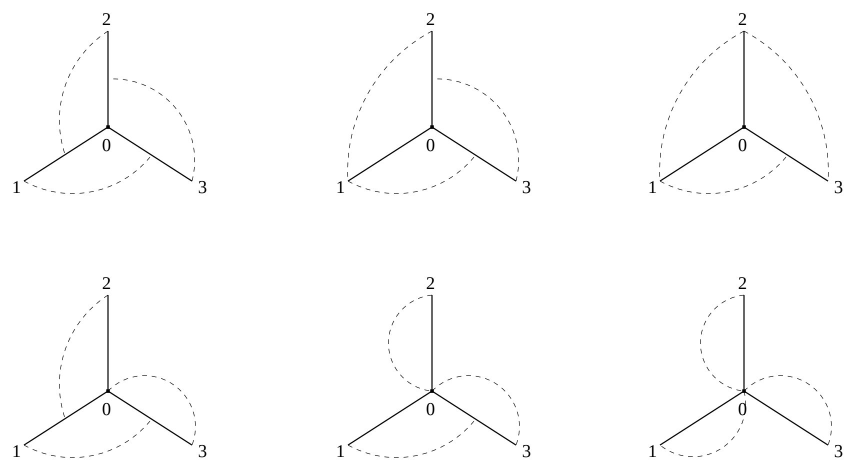


Figure 6.3: Some cyclic laces containing only 3 bonds

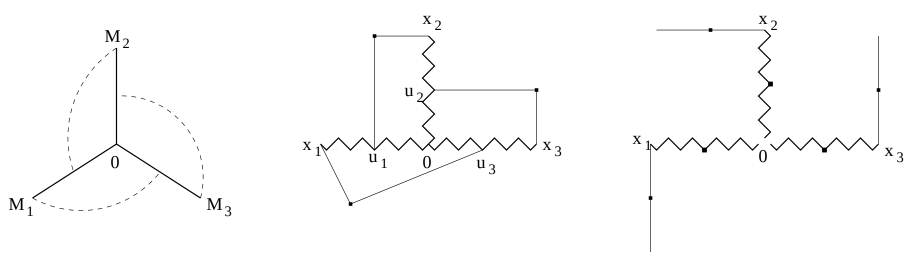


Figure 6.4: A basic cyclic lace $L$ containing only 3 bonds, its corresponding diagram $F(L)$ and its decomposition into 3 subdiagrams, $F_1(L)$, $F_2(L)$, $F_3(L)$.

where we use the convention that $x_4 = x_1$ (see Figure 6.4). We use the expression "opening up" a diagram informally to mean that we drop the restriction that two specific lines have a common endvertex and take the sup over the displacement of their endvertices. For example, the diagram corresponding to $\sum_x M_{\vec{m}}^{(N)}(0,0,x,0)$ (see Definition 5.1.5) with both ends opened up (i.e. opened up at 0 and $x$) would be $\sup_{b,y} \sum_x M_{\vec{m}}^{(N)}(0,b,x,y)$.

Opening up the diagram (which we denote $F(L)$) expressed in (6.19) at the vertices $n_j$, Equation (6.19) is bounded by

$$\begin{aligned} &\sup_{b_1,b_2,b_3} \sum_{\vec{x}} \sum_{\vec{u}} \prod_{j=1}^{3} h_{n_j}(u_j) h_{M_j-n_j}(x_j - u_j) \rho^{(2)}(x_{j+1} - b_{j+1}) \\ &= \prod_{j=1}^{3} \sup_{b_j} \sum_{x_j} \sum_{u_j} h_{n_j}(u_j) h_{M_j-n_j}(x_j - u_j) \rho^{(2)}(x_j - b_j), \end{aligned} \tag{6.20}$$

which is a product over 3 separate diagrams, $F_1(L)$, $F_2(L)$, $F_3(L)$, each corresponding to $M^{(1)}_{M_j}(0, b_j, x_j, 0)$ with an extra vertex on the backbone (see 5.7).

Suppose now that $m_1 = M_1$ (this is possible depending on the relative size of the $M_i$). Then two of the bonds have an endpoint at the endvertex of branch 1 and the contribution to $\sum_{\vec{x}} \pi^{(3)}_{\vec{M}}(\vec{x})$ from this lace $L'$ is bounded by

$$\begin{aligned}
&\sum_{\vec{x}} \sum_{\vec{u}} h_{M_1}(x_1)\rho(u_1 - x_1)\rho^{(2)}(x_2 - u_1)h_{n_2}(u_2) \\
&\quad \times h_{M_2-n_2}(x_2 - u_2)\rho^{(2)}(x_3 - u_2)h_{n_3}(u_3)h_{M_3-n_3}(x_3 - u_3)\rho^{(2)}(u_1 - u_3) \\
&\le \sup_{b_1,b_2,b_3} \sum_{\vec{x}} \sum_{\vec{u}} h_{M_1}(x_1)\rho(u_1 - x_1)\rho^{(2)}(x_2 - b_2)h_{n_2}(u_2) \\
&\quad \times h_{M_2-n_2}(x_2 - u_2)\rho^{(2)}(x_3 - b_3)h_{n_3}(u_3)h_{M_3-n_3}(x_3 - u_3)\rho^{(2)}(u_1 - b_1) \\
&= \sup_{b_1} \sum_{x_1} \sum_{u_1} h_{M_1}(x_1)\rho(u_1 - x_1)\rho^{(2)}(u_1 - b_1) \\
&\quad \times \prod_{j=2}^{3} \sup_{b_j} \sum_{x_j,u_j} h_{n_j}(u_j)h_{M_j-n_j}(x_j - u_j)\rho^{(2)}(x_j - b_j) \\
&= \sup_{b_1} \sum_{x_1} h_{M_1}(x_1)\rho^{(3)}(x_1 - b_1) \\
&\quad \times \prod_{j=2}^{3} \sup_{b_j} \sum_{x_j,u_j} h_{n_j}(u_j)h_{M_j-n_j}(x_j - u_j)\rho^{(2)}(x_j - b_j) \\
&.
\end{aligned} \tag{6.21}$$

Once again this is a product of 3 separate diagrams $F_1(L')$, $F_2(L')$, $F_3(L')$, each corresponding to $M^{(1)}_{M_j}(0, b_j, x_j, 0)$ with an extra vertex (two on backbones and one on a $\rho$).

- Consider now the case where one or more of the 3 bonds has the branch point as an endvertex. The diagrams arising from such laces depend on how many of the 3 bonds have this property, and each case is treated slightly differently. We present the most complicated case, where all 3 bonds in the lace $L$ have the branch point as an endvertex, as in Figure 6.5. The contribution to $\sum_{\vec{x}} \pi^{(3)}_{\vec{M}}(\vec{x})$ from this lace $L$ is bounded by

$$\sum_{\vec{x}} \sum_{w,z} \rho(w)\rho^{(2)}(x_1 - w)\rho(w - z)\rho^{(2)}(x_2 - z)\rho^{(2)}(x_3 - z) \prod_{j=1}^{3} h_{M_i}(x_i), \tag{6.22}$$

plus two other terms (see Figure 6.5) of similar form arising from the possible shapes of a lattice tree containing 4 fixed vertices. Equation 6.22 is the first diagram in Figure 6.5 and is bounded by

$$\begin{aligned}
&\sup_{b_1,b_2,b_3} \sum_{\vec{x}} \sum_{w,z} \rho(w-a_1)\rho^{(2)}(x_1-w)\rho(b_2-z) \\
&\quad \times \rho^{(2)}(x_2-z)\rho^{(2)}(x_3-b_3)\prod_{j=1}^{3} h_{M_i}(x_i) \\
&= \sup_{b_1} \sum_{x_1,w} \rho(w-b_1)\rho^{(2)}(x_1-w)h_{M_1}(x_1) \\
&\quad \times \sup_{b_2} \sum_{x_2,z} \rho(b_2-z)\rho^{(2)}(x_2-z)h_{M_2}(x_2) \\
&\quad \times \sup_{b_3} \sum_{x_3} \rho^{(2)}(x_3-b_3)h_{M_3}(x_3) \\
&= \sup_{b_1} \sum_{x_1} \rho^{(3)}(x_1-b_1)h_{M_1}(x_1) \\
&\quad \times \sup_{b_2} \sum_{x_2} \rho^{(3)}(x_2-b_2)h_{M_2}(x_2) \\
&\quad \times \sup_{b_3} \sum_{x_3} \rho^{(2)}(x_3-b_3)h_{M_3}(x_3).
\end{aligned} \tag{6.23}$$

Again this is a product of 3 separate diagrams $F_1(L)$, $F_2(L)$, $F_3(L)$, each corresponding to $M^{(1)}_{M_j}(0, b_j, x_j, 0)$, two of which have an extra vertex on some $\rho$. The two other terms give rise to the same bounds up to permutation of the indices.

We have already bounded the contribution from diagrams with an extra vertex in Section 5.2.1. By (5.45) we have,

$$\sum_{\vec{x}} \overset{c}{\pi}{}^{3}_{\vec{M}}(\vec{x}) \le \prod_{j=1}^{3} \frac{C\beta^{2-\frac{8\nu}{d}}}{M_j^{\frac{d-6}{2}}}, \tag{6.24}$$

which satisfies (6.4) with $N = 3$ and $q = 0$. Similarly by (5.49) we have,

$$\begin{aligned}
\sum_{\vec{x}} |x_j|^2 \overset{c}{\pi}{}^{3}_{\vec{M}}(\vec{x}) &\le \sum_{i=1}^{3} \frac{C\sigma^2\beta^{2-\frac{8\nu}{d}}}{M_i^{\frac{d-8}{2}}} \prod_{j\neq i} \frac{C\beta^{2-\frac{6\nu}{d}}}{M_j^{\frac{d-6}{2}}} \\
&\le \|M\|_\infty \sigma^2 (C\beta^{2-\frac{8\nu}{d}})^3 \prod_{j=1}^{3} \frac{1}{M_j^{\frac{d-6}{2}}},
\end{aligned} \tag{6.25}$$

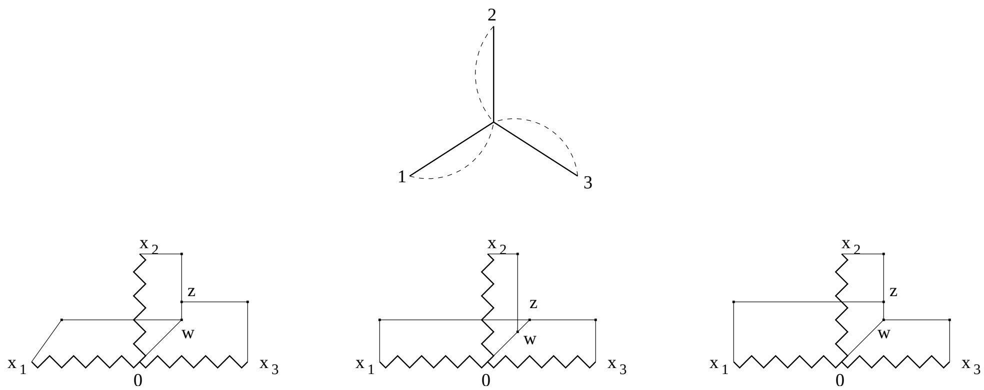

Figure 6.5: The diagrams arising from a lace where all three bonds associated to a branch have the branch point as one of their endpoints.

which satisfies (6.4) with $N = 3$ and $q = 1$. Therefore we have proved Lemma 6.1.2 for $N = 3$.

At this point we know how to bound the diagrams arising from cyclic laces containing only three bonds. A cyclic lace $L$ that contains $N > 3$ bonds has $N - 3$ additional bonds that do not cover the branch point. As such, each of the additional $N - 3$ bonds has both endvertices strictly on some branch $j$. Suppose that the number of additional bonds on branch $j$ is $N_j - 1$, so that $\sum_{j=1}^{3} N_j = N$. We perform the same operation of breaking up the diagram $F(L)$ at the branch point and opening the diagram at each $n_j$ to get three separate diagrams $F_1(L)$, $F_2(L)$, $F_3(L)$, each (except in some degenerate cases that satisfy stronger bounds) corresponding to $M_{M_j}^{(N_j)}(0, b_j, x_j, 0)$ with an extra vertex. This can be proved explicitly by induction on $N_1$, $N_2$ and $N_3$. The degenerate cases are when $n_j$ is the endpoint of more than one bond for some $j$.

By (5.45) we have,

$$\begin{aligned} \sum_{\vec{x}} \overset{c}{\pi}{}_{\vec{M}}^{N}(\vec{x}) \leq & \sum_{\substack{N_1, N_2, N_3 : \\ \sum N_i = N}} \prod_{j=1}^{3} \frac{(C\beta^{2-\frac{8\nu}{d}})^{N_i}}{M_j^{\frac{d-6}{2}}} \\ & \leq N^3 (C\beta^{2-\frac{8\nu}{d}})^N \prod_{j=1}^{3} \frac{1}{M_j^{\frac{d-6}{2}}}, \end{aligned} \tag{6.26}$$

which satisfies (6.4) with $q = 0$. By (5.49)

$$\sum_{\vec{x}} |x_j|^2 \overset{c}{\pi}{}^N_{\vec{M}}(\vec{x}) \leq \sum_{\substack{N_1, N_2, N_3 : \\ \sum N_i = N}} \sum_{i=1}^{3} \frac{\sigma^2 (C\beta^{2-\frac{8\nu}{d}})^{N_i}}{M_i^{\frac{d-8}{2}}} \prod_{j \neq i} \frac{(C\beta^{2-\frac{6\nu}{d}})^{N_j}}{M_j^{\frac{d-6}{2}}}$$
$$\leq N^3 \|M\|_\infty \sigma^2 (C\beta^{2-\frac{8\nu}{d}})^N \prod_{j=1}^{3} \frac{1}{M_j^{\frac{d-6}{2}}}, \tag{6.27}$$

which satisfies (6.4) with $q = 1$. This completes the proof of Lemma 6.1.2. □

## 6.4 Proof of Lemma 6.1.3

In this chapter we prove Lemma 6.1.3. We prove the Lemma by considering separately the contribution to $\overset{a}{\pi}{}^N_{\vec{M}}(\vec{u})$ from laces with two bonds covering the branch point and from laces with three bonds covering the branch point. We write $\overset{a*}{\pi}{}^N_{\vec{M}}(\vec{u})$ for the contribution to $\overset{a}{\pi}{}^N_{\vec{M}}(\vec{u})$ with two bonds covering the branch point and $\overset{a+}{\pi}{}^N_{\vec{M}}(\vec{u})$ for the contribution to $\overset{a}{\pi}{}^N_{\vec{M}}(\vec{u})$ with two bonds covering the branch point.

### 6.4.1 Acyclic laces with 2 bonds covering the branch point

In this subsection we prove the following Lemma

**Lemma 6.4.1.** *For $q \in \{0, 1\}$,*

$$\sum_{\vec{u} \in \mathbb{Z}^{3d}} |u_j|^{2q} \overset{a*}{\pi}{}^N_{\vec{M}}(\vec{u}) \leq N^3 (N^2 \sigma^2 \|\vec{M}\|_\infty)^q \left(C\beta^{2-\frac{8\nu}{d}}\right)^N$$
$$\times \sum_{i=1}^{3} \frac{1}{[M_i]^{\frac{d-6}{2}}} \sum_{j \neq i} \sum_{m_j \leq M_j} \frac{1}{[M_j - m_j]^{\frac{d-6}{2}}} \frac{1}{[M_k + m_j]^{\frac{d-4}{2}}}, \tag{6.28}$$

*where $k \neq i, j$.*

*Proof.* As in the case of the cyclic laces, our strategy is to decompose the resulting diagrams into subdiagrams (3 in general) that we have already bounded in Chapter 5.

Consider the acyclic lace $L \in \mathcal{L}_a^2$ containing only two bonds. An acyclic lace contains a *special branch* with the property that there is only one bond covering the branch point with an endpoint on that branch. Without loss of generality we suppose that the (although there may be more than one) special branch determined by the acyclic lace $L$ is branch 3, and that the bond $e_3$ associated to branch 3 has

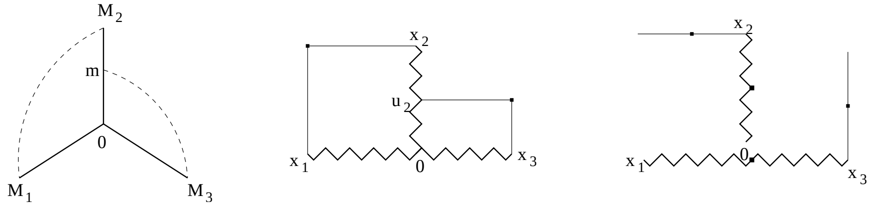


Figure 6.6: An acyclic lace containing only two bonds, its associated diagram and the decomposition into subdiagrams.

its other endpoint on branch 2. We let $m$ denote the endpoint of $e_3$ on branch 2 so that $0 \leq m \leq M_2$. In addition we suppose that $0 < m < M_2$, so that the lace appears as in Figure 6.6. It is easy to show that the contribution to $\sum_{\vec{x}} \pi^{(2)}_{\vec{M}}(\vec{x})$ from this lace is bounded by

$$\begin{aligned}
&\sum_{\vec{x}} \sum_{u_2} h_{M_1}(x_1) h_m(u_2) h_{M_2-m}(x_2-u_2) h_{M_3}(x_3) \rho^{(2)}(x_2-x_1) \rho^{(2)}(u_2-x_3) \\
&\leq \sup_{b_2,b_3} \sum_{\vec{x}} \sum_{u_2} h_m(u_2) h_{M_2-m}(x_2-u_2) h_{M_1}(x_1) h_{M_3}(x_3) \\
&\quad \times \rho^{(2)}(b_2-x_2) \rho^{(2)}(b_3-x_3) \qquad (6.29)\\
&= \sup_{b_2} \sum_{x_2,u_2} h_m(u_2) h_{M_2-m}(x_2-u_2) \rho^{(2)}(b_2-x_2) \\
&\quad \times \sup_{b_3} \sum_{x_1,x_3} h_{M_1}(x_1) h_{M_3}(x_3) \rho^{(2)}(b_3-x_3).
\end{aligned}$$

Using translation invariance on the second term, this is a product of two subdiagrams, $M^{(1)}_{M_2}(0, b_j, x_j, 0)$ with an extra vertex (that we bounded in Section 5.2.1) and $M^{(1)}_{M_1+M_3}(0, b_j, x_j, 0)$ with $h_{M_1+M_3}(x)$ replaced with $(h_{M_1} * h_{M_3})(x)$ (which we bounded in Section 6.2). Using (5.45) and (6.17) and summing over the permutations of branch labels we have

$$\sum_{\vec{u} \in \mathbb{Z}^{3d}} \overset{a*2}{\pi}_{\vec{M}}(\vec{u}) \leq \sum_{i=1}^{3} \sum_{j \neq i} \frac{C\beta^{2-\frac{6\nu}{d}}}{[M_j]^{\frac{d-6}{2}}} \frac{C\beta^{2-\frac{4\nu}{d}}}{[M_i+M_k]^{\frac{d-4}{2}}}, \qquad (6.30)$$

where $k \neq i, j$. This obeys the bound (6.28) with $N = 2$, $q = 0$. Similarly using (5.49) and (6.18) we have

$$\sum_{\vec{u} \in \mathbb{Z}^{3d}} \overset{a*2}{\pi}_{\vec{M}}(\vec{u}) \leq \sigma^2 \|\vec{M}\|_\infty \sum_{i=1}^{3} \sum_{j \neq i} \frac{C\beta^{2-\frac{8\nu}{d}}}{[M_j]^{\frac{d-6}{2}}} \frac{C\beta^{2-\frac{8\nu}{d}}}{[M_i+M_k]^{\frac{d-4}{2}}}, \qquad (6.31)$$

which obeys the bound (6.28) with $N = 2$, $q = 1$.

For general acyclic laces $L \in \mathcal{L}_a^N$ for which only two bonds cover the branch point, we again suppose that the special branch is branch 3, and that the bond $e_3$ associated to branch 3 has its other endpoint on branch 2. As before we let $m$ denote the endpoint of $e_3$ on branch 2 so that $0 \leq m \leq M_2$. Let $e$ denote the other bond covering the branchpoint. We suppose that $L$ has $N_1$ bonds (other than the ones covering the branch point) strictly on branch 1 and $N_j - 1$ bonds strictly on branch $j$, for $j = 2, 3$ respectively. Thus $2 + N_1 + N_2 - 1 + N_3 - 1 = N$. We also let $m_1$ denote the first vertex (from the branch point) strictly on branch 1 that is an endvertex of some bond in $L$.

The reader should refer to Figures 6.8 to 6.11 when digesting the bounds that follow. We bound the Feynmann diagram $F(L)$ for $L$ by doing the following:

1. We define $F_1$ to be the part of the diagram consisting of the backbone corresponding to the interval from $m_1$ to $M_1$ of branch 1, together with any $\rho$ obtained from a bond with both endvertices on branch 1 (such a bond must have both endvertices strictly on branch 1, otherwise it would be a third bond covering the branch point). In the degenerate case that $m_1$ is also an endpoint of $e$, the $\rho$ incident to the backbone at $m_1$ is also considered part of $F_1$. We open up $F_1$ at $m_1$. Note that if $m_1 = M_1$ then $F_1$ is defined to be empty (compare with the $N = 2$ case). Note further that (except in the degenerate case already discussed) the convolution of two $\rho$'s obtained from bond $e$ is not considered part of $F_1$, and thus $F_1$ contains either an extra vertex on the backbone (if the endpoint of $e$ on branch 1 is not the endpoint of any other bond) or on a $\rho$ (if the endpoint of $e$ on branch 1 is the endpoint of some other bond).

2. We define $F_3$ to be the backbone corresponding to branch 3, together with the backbone corresponding to the interval 0 to $m_1$ on branch 1 and with any $\rho$ derived from a bond with an endpoint strictly on branch 3. In particular we take the $\rho * \rho$ obtained from bond $e_3$ as part of $F_3$. We open up the diagram $F_3$ at $m$ (leaving a extra vertex on $F_2$).

3. We define $F_2$ to be the backbone of branch 2 along with any $\rho$ corresponding to a bond with an endpoint strictly on branch 2 (except for the bond $e_3$). We open up the diagram $F_2$ where it meets $F_1$ so that the $\rho * \rho$ corresponding to $e$ is part of $F_2$.

Note that the only properties of the acyclic lace $L$ (that has 2 bonds covering the branchpoint) that are important when constructing of $F_1, F_2$ and $F_3$ are the bonds that cover the branchpoint, and more specifically, whether or not the endpoints of

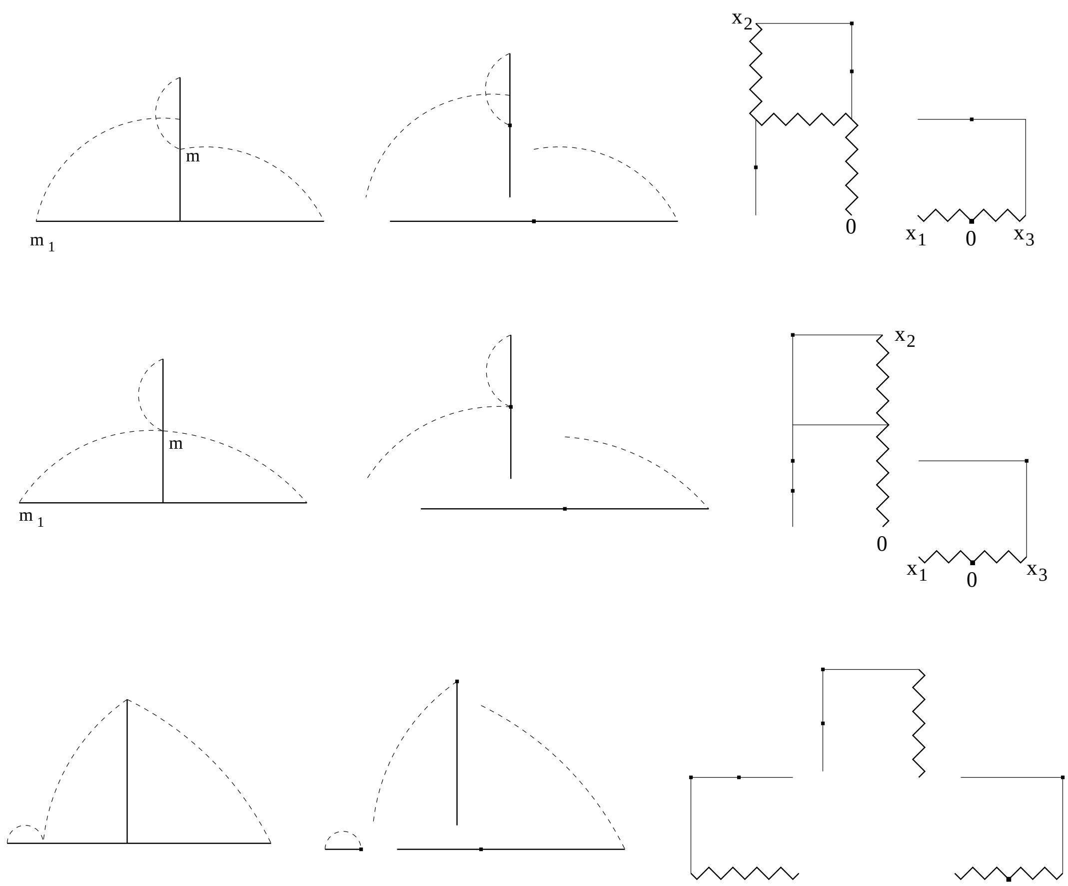


Figure 6.7: Examples of acyclic laces with only 2 bonds covering the branch point, and their decomposition into opened subdiagrams.

those bonds on branches 2 and 3 are also endpoints of some other bonds. To help the readers understanding of this construction we give 3 figures giving examples of the different possibilities which may arise depending on $e$ and $e_3$.

As in Figures 6.7 and 6.8, this leaves us with 2 (if $m_1 = M_1$) or 3 subdiagrams (in general $F_1(L)$ and $F_2(L)$ contain an extra vertex), that we have already bounded. By (5.45) and (6.17) and summing over the possible locations of $m_1$ and permutations of branch labels,

$$\begin{aligned}\sum_{\vec{u}\in\mathbb{Z}^{3d}} \overset{a*N}{\pi}_{\vec{M}}(\vec{u}) \leq & \sum_{N_1,N_2,N_3}\sum_{i=1}^{3}\sum_{j\neq i}\frac{\left(C\beta^{2-\frac{8\nu}{d}}\right)^{N_i}}{[M_i]^{\frac{d-6}{2}}} \\ & \times \sum_{m_j\leq M_j}\frac{\left(C\beta^{2-\frac{8\nu}{d}}\right)^{N_j}}{[M_j-m_j]^{\frac{d-6}{2}}}\frac{\left(C\beta^{2-\frac{8\nu}{d}}\right)^{N_k}}{[M_k+m_j]^{\frac{d-4}{2}}} \\ & \leq N^3\left(C\beta^{2-\frac{8\nu}{d}}\right)^{N}\sum_{i=1}^{3}\sum_{j\neq i}\frac{1}{[M_i]^{\frac{d-6}{2}}}\sum_{m_j\leq M_j}\frac{1}{[M_j-m_j]^{\frac{d-6}{2}}}\frac{1}{[M_k+m_j]^{\frac{d-4}{2}}},\end{aligned} \tag{6.32}$$

which satisfies (6.28) with $q=0$. Similarly by (5.49) and (6.18)

$$\begin{aligned}\sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^2 \overset{a*N}{\pi}_{\vec{M}}(\vec{u}) \leq & N^2\sigma^2\|\vec{M}\|_\infty N^3\left(C\beta^{2-\frac{8\nu}{d}}\right)^{N} \\ & \times\sum_{i=1}^{3}\frac{1}{[M_i]^{\frac{d-6}{2}}}\sum_{j\neq i}\sum_{m_j\leq M_j}\frac{1}{[M_j-m_j]^{\frac{d-6}{2}}}\frac{1}{[M_k+m_j]^{\frac{d-4}{2}}},\end{aligned} \tag{6.33}$$

which satisfies (6.28) with $q=1$, and completes the proof of Lemma 6.4.1. □

### 6.4.2 Acyclic laces with 3 bonds covering the branchpoint

In this section we bound the contribution to $\overset{a}{\pi}$ from (acyclic) laces that have 3 bonds covering the branchpoint. The idea is similar to that of acyclic laces with 2 bonds covering the branchpoint, but the contribution from non-minimal acyclic laces requires careful treatment. In particular for non-minimal laces we need the following two definitions and lemmas. We refer to these lemmas as 4-star lemmas, and they are proved later in this section.

Define

$$\begin{aligned}g_{M_1,M_2}(x_1,x_2,w) \equiv & \sum_{m_1\leq M_1}\sum_{m_2\leq M_2}\sum_{u_1,u_2} h_{m_1}(u_1)h_{M_1-m_1}(x_1-u_1) \\ & \times\rho^{(2)}(u_2-u_1)h_{m_2}(u_2-w)h_{M_2-m_2}(x_2+w-u_2).\end{aligned} \tag{6.34}$$

This can be seen diagrammatically in Figure 6.9.

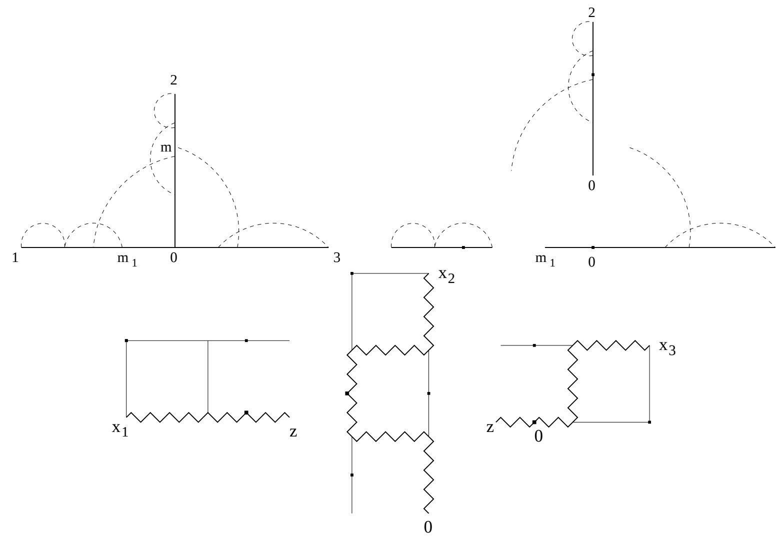


Figure 6.8: An acyclic lace with only 2 bonds covering the branch point, and its decomposition into opened subdiagrams for which we have existing bounds. The branches are labelled 1 to 3 from left to right.

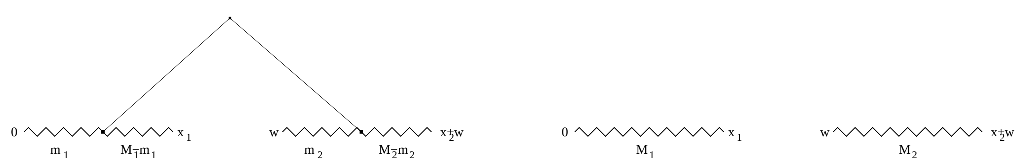


Figure 6.9: On the left is a so called 4-star diagram of (6.34) for the case $l = 2$ which is shown in Lemma 6.4.2 to obey the $\|.\|_1$ and $\|.\|_\infty$ bounds of the diagram on the right times a factor $C\beta^{2-\frac{4\nu}{d}}$.

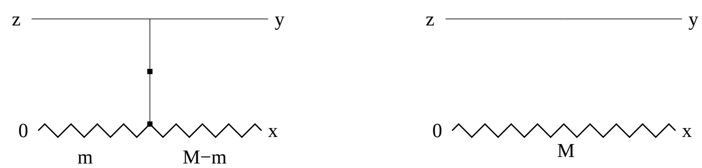


Figure 6.10: On the left is another called 4-star diagram which is shown in Lemma 6.4.3 to obey the $\|.\|_1$ and $\|.\|_\infty$ bounds of the diagram on the right times a factor $C\beta^{2-\frac{6\nu}{d}}$.

**Lemma 6.4.2.** *The 4-star diagram $g_{M_1,M_2}(x_1,x_2,w)$ satisfies the following bounds*

$$\begin{aligned}
\sup_{x_1}\sup_{x_2} g_{M_1,M_2}(x_1,x_2,w) &\le C\beta^{2-\frac{6\nu}{d}}\frac{C\beta^2}{M_1^{\frac{d}{2}}}\frac{C\beta^2}{M_2^{\frac{d}{2}}},\\
\sum_{x_2}\sup_{x_1} g_{M_1,M_2}(x_1,x_2,w) &\le C\beta^{2-\frac{6\nu}{d}}\frac{c\beta^2}{M_1^{\frac{d}{2}}},\\
\sup_{x_1}\sum_{x_2} g_{M_1,M_2}(x_1,x_2,w) &\le C\beta^{2-\frac{6\nu}{d}}\frac{c\beta^2}{M_1^{\frac{d}{2}}},\\
\sum_{x_1}\sum_{x_2} g_{M_1,M_2}(x_1,x_2,w) &\le C\beta^{2-\frac{6\nu}{d}}.
\end{aligned} \tag{6.35}$$

For the second 4-star Lemma we let $b(x) = I_{\{x=0\}} + \frac{I_{\{x\neq 0\}}}{L^{2-\nu}|x|^{d-2}}$. Clearly $|x| \le |y|$ implies that $b(x) \ge b(y)$. Note from (1.13) that $\rho(x) \le Cb(x)$, and from Remark 5.4.4 that the only information about $\rho$ that we used in bounding the diagrams was that $\rho(x) \le Cb(x)$.

**Lemma 6.4.3.**

$$\begin{aligned}
\sup_x \sum_{m\le M}\sum_{u,v} h_m(u)h_{M-m}(x-u)\rho^{(2)}(v-u)\rho(v-y)\rho(z-v) &\le \frac{C\beta^2}{M^{\frac{d}{2}}} b(z-y)\beta^{2-\frac{6\nu}{d}},\\
\sum_x \sum_{m\le M}\sum_{u,v} h_m(u)h_{M-m}(x-u)\rho^{(2)}(v-u)\rho(v-y)\rho(z-v) &\le Cb(z-y)\beta^{2-\frac{6\nu}{d}}.
\end{aligned} \tag{6.36}$$

We prove the following Lemma, assuming Lemmas 6.4.2 and 6.4.3.

**Lemma 6.4.4.** *For $q \in \{0,1\}$,*

$$\begin{aligned}
\sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^{2q} a^{+N}_{\pi\,\vec{M}}(\vec{u}) \le & N^3 (N^2\sigma^2 \|\vec{M}\|_\infty)^q \left(C\beta^{2-\frac{8\nu}{d}}\right)^N \\
& \times \sum_{i=1}^{3} \frac{1}{[M_i]^{\frac{d-6}{2}}} \sum_{j\neq i}\sum_{m_j\le M_j} \frac{1}{[M_j-m_j]^{\frac{d-6}{2}}}\frac{1}{[M_k+m_j]^{\frac{d-4}{2}}},
\end{aligned} \tag{6.37}$$

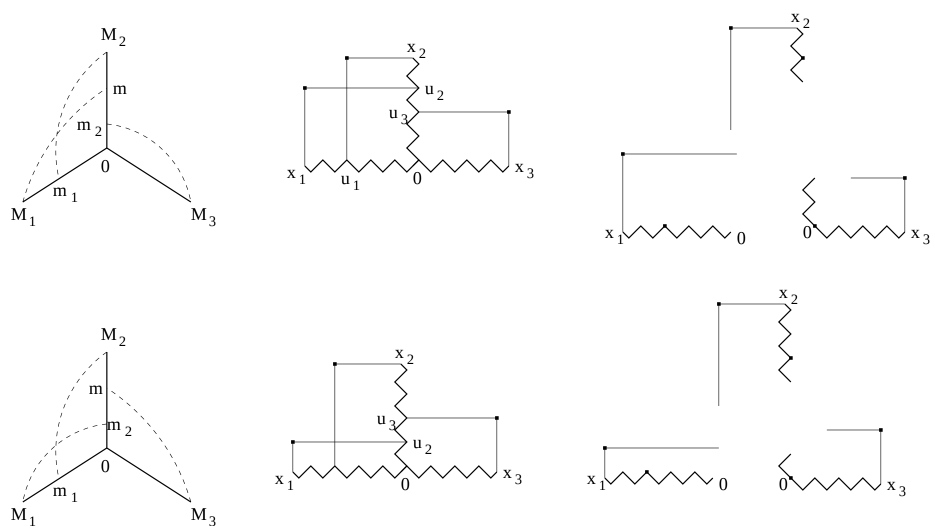


Figure 6.11: Basic acyclic laces with only 3 bonds covering the branch point, and their decomposition into opened subdiagrams.

*where* $k \neq i, j$.

*Proof.* We begin with an acyclic lace $L \in \mathcal{L}_a^3$ containing only 3 bonds, all of which cover the branch point. We suppose that the special branch is branch 3 and that the bond $e_3$ associated to branch 3 has its other endvertex on branch 2. In general (see for example Figure 6.11) this means that each of the bonds associated to branch 1 and 3 have an endvertex on branch 2. We let $m_2 \leq M_2$ denote the first vertex (from the branch point) strictly on branch 2 that is the endvertex of some bond $f$ in $L$. We will also assume that no endvertices of the 3 bonds coincide. When some such endvertices do coincide we must use a decomposition similar to what follows with adjustments as we did for the cyclic laces.

Consider the first lace of Figure 6.11. Let $m_2 \leq M_2$ denote the first vertex (from the root) strictly on branch 2 that is the endvertex of some bond $f$ in $L$. Here $m_2$ is an endvertex of the bond associated to branch 3 and the endvertex of the bond associated to branch 1 is therefore at some $m$ with $m_2 < m < M_2$. It is easy to show that the contribution to $\sum_{\vec{x}} \pi^{(3)}_{\vec{M}}(\vec{x})$ from this lace is bounded by

$$\begin{aligned}
&\sum_{\vec{x}} \sum_{\vec{u}} h_{m_1}(u_1) h_{M_1-m_1}(x_1-u_1) h_{m_2}(u_3) h_{m-m_2}(u_2-u_3) h_{M_2-m}(x_2-u_2) h_{M_3}(x_3) \\
&\quad \times \rho^{(2)}(x_1-u_2)\rho^{(2)}(x_2-u_1)\rho^{(2)}(x_3-u_3) \\
&\le \sup_{b_1} \sum_{u_1,x_1} h_{m_1}(u_1) h_{M_1-m_1}(x_1-u_1)\rho^{(2)}(x_1-b_1) \\
&\quad \times \sup_{u,b_2} \sum_{u_2,x_2} h_{m-m_2}(u_2-u) h_{M_2-m}(x_2-u_2)\rho^{(2)}(x_2-b_2) \\
&\quad \times \sup_{b_3} \sum_{u_3,x_3} h_{m_2}(u_3) h_{M_3}(x_3)\rho^{(2)}(x_3-a_3).
\end{aligned} \tag{6.38}$$

This is a product of three diagrams, two of which contain an extra vertex and the other with $h_{m_2+M_3}$ replaced with $h_{m_2} * h_{M_3}$. The other lace of Figure 6.11 gives a similar product. By (6.17) and (6.18), and summing over the permutations of branch labels we have

$$\sum_{\vec{u}\in\mathbb{Z}^{3d}} \overset{a+3}{\pi}_{\vec{M}}(\vec{u}) \le \sum_{i=1}^{3} \frac{C\beta^{2-\frac{6\nu}{d}}}{[M_k]^{\frac{d-6}{2}}} \sum_{j\neq i} \frac{C\beta^{2-\frac{6\nu}{d}}}{[M_j-m_j]^{\frac{d-6}{2}}} \frac{C\beta^{2-\frac{4\nu}{d}}}{[M_i+m_j]^{\frac{d-4}{2}}}, \tag{6.39}$$

where $k\neq i,j$. This obeys the bound (6.37) with $N=3$, $q=0$. Similarly using (5.49) and (6.18) we have

$$\sum_{\vec{u}\in\mathbb{Z}^{3d}} \overset{a*3}{\pi}_{\vec{M}}(\vec{u}) \le \sigma^2 \|\vec{M}\|_\infty \sum_{i=1}^{3}\sum_{j\neq i} \frac{C\beta^{2-\frac{8\nu}{d}}}{[M_k]^{\frac{d-6}{2}}} \frac{C\beta^{2-\frac{8\nu}{d}}}{[M_j-m_j]^{\frac{d-6}{2}}} \frac{C\beta^{2-\frac{8\nu}{d}}}{[M_i+m_j]^{\frac{d-4}{2}}}, \tag{6.40}$$

which obeys the bound (6.37) with $N=3$, $q=1$.

For general $L\in\mathcal{L}_a^N$ with 3 bonds covering the branch point, if $L$ is a minimal lace (see Definition 2.1.7) then we proceed as before with $m_2\le M_2$ denoting the first vertex (from the branch point) strictly on branch 2 that is the endvertex of some bond $f$ in $L$. We leave it as an exercise for the reader that by breaking the diagram at $m_2$ and 0 we obtain a product of three diagrams that we have already bounded, giving a bound on the contribution to $\sum_{\vec{u}\in\mathbb{Z}^{3d}} |u_j|^{2q} \overset{a+N}{\pi}_{\vec{M}}(\vec{u})$ from minimal laces of

$$\begin{aligned}
&N^3(N^2\sigma^2\|\vec{M}\|_\infty)^q \left(C\beta^{2-\frac{8\nu}{d}}\right)^N \\
&\times \sum_{i=1}^{3} \frac{1}{[M_i]^{\frac{d-6}{2}}} \sum_{j\neq i} \sum_{m_j\le M_j} \frac{1}{[M_j-m_j]^{\frac{d-6}{2}}} \frac{1}{[M_k+m_j]^{\frac{d-4}{2}}}.
\end{aligned} \tag{6.41}$$

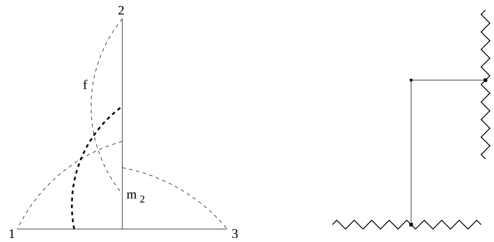


Figure 6.12: An application of Lemma (6.4.2) to remove the bond associated to branch 2.

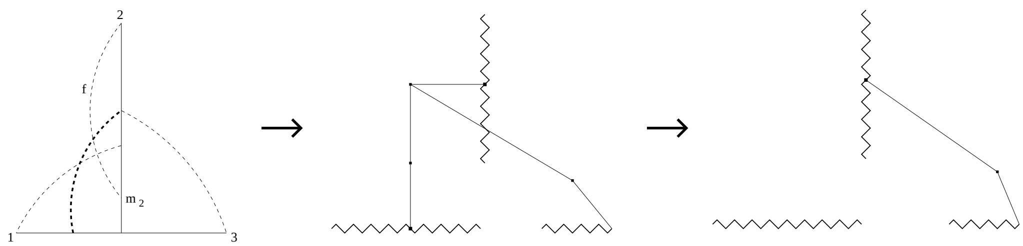


Figure 6.13: An application of Lemma (6.4.3) to remove the bond associated to branch 2.

Therefore we are left to prove a bound of the form (6.37) for the contribution from non-minimal laces. We will argue that such a lace has a bond that we can "remove" in such a way that the resulting diagrams are diagrams arising from an acyclic lace $L' \in \mathcal{L}_a^{N-1}$ (with two bonds covering the branch point) that we have already bounded, together with an extra factor of $\beta^{2-\frac{8\nu}{d}}$.

There are many different cases to consider, depending on which bond ($e_2$ or $e_1$) is removable and how many endvertices of that bond are an endvertex of some other bond in $L$. We will present the argument for the three cases where $e_2$ is removable and leave the others as an exercise. From this point we assume that $e_2$ is a removable bond.

Case (0). Suppose that neither of the endvertices of $e_2$ are the endvertices of any other bond in $L$. Then we use Lemma 6.4.2 to remove the bond $e_2$ and obtain the extra factor $\beta^{2-\frac{8\nu}{d}}$, as in Figure 6.12. This is a non-trivial consequence of Lemma 6.1.4 and so we give further explanation. However this explanation is one of the most notationally difficult parts of this thesis, so we don't give every detail.

Removing the bond $e_2$ from the lace $L$ leaves an acyclic lace $L' = L \backslash e_2 \in \mathcal{L}^{N-1}$

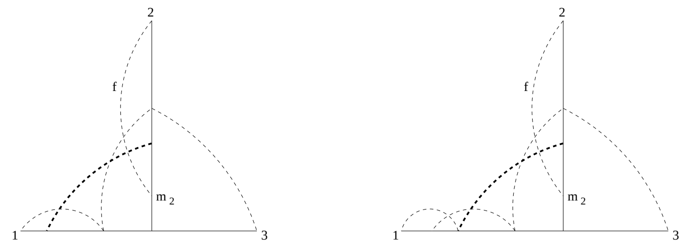


Figure 6.14: Another application of Lemma (6.4.3) to remove the bond associated to branch 1.

with two bonds covering the branch point, which we analysed in Section 6.4.1. Recall that we bounded the contribution to $\sum_{\vec{u}\in\mathbb{Z}^{3d}}|u_j|^{2q}\pi^N_{\vec{M}}(\vec{u})$ from such laces by breaking up the diagram $F(L')$ for each $L'$ at $m_1$ and 0 into three subdiagrams $F_1(L')$, $F_2(L')$ and $F_3(L')$. The bounds on those diagrams relied only on the bounds $\rho(x)\leq b(x)$, $\|h_m\|_\infty\leq\frac{C\beta^2}{m^{\frac{d}{2}}}$ and $\|h_m\|_1\leq C$ when $q=0$, and in addition the bounds $\sup_x|x|^2h_m(x)\leq\frac{\sigma^2mC\beta^2}{m^{\frac{d}{2}}}$ and $\sum_x|x|^2h_m(x)\leq Cm\sigma^2$ when $q=1$. Let $N_i$ denote the number of bonds that contribute to diagram $F_i$ in this decomposition of $F(L')$.

Let $m_1$ be defined as in Section 6.4.1 as the first vertex from the root on branch 1 that is the endvertex of some bond in $L'$ that has an endvertex strictly on branch 1. Either the endvertex $m^*$ of $e_2$ on branch 1 is greater than $m_1$ or less than $m_1$ (it is not equal to $m_1$ by definition of $m_1$ and the fact that neither of the endvertices of $e_2$ are the endvertices of any other bond in $L$).

We can write an explicit bound for the contribution to $\sum_{\vec{x}}\pi^N_{\vec{M}}(\vec{x})$ from the lace $L$ in terms of a diagram $F(L)$ consisting of various convolutions of $\rho$'s and $h_m$'s. In particular that diagram contains a term $\rho^{(2)}(u-u')$ obtained from the bond $e_2$. We break up this diagram at $m_1$ and obtain a product of *two* subdiagrams, which we denote by $F_1(L')$ and $F'(L)$ if $m^*<m_1$ and $F_3(L')$ and $F'(L)$ if $m^*>m_1$. We consider only the case $m^*<m_1$, as the proof of the other case is very similar. When $m^*<m_1$ (see Figure 6.15) the diagram $F_1(L')$ is the same diagram we obtain when estimating $F(L')$ and is bounded by $\sup_{a_1,b_1}\sum_{x_1}M^{(N_1),n_1}_{\vec{m}}(a_1,b_1,x_1,0)$, where $\vec{m}\in\mathcal{H}_{M_1-m_1,N_1}$, and $n_1$ denotes the location of the extra vertex.

As in Figure 6.15, $F'(L)$ is the diagram $F_3(L')$ with the first factor $h_k(v-v')$ of the backbone being replaced by a diagram $F_2'(L)(x-w)$. Thus $F'(L)$ is bounded by $\sup_{a_3,b_3}\sum_{x_3}M^{(N_3),*}_{\vec{m}}(a_3,b_3,x_3,0)$, where $M^{(N_3),*}_{\vec{m}}$ is the dia-

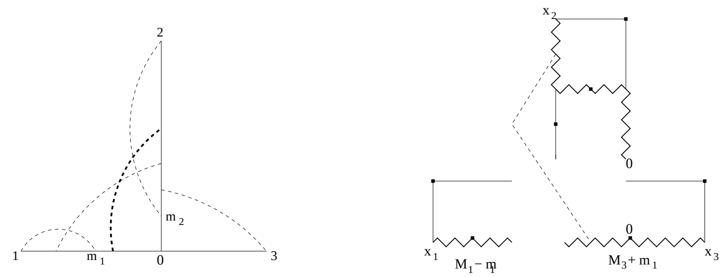


Figure 6.15: A non-minimal lace $L$ and the subdiagrams $F_1(L')$ (the bottom left subdiagram in the second figure) and $F'(L)$.

gram $M^{(N_3)}_{\vec{m}}$ with one specific factor $h_k(v-v')$ being replaced by the diagram $F_2'(L)(v-v')$. Furthermore $F_2'(L)(v-v')$ is itself the diagram $F_2(L')$ with one of the first three factors $h_l(u-u')$ on the backbone being replaced by $g_{k,l}(u-u', v-v', v')$, and with an extra vertex at $n_2$. Therefore $F_2'(L)(v-v')$ is bounded by $\sup_{a_2,b_2}\sum_{x_2} M^{(N_2),n_2,g}_{\vec{m}}(a_2,b_2,x_2,0)(v-v')$, where $M^{(N_2),n_2,g}_{\vec{m}}(\bullet)(v-v')$ is the diagram $M^{(N_2)}_{\vec{m}}(\bullet)$ with one of three factors $h_l(u-u')$ being replaced by $g_{k,l}(u-u',v-v',v')$, and with an extra vertex at $n_2$.

It follows that the contribution to $\sum_{\vec{x}} \overset{a+N}{\pi}_{\vec{M}}(\vec{x})$ from non-minimal acyclic laces such that: the special branch is branch 3, $e_3$ has its other endvertex on branch 2, $e_2$ is removable and has no endvertices in common with any other bond, and $m^* < m_1$ is bounded by

$$C \sum_{\substack{N_1,N_2,N_3:\\ \sum N_i = N-1}} \sum_{m_1\le M_1} \sum_{\vec{m}_1\in\mathcal{H}_{M_1-m_1,N_1}} \sum_{n_1\le M_1-n_1} \sup_{a_1,b_1}\sum_{x_1} M^{(N_1),n_1}_{\vec{m}_1}(a_1,b_1,x_1,0)$$
$$\times \sum_{\vec{m}_3\in\mathcal{H}_{M_3-m_1,N_3}} \sup_{a_3,b_3}\sum_{x_3} M^{(N_3),*}_{\vec{m}_3}(a_3,b_3,x_3,0). \tag{6.42}$$

Here $M^{(N_3),*}_{\vec{m}_3}(a_3,b_3,x_3,0)$ denotes the diagram $M^{(N_3)}_{\vec{m}_3}(a_3,b_3,x_3,0)$ with the first factor $h_m(v-v')$ of the backbone being replaced by

$$\sum_{\vec{m}_2\in\mathcal{H}_{M_2,N_2}} \sum_{n_2\le M_2} \sup_{a_2,b_2}\sum_{x_2} M^{(N_2),n_2,g}_{\vec{m}_2}(a_2,b_2,x_2,0)(v-v'), \tag{6.43}$$

and $M^{(N_2),n_2,g}_{\vec{m}_2}(a_2,b_2,x_2,0)(v-v')$ denotes the diagram $M^{(N_2),n_2}_{\vec{m}_2}(a_2,b_2,x_2,0)$ with a specific factor $h_l(u-u')$ being replaced by $g_{k,l}(u-u',v-v',v')$.

We prove that (6.42) is bounded above by

$$C\beta^{2-\frac{6\nu}{d}} \sum_{\substack{N_1, N_2, N_3 : \\ \sum N_i = N-1}} \sum_{m_1 \le M_1} \frac{(C\beta^{2-\frac{6\nu}{d}})^{N_1}}{[M_1 - m_1]^{\frac{d-6}{2}}} \frac{(C\beta^{2-\frac{6\nu}{d}})^{N_2}}{M_2^{\frac{d-6}{2}}} \frac{(C\beta^{2-\frac{6\nu}{d}})^{N_3}}{[M_3 + m_1]^{\frac{d-4}{2}}}, \tag{6.44}$$

which satisfies the bound (6.37) of Lemma 6.4.4) with $q = 0$.

By (5.45) we have

$$\sum_{\vec{m} \in \mathcal{H}_{M_1 - m_1, N_1}} \sum_{n \le M_1 - m_1} \sup_{a_1, b_1, y_1} \sum_{x_1} M_{\vec{m}}^{(N_1), n}(a_1, b_1, x_1, y_1) \le \frac{(C\beta^{2-\frac{8\nu}{d}})^{N_1}}{[M_1 - m_1]^{\frac{d-6}{2}}}. \tag{6.45}$$

By Remarks 5.2.1–5.4.5 and (6.17), the bounds $\sup_{v-v'} h_l(v - v') \le \frac{C\beta^2}{l^{\frac{d}{2}}}$ and $\sum_{v-v'} h_l(v - v') \le K$, together with bounds on the rest of the lines in $F_3(L')$ imply that

$$\sum_{\vec{m} \in \mathcal{H}_{M_3 + m_1, N_3}} \sup_{a_3, b_3} \sum_{x_3} M_{\vec{m}}^{(N_3)}(a_3, b_3, x_3, 0) \le \frac{(C\beta^{2-\frac{8\nu}{d}})^{N_3}}{[M_3 + m_1]^{\frac{d-4}{2}}}. \tag{6.46}$$

Therefore by Lemma 6.1.4, to prove that (6.42) is bounded by (6.44) it is enough to show:

$$\begin{aligned} &\sup_{v-v'} \sum_{\vec{m} \in \mathcal{H}_{M_2, N_2}} \sum_{n_2 \le M_2} \sup_{a_2, b_2} \sum_{x_2} M_{\vec{m}}^{(N_2), n_2, g}(a_2, b_2, x_2, 0)(v - v') \\ &\le \frac{C\beta^2}{k^2} C\beta^{2-\frac{6\nu}{d}} \frac{(C\beta^{2-\frac{8\nu}{d}})^{N_2}}{M_2^{\frac{d-6}{2}}}, \end{aligned} \tag{6.47}$$

and

$$\begin{aligned} &\sum_{v-v'} \sum_{\vec{m} \in \mathcal{H}_{M_2, N_2}} \sum_{n_2 \le M_2} \sup_{a_2, b_2} \sum_{x_3} M_{\vec{m}}^{(N_2), g, n}(a_3, b_3, x_3, 0)(v - v') \\ &\le C\beta^{2-\frac{6\nu}{d}} \frac{(C\beta^{2-\frac{8\nu}{d}})^{N_2}}{M_2^{\frac{d-6}{2}}}. \end{aligned} \tag{6.48}$$

Again by Lemma 6.1.4, to show (6.47) it is sufficient to prove

$$\sup_{v-v'} \sup_{u-u'} g_{k,l}(u - u', v - v', w) \le \frac{C\beta^2}{k^2} C\beta^{2-\frac{6\nu}{d}} \frac{C\beta^2}{l^2}, \tag{6.49}$$

and

$$\sup_{v-v'} \sum_{u-u'} g_{k,l}(u - u', v - v', w) \le \frac{C\beta^2}{k^2} C\beta^{2-\frac{6\nu}{d}}. \tag{6.50}$$

Similarly, to show (6.48) it is sufficient to prove

$$\sum_{v-v'} \sup_{u-u'} g_{k,l}(u-u', v-v', w) \le C\beta^{2-\frac{6\nu}{d}} \frac{C\beta^2}{l^2}, \tag{6.51}$$

and

$$\sum_{v-v'} \sum_{u-u'} g_{k,l}(u-u', v-v', w) \le C\beta^{2-\frac{6\nu}{d}}. \tag{6.52}$$

But (6.49)–(6.52) are exactly the statements of Lemma 6.4.2.

This proves that (6.42) is bounded above by (6.44) as required.

Case (1). If exactly one endvertex of $e_2$ is also the endvertex of some other bond (by definition of a lace, in the case we are considering here the endvertex of $e_2$ strictly on branch 2 could only be the endvertex of $e_3$) then we proceed as in case (1) except that we use Lemma 6.4.3 instead of Lemma 6.4.2 to remove the bond $e_2$ and obtain the extra factor $\beta^{2-\frac{6\nu}{d}}$, (see Figure 6.13).

Case (2). Finally suppose both endvertices of $e_2$ are also the endvertices of other bonds in $L$. Then (exercise left for the reader) $e_1$ is a bond with the properties that at least one of the endvertices of $e_1$ is not the endvertex of any other bond in $L$, and $L \setminus e_1$ is a lace. Then, depending on whether or not one endvertex of $e_1$ is the endvertex of another bond in $L$, we use Lemma 6.4.2 or 6.4.3 to remove $e_1$ and obtain the extra factor $\beta^{2-\frac{6\nu}{d}}$.

We have now proved that the contribution (up to permutation of branch labels) to $\sum_{\vec{x}} \pi^{(N)}_{\vec{M}}(\vec{x})$ from non-minimal acyclic laces with 3 bonds covering the branch point is at most

$$N^3 (C\beta^{2-\frac{8\nu}{d}})^N \frac{1}{M_1^{\frac{d-6}{2}}} \sum_{m_2 \le M_2} \frac{1}{[M_2 - m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3 + m_2]^{\frac{d-4}{2}}}. \tag{6.53}$$

For the $q = 1$ case of Lemma 6.4.4 we use $|x_j|^2 \le 2N' \sum_1^{2N'-1} |u_{j,l}|^2$ (this gives the $N^2$ factor) where the $u_{j,l}$ denote the displacements along the backbone of diagram $F_j(L')$. If the extra factor $|u_{j,l}|^2$ occurs on a part of the diagram $F'(L)$ where $F_1(L')$ and $F_2(L')$ are joined by $g_{k,l}$ then we can use Lemma 6.1.4 to include a factor $|u|^2$ on one of the lines in the 4-star lemmas, and proceed to get the extra factor $\sigma^2 \|M\|_\infty$. Otherwise the extra factor $\sigma^2 \|M\|_\infty$ comes by applying Lemma 6.1.4 to the diagram $F_i(L')$ where the $|u_{j,l}|^2$ is attached.

This proves that the contribution (up to permutation of branch labels) to $\sum_{\vec{x}} |x_j|^2 \pi^{(N)}_{\vec{M}}(\vec{x})$ from non-minimal acyclic laces with 3 bonds covering the branch point is at most

$$N^3(N^2\sigma^2\|\vec{M}\|_\infty)(C\beta^{2-\frac{8\nu}{d}})^N \frac{1}{M_1^{\frac{d-6}{2}}} \sum_{m_2\le M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}}. \qquad (6.54)$$

This completes the proof of Lemma 6.4.4. □

**Proof of Lemma 6.4.2.**

For the first bound, note that either $m_i \le \frac{M_i}{2}$ or $M_i - m_i \le \frac{M_i}{2}$. Breaking up the sums over $m_1$ and $m_2$ according to these restrictions gives rise to 4 terms. One such term is

$$\begin{aligned} &\sum_{m_1\le\frac{M_1}{2}} \sum_{m_2\le\frac{M_1}{2}} \sum_{u_1,u_2} h_{m_1}(u_1)h_{M_1-m_1}(x_1-u_1)\rho^{(2)}(u_2-u_1) \\ &\quad \times h_{m_2}(u_2-w)h_{M_2-m_2}(x_2+w-u_2) \\ &\le \sum_{m_1\le\frac{M_1}{2}} \sum_{m_2\le\frac{M_1}{2}} \sup_{u_1} h_{M_1-m_1}(x_1-u_1) \\ &\quad \times \sup_{u_2} h_{M_2-m_2}(x_2+w-u_2)\left(h_{m_1} * \rho^{(2)} * h_{m_2}\right)(w) \\ &\le \frac{C\beta^2}{M_1^{\frac{d}{2}}}\frac{C\beta^2}{M_2^{\frac{d}{2}}} \sum_{m_1\le\frac{M_1}{2}} \sum_{m_2\le\frac{M_1}{2}} \frac{C\beta^{2-\frac{4\nu}{d}}}{(m_1+m_2)^{\frac{d-4}{2}}} \le C\beta^{2-\frac{4\nu}{d}}\frac{C\beta^2}{M_1^{\frac{d}{2}}}\frac{C\beta^2}{M_2^{\frac{d}{2}}}, \end{aligned} \qquad (6.55)$$

where we have used 5.5 with $l=1$ and $k=0$ with the fact that $M_i - m_i \ge \frac{M_i}{2}$ on the $h_{M_i-m_i}$'s and with $l=2$ and $k=2$ on the convolution of $h$'s and $\rho$'s. By similar arguments we get the result for the other 3 terms which proves the first bound.

For the second bound we again split the sums over $m_1$ and $m_2$ to leave us with 4 terms. The most difficult to bound is

$$\begin{aligned}
&\sum_{x_2}\sup_{x_1}\sum_{m_1>\frac{M_1}{2}}\sum_{m_2\le\frac{M_1}{2}}\sum_{u_1,u_2} h_{m_1}(u_1)h_{M_1-m_1}(x_1-u_1)\rho^{(2)}(u_2-u_1)\\
&\quad\times h_{m_2}(u_2-w)h_{M_2-m_2}(x_2+w-u_2)\\
\le&\frac{C\beta^2}{M_1^{\frac{d}{2}}}\sum_{x_2}\sup_{x_1}\sum_{m_1>\frac{M_1}{2}}\sum_{m_2\le\frac{M_1}{2}}\sum_{u_1,u_2} h_{M_1-m_1}(x_1-u_1)\rho^{(2)}(u_2-u_1)\\
&\quad\times h_{m_2}(u_2-w)h_{M_2-m_2}(x_2+w-u_2)\\
\le&\frac{C\beta^2}{M_1^{\frac{d}{2}}}\sup_{u}\sum_{x_2} h_{M_2-m_2}(x_2+w-u)\\
&\quad\times\sup_{x_1}\sum_{m_2\le\frac{M_1}{2}}\sum_{u_1,u_2}\left(\sum_{m_1>\frac{M_1}{2}} h_{M_1-m_1}(x_1-u_1)\right)\rho^{(2)}(u_2-u_1)h_{m_2}(u_2-w) \qquad (6.56)\\
\le&\frac{C\beta^2}{M_1^{\frac{d}{2}}}\sup_{u}\sum_{x_2} h_{M_2-m_2}(x_2+w-u)\\
&\quad\times\sup_{x_1}\sum_{m_2\le\frac{M_1}{2}}\sum_{u_1,u_2}\rho(x_1-u_1)\rho^{(2)}(u_2-u_1)h_{m_2}(u_2-w)\\
\le&\frac{C\beta^2}{M_1^{\frac{d}{2}}}\sup_{x_1}\sum_{m_2\le\frac{M_1}{2}}\sum_{u_2}\rho^{(3)}(u_2-x_1)h_{m_2}(u_2-w)\\
\le&\frac{C\beta^2}{M_1^{\frac{d}{2}}}\sup_{x_1}\sum_{m_2\le\frac{M_1}{2}}\frac{C\beta^{2-\frac{6\nu}{d}}}{m_2^{\frac{d-6}{2}}}\le\frac{C\beta^2}{M_1^{\frac{d}{2}}}KC\beta^{2-\frac{6\nu}{d}},
\end{aligned}$$

where we used Proposition 5.1.4 in the penultimate step.

The third bound follows from the second by symmetry and taking the sup outside the sum.

For the fourth bound we see that

$$
\begin{aligned}
&\sum_{x_1,x_2}\sum_{m_1\le M_1}\sum_{m_2\le M_2}\sum_{u_1,u_2} h_{m_1}(u_1)h_{M_1-m_1}(x_1-u_1)\rho^{(2)}(u_2-u_1)\\
&\times h_{m_2}(u_2-w)h_{M_2-m_2}(x_2+w-u_2)\\
=&C\sum_{m_1\le M_1}\sum_{m_2\le M_2}\sum_{u_1,u_2} h_{m_1}(u_1)\rho^{(2)}(u_2-u_1)h_{m_2}(u_2-w)\\
\le&C\sum_{m_1\le M_1}\sum_{m_2\le M_2}\sum_{u_1,u_2} h_{m_1}(u_1)h_{M_1-m_1}(x_1-u_1)\rho^{(2)}(u_2-u_1)h_{m_2}(u_2-w)\\
\le&C\sum_{m_1\le M_1}\sum_{m_2\le M_2}\frac{C\beta^{2-\frac{4\nu}{d}}}{[m_1+m_2]^{\frac{d-4}{2}}}\\
\le&C\beta^{2-\frac{4\nu}{d}}.
\end{aligned}
\tag{6.57}
$$

□

**Proof of Lemma 6.4.3.**

Firstly since $|z-y|\le|v-y|+|v-z|\le 2(|v-y|\vee|v-z|)$ we have

$$
\begin{aligned}
&\sum_v \rho(v-u)^{(2)}\rho(v-y)\rho(z-v)\\
&\le C\sum_{v:|v-y|\ge\frac{|z-y|}{2}}\rho^{(2)}(v-u)b(v-y)\rho(z-v)+C\sum_{v:|v-z|\ge\frac{|z-y|}{2}}\rho^{(2)}(v-u)\rho(v-y)b(z-v)\\
&\le C\sum_{v:|v-y|\ge\frac{|z-y|}{2}}\rho^{(2)}(v-u)b(z-y)\rho(z-v)+C\sum_{v:|v-z|\ge\frac{|z-y|}{2}}\rho^{(2)}(v-u)\rho(v-y)b(z-y)\\
&\le Cb(z-y)\left(\rho^{(3)}(z-u)+\rho^{(3)}(y-u)\right),
\end{aligned}
\tag{6.58}
$$

Therefore for the first bound,

$$
\begin{aligned}
&\sup_x\sum_{m\le M}\sum_u h_m(u)h_{M-m}(x-u)\sum_v\rho^{(2)}(v-u)\rho(v-y)\rho(z-v)\\
&\le Cb(z-y)\sup_x\sum_{m\le M}\sum_u h_m(u)h_{M-m}(x-u)\left(\rho^{(3)}(z-u)+\rho^{(3)}(y-u)\right)\\
&\le C\frac{\beta^2}{M^{\frac{d}{2}}}b(z-y)\sum_{m\le\frac{M}{2}}\frac{\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}}\le\frac{C\beta^2}{M^{\frac{d}{2}}}b(z-y)\beta^{2-\frac{6\nu}{d}},
\end{aligned}
\tag{6.59}
$$

where we have used Proposition 5.1.4 and the fact that either $m\ge\frac{M}{2}$ or $M-m\ge\frac{M}{2}$.

For the second bound, we again use Proposition 5.1.4 to get,

$$\begin{aligned}
&\sum_x \sum_{m\le M} \sum_u h_m(u) h_{M-m}(x-u) \sum_v \rho^{(2)}(v-u)\rho(v-y)\rho(z-v) \\
&\le Cb(z-y) \sum_x \sum_{m\le M} \sum_u h_m(u) h_{M-m}(x-u) \left(\rho^{(3)}(z-u) + \rho^{(3)}(y-u)\right) \\
&\le Cb(z-y) \sum_{m\le M} \sum_u h_m(u) \left(\rho^{(3)}(z-u) + \rho^{(3)}(y-u)\right) \\
&\le Cb(z-y) \sum_{m\le M} \frac{\beta^{2-\frac{6\nu}{d}}}{m^{\frac{d-6}{2}}} \le Cb(z-y)\beta^{2-\frac{6\nu}{d}}.
\end{aligned} \tag{6.60}$$

□

## 6.5 Proof of Lemma 4.3.3

We now prove Lemma 4.3.3, the companion of Proposition 4.3.2. Recall the definition of $B_N(\vec{M})$ from 6.2.

**Lemma (4.3.3).** *There is a constant $C'$ independent of $L$ such that*

$$\sum_N \sum_{\vec{M}:M_j\ge n_j} N^3 B_N(\vec{M}) \le \frac{C'\beta^{2-\frac{8\nu}{d}}}{[n_j]^{\frac{d-8}{2}}}, \qquad \textit{and} \tag{6.61}$$

$$\sum_N \sum_{\vec{M}\le\vec{n}} N^5 \|\vec{M}\|_\infty B_N(\vec{M}) \le \begin{cases} \|\vec{n}\|_\infty^{\frac{10-d}{2}\vee 0}, & \textit{if } d\neq 10 \\ \log\|n\|_\infty, & \textit{if } d=10. \end{cases} \tag{6.62}$$

*Proof.* Summing over $N$ first gives the factor $C\beta^{2-\frac{8\nu}{d}}$, for small enough $\beta$. Summing over each $M_j$ separately we have,

$$\sum_{\vec{M}:M_j\ge n_j} \frac{1}{M_j^{\frac{d-6}{2}}} \le \frac{C}{[n_j]^{\frac{d-8}{2}}}, \tag{6.63}$$

and

$$\sum_{\vec{M}\le\vec{n}} \|\vec{M}\|_\infty \prod_{j=1}^3 \frac{1}{M_j^{\frac{d-6}{2}}} \le \begin{cases} \|n\|_\infty^{\frac{10-d}{2}\vee 0}, & \text{if } d\neq 10 \\ \log\|n\|_\infty, & \text{if } d=10, \end{cases} \tag{6.64}$$

as required. This verifies the Lemma for the first component of $B_N(\vec{M})$.

For the second component of $B_N(\vec{M})$ (ignoring the sum over $j$) we sum over $M_1$ separately to get

$$\begin{aligned} &\sum_{\vec{M}:M_1\geq n_1} \frac{1}{M_1^{\frac{d-6}{2}}} \sum_{m_2\leq M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ &\leq \frac{1}{[n_1]^{\frac{d-8}{2}}} \sum_{M_2,M_3} \sum_{m_2\leq M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}}. \end{aligned} \tag{6.65}$$

We leave it as an exercise for the reader to show (by summing separately over $m_2 \leq \frac{M_2}{2}$ and $m_2 > \frac{M_2}{2}$) that

$$\sum_{M_2,M_3} \sum_{m_2\leq M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \leq C. \tag{6.66}$$

Furthermore

$$\begin{aligned} &\sum_{\vec{M}:M_2\geq n_2} \frac{1}{M_1^{\frac{d-6}{2}}} \sum_{m_2\leq M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ \leq &C \sum_{M_2\geq n_2} \sum_{M_3} \sum_{m_2\leq M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ \leq &C \sum_{M_2\geq n_2} \sum_{M_3} \sum_{m_2<\frac{M_2}{2}} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ &+ C \sum_{M_2\geq n_2} \sum_{M_3} \sum_{m_2\geq\frac{M_2}{2}} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ \leq &C \sum_{M_2\geq n_2} \frac{1}{[M_2]^{\frac{d-6}{2}}} \sum_{M_3} \sum_{m_2<\frac{M_2}{2}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ &+ C \sum_{m_2\geq\frac{n_2}{2}} \sum_{M_3} \left( \sum_{M_2\geq m_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \right) \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ &\leq \frac{C}{n_2^{\frac{d-8}{2}}} + C \sum_{m_2\geq\frac{n_2}{2}} \sum_{M_3} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \leq \frac{C}{n_2^{\frac{d-8}{2}}} + C \sum_{m_2\geq\frac{n_2}{2}} \frac{1}{[m_2]^{\frac{d-6}{2}}} \leq \frac{C}{n_2^{\frac{d-8}{2}}}. \end{aligned} \tag{6.67}$$

Similarly we get

$$\sum_{\vec{M}:M_3\geq n_3} \frac{1}{M_1^{\frac{d-6}{2}}} \sum_{m_2\leq M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \leq \frac{C}{n_2^{\frac{d-8}{2}}}. \tag{6.68}$$

After permuting the labels $1, 2, 3$, this verifies the first claim of the Lemma for the second component of $B_N(\vec{M})$.

For the second claim we need to show that

$$\sum_{\vec{M}\le\vec{n}} \|\vec{M}_\infty\| \frac{1}{M_1^{\frac{d-6}{2}}} \sum_{m_2\le M_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \le \begin{cases} \|\vec{n}\|_\infty^{\frac{10-d}{2}\vee 0}, & \text{if } d\neq 10 \\ \log\|n\|_\infty, & \text{if } d=10. \end{cases} \tag{6.69}$$

If $\|M_\infty\|_\infty = M_1$ this follows easily by summing first over $M_1$ and using (6.66). If $\|M\|_\infty = M_2$, as in (6.67) we get a bound of

$$\begin{aligned} &C\sum_{M_2} \frac{M_2}{[M_2]^{\frac{d-6}{2}}} \sum_{M_3} \sum_{m_2<\frac{M_2}{2}} \frac{1}{[M_3+m_2]^{\frac{d-4}{2}}} \\ &+C\sum_{m_2}\sum_{M_3}\left(\sum_{M_2\ge m_2} \frac{1}{[M_2-m_2]^{\frac{d-6}{2}}}\right) \frac{m_2}{[M_3+m_2]^{\frac{d-4}{2}}}, \end{aligned} \tag{6.70}$$

and the result follows by the same methods that we used for (6.67). Similarly we get the result if $\|M\|_\infty = M_3$. After permuting the labels $1,2,3$, this verifies the second claim of the Lemma for the second component of $B_N(\vec{M})$, and thus proves the Lemma. □

## 6.6 Proof of Lemma 4.2.1

In this section we prove the three bounds of Lemma 4.2.1, and Lemma 4.4.1. Fix a skeleton network $\mathcal{N}(\alpha,\vec{n})$, with $\alpha\in\Sigma_r$ and recall Definition 2.1.1, where $b$ is the branch point neighbouring the root of $\mathcal{N}$. Let $\mathcal{M}\subseteq\mathcal{N}(\alpha,\vec{n})$. If $U_{st}\in\{-1,0\}$ for each $st$, then trivially for any $A\subset\mathbf{E}_\mathcal{M}$,

$$\prod_{st\in\mathbf{E}_\mathcal{M}} [1+U_{st}] \le \prod_{st\in A}[1+U_{st}], \tag{6.71}$$

so that in particular for any finite collection of disjoint sets $G_i\subset\mathbf{E}_\mathcal{M}$,

$$\prod_{st\in\mathbf{E}_\mathcal{M}} [1+U_{st}] \le \prod_i \prod_{st\in G_i}[1+U_{st}]. \tag{6.72}$$

We will use these bounds frequently without explicit reference.

Before we proceed with our analysis of certain error terms appearing in Lemma 4.2.1 we quickly verify a trivial result, Lemma 4.4.1, where $\#_\mathcal{M}$ is the number of branches in $\mathcal{M}$.

**Lemma (4.4.1).** *There exists a constant $K$, independent of $L$, $\mathcal{M}$ and $\vec{\kappa}$ such that for any network $\mathcal{M}$*

$$\widehat{t}_\mathcal{M}(\vec{\kappa}) \le K^{\#_\mathcal{M}}. \tag{6.73}$$

*Proof.* Label the branches of $\mathcal{M}$, $1, \ldots, \#_{\mathcal{M}}$ and write $n_i$ for the length of branch $i$. Then the vertices of $\mathcal{M}$ can be relabelled by $(i, m_i)$ where $i$ is a branch and $0 \leq m_i \leq n_i$ is distance along the branch $i$. Note that branch points of $\mathcal{M}$ will receive multiple labels. Using

$$\prod_{st \in \mathcal{M}} [1 + U_{st}] \leq \prod_{i=1}^{\#_{\mathcal{M}}} \prod_{\substack{st \in \mathcal{M}_i : \\ 0 < s < t < n_i}} [1 + U_{st}], \tag{6.74}$$

we have that

$$\sum_{\vec{y} \in \mathbb{Z}^{d(\#_{\mathcal{M}})}} \hat{t}_{\mathcal{M}}(\vec{y}) \leq \rho(0)^{2\#_{\mathcal{M}}} \prod_{i=1}^{\#_{\mathcal{M}}} \sum_{y_i} h_{n_i}(y_i). \tag{6.75}$$

The result now follows from Proposition 5.1.4 with $l = 1$ and $k = 0$. $\square$

### 6.6.1 Proof of the first bound of Lemma 4.2.1

Recall that $\phi_{\mathcal{N}}^{\mathcal{R}}(\vec{y}) \geq 0$ was defined in (4.18) as

$$\sum_{\omega \in \Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s \in \mathcal{N}} \sum_{R_s \in \mathcal{T}(\omega(s))} W(R_s) \left( \prod_{\mathfrak{b} \in \mathcal{R}^c} [1 + U_{\mathfrak{b}}] \right) \left( 1 - \prod_{\mathfrak{b} \in \mathcal{R}} [1 + U_{\mathfrak{b}}] \right), \tag{6.76}$$

where $U_{st}$ is given by (4.15). In this section we prove that

$$\sum_{\vec{y}} \phi_{\mathcal{N}}^{\mathcal{R}}(\vec{y}) \leq \sum_{e \in E} \frac{C \beta^{2 - \frac{4\nu}{d}}}{n_e^{\frac{d-8}{2}}}, \tag{6.77}$$

Let $\mathcal{N}_e$ denote the branch of $\mathcal{N}$ corresponding to edge $e$ of $\alpha$ and let $\mathcal{R}^{e,e'} = \{st \in \mathcal{R} : s \in \mathcal{N}_e, t \in \mathcal{N}_{e'}\}$. We claim that when $U_{st} \in \{-1, 0\}$ for all $st$,

$$\begin{aligned} 1 - \prod_{st \in \mathcal{R}} [1 + U_{st}] &\leq \sum_{\substack{e, e' \in \mathcal{B}_{\mathcal{N}} : \\ \mathcal{N}_e \cap \mathcal{N}_{e'} = \emptyset}} \left( 1 - \prod_{st \in \mathcal{R}^{e,e'}} [1 + U_{st}] \right) \\ &\leq \sum_{\substack{e, e' \in \mathcal{B}_{\mathcal{N}} : \\ \mathcal{N}_e \cap \mathcal{N}_{e'} = \emptyset}} \sum_{\substack{m_e \leq n_e \\ m_{e'} \leq n_{e'}}} -U_{m_e, m'_e}^{e,e'}, \end{aligned} \tag{6.78}$$

where the sum over $e, e'$ is a sum over pairs of edges of $\alpha$ that do not have an endvertex in common (which can be expressed as $\mathcal{N}_e \cap \mathcal{N}_{e'} = \emptyset$). To verify (6.78), observe that each of the quantities

$$1 - \prod_{st \in \mathcal{R}} [1 + U_{st}], \qquad 1 - \prod_{st \in \mathcal{R}^{e,e'}} [1 + U_{st}], \qquad -U_{(e, m_e), (e', m'_e)}, \tag{6.79}$$

are either zero or one. Suppose the left hand side of (6.78) is non-zero. Then there exists some $st \in \mathcal{R}$ with $U_{st} = -1$. By definition of $\mathcal{R}$, $st$ covers two branch points of $\mathcal{N}$ so that $st \in \mathcal{R}^{e,e'}$ for some $e, e'$ that do not have a common endvertex. For this $e$ and $e'$, we have $1 - \prod_{st \in \mathcal{R}^{e,e'}} [1 + U_{st}] = 1$ and the first inequality is verified. Now for fixed $e, e'$, if $1 - \prod_{st \in \mathcal{R}^{e,e'}} [1 + U_{st}]$ is non-zero then there exists $st \in \mathcal{R}^{e,e'}$ with $U_{st} = -1$. But $s = (e, m_e)$, $t = (e', m_{e'})$ for some $m_e \leq n_e$, $m_{e'} \leq n_{e'}$ so that for this $m_e$ and $m_{e'}$, $-U_{(e,m_e),(e',m_e')} = 1$. This proves the second inequality.

Examining the second quantity in (2.4) when $U_{st} \in \{-1, 0\}$ for all $st$ we have,

$$\begin{aligned}
0 \leq & \prod_{st \in \mathbf{E}_{\mathcal{N}} \setminus \mathcal{R}} [1 + U_{st}] \left( 1 - \prod_{st \in \mathcal{R}} [1 + U_{st}] \right) \\
\leq & \sum_{\substack{e, e' \in \mathcal{B}_{\mathcal{N}} : \\ \mathcal{N}_e \cap \mathcal{N}_{e'} = \emptyset}} \sum_{\substack{m_e \leq n_e \\ m_{e'} \leq n_{e'}}} [-U^{e,e'}_{m_e, m_e'}] \prod_{st \in \mathbf{E}_{\mathcal{M}} \setminus \mathcal{R}} [1 + U_{st}] \\
\leq & \sum_{\substack{e, e' \in \mathcal{B}_{\mathcal{N}} : \\ \mathcal{N}_e \cap \mathcal{N}_{e'} = \emptyset}} \sum_{\substack{m_e \leq n_e \\ m_{e'} \leq n_{e'}}} [-U^{e,e'}_{m_e, m_e'}] \prod_{f \neq e, e'} \prod_{\substack{s, t \in \mathcal{N}_f : \\ 0 < s < t < n_f}} [1 + U_{st}] \\
\times & \prod_{\substack{s, t \in \mathcal{N}_e : \\ 0 < s < t < m_e}} [1 + U_{st}] \prod_{\substack{s, t \in \mathcal{N}_e : \\ m_e < s < t < n_e}} [1 + U_{st}] \\
\times & \prod_{\substack{s, t \in \mathcal{N}_{e'} : \\ 0 < s < t < m_{e'}}} [1 + U_{st}] \prod_{\substack{s, t \in \mathcal{N}_{e'} : \\ m_e < s < t < n_{e'}}} [1 + U_{st}],
\end{aligned} \tag{6.80}$$

where we have used (6.72) in the final step.

Breaking up $\omega$ (in 6.76) at every branch point and at $(e, m_e)$ and $(e', m_{e'})$ and applying inequality (6.80) we obtain

$$\begin{aligned}
& \sum_{\vec{y}} \phi^{\mathcal{R}}_{\mathcal{N}}(\vec{y}) \leq \rho(0)^{2r-2} \sum_{\vec{y}} \sum_{\substack{e, e' \in \mathcal{B}_{\mathcal{N}} : \\ \mathcal{N}_e \cap \mathcal{N}_{e'}}} \sum_{\substack{m_e \leq n_e \\ m_{e'} \leq n_{e'}}} \prod_{f \neq e, e'} h_{n_f}(y_f) \\
& \times \sum_{u, u'} h_{m_e}(u - v_e(\vec{y})) h_{n_e - m_e}(v_e(\vec{y}) + y_e - u) \\
& \times h_{m_{e'}}(u' - v_{e'}(\vec{y})) h_{n_{e'} - m_{e'}}(y_{e'} + v_{e'}(\vec{y}) - u') \rho^{(2)}(u - u'),
\end{aligned} \tag{6.81}$$

where $v_e(\vec{y}) = \sum_{f \overset{\alpha}{\rightsquigarrow} e} y_f$ and the notation $f \overset{\alpha}{\rightsquigarrow} e$ denotes the set of edges in $\alpha$ on the path from the root to edge $e$ (not including $e$). Rearranging sums we get that

(6.81) is bounded by a constant times

$$\sum_{\substack{e,e' \in \mathcal{B}_{\mathcal{N}}: \\ \mathcal{N}_e \cap \mathcal{N}_{e'}}} \prod_{f \neq e,e'} \sum_{y_f} h_{n_f}(y_f)$$
$$\times \sup_{v,w} \sum_{\substack{m_e \leq n_e \\ m_{e'} \leq n_{e'}}} \sum_{y_e, y_{e'}, u, u'} h_{m_e}(u-v) h_{n_e - m_e}(v + y_e - u) \tag{6.82}$$
$$\times h_{m_{e'}}(u' - w) h_{n_{e'} - m_{e'}}(w + y_{e'} - u') \rho^{(2)}(u - u').$$

By translation invariance, the last two lines of (6.82) are equal to the sup over $z$ of $\sum_{x_1,x_2} g_{n_e,n_{e'}}(x_1, x_2, z)$, one of the quantities that we bounded in Lemma 6.4.2. However we now need to prove a stronger bound than that appearing in Lemma 6.4.2. Break up the sums over $m_1$ into the two terms $m_1 \leq \frac{n_1}{2}$, and $m_1 > \frac{n_1}{2}$ and similarly for $m_2$. Then we are left with 4 terms, one of which is

$$\sum_{x_1,x_2,u_1,u_2} \sum_{\substack{m_e \geq \frac{n_e}{2} \\ m_{e'} \geq \frac{n_{e'}}{2}}} h_{m_e}(u_1) h_{n_e - m_e}(x_1 - u_1)$$
$$\times h_{m_{e'}}(u - w) h_{n_e - m_e}(x_1 - u) \rho^{(2)}(u - u')$$
$$= \sum_{\substack{m_e \geq \frac{n_e}{2} \\ m_{e'} \geq \frac{n_{e'}}{2}}} \sum_{u_1,u_2} h_{m_e}(u_1) h_{m_{e'}}(u - w) \rho^{(2)}(u - u')$$
$$\times \sum_{x_1} h_{n_e - m_e}(x_1 - u_1) \sum_{x_2} h_{n_{e'} - m_{e'}}(x_2 + w - u_2) \tag{6.83}$$
$$\leq \sum_{\substack{m_e \geq \frac{n_e}{2} \\ m_{e'} \geq \frac{n_{e'}}{2}}} \frac{C\beta^{2-\frac{4\nu}{d}}}{[m_e + m_{e'}]^{\frac{d-4}{2}}},$$

where in the last line we have applied Proposition 5.1.4 multiple times.

By first summing over the minimum of $m_e$ and $m_e'$ this is bounded by a constant times

$$\sum_{m_e \geq \frac{n_e}{2}} \frac{m_e C\beta^{2-\frac{4\nu}{d}}}{[m_e]^{\frac{d-4}{2}}} + \sum_{m_{e'} \geq \frac{n_{e'}}{2}} \frac{m_{e'} C\beta^{2-\frac{4\nu}{d}}}{[m_{e'}]^{\frac{d-4}{2}}}$$
$$\leq \frac{C\beta^{2-\frac{4\nu}{d}}}{n_e^{\frac{d-8}{2}}} + \frac{C\beta^{2-\frac{4\nu}{d}}}{n_{e'}^{\frac{d-8}{2}}}. \tag{6.84}$$

The other 3 terms give the same bounds by symmetry.

We may now sum over each $y_f$ in (6.82) separately, and using Proposition 5.1.4 with $l = 1$ and $k = 0$ gives

$$\sum_{\vec{y}} \phi_{\mathcal{N}}^{\mathcal{R}}(\vec{y}) \leq \sum_{e \in E} \frac{C\beta^{2-\frac{4\nu}{d}}}{n_e^{\frac{d-8}{2}}}, \tag{6.85}$$

where the constant also depends on $r$. This proves the first bound of Lemma 4.2.1. □

### 6.6.2 Proof of the third bound of Lemma 4.2.1

Recall that $\phi_{\mathcal{N}}^{\pi}(\vec{y})$ was defined in (4.23) as

$$\phi_{\mathcal{N}}^{\pi}(\vec{y}) = \sum_{\omega \in \Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s \in \mathcal{N}} \sum_{R_s \in \mathcal{T}(\omega(s))} W(R_s) \sum_{\vec{m} \in \overline{\mathcal{H}}_{\vec{n}_b}} J(\mathcal{S}^{\Delta}(\vec{m})) \prod_{i=1}^{3} K\left((\mathcal{N} \setminus \mathcal{S}^{\Delta}(\vec{m}))_i\right), \tag{6.86}$$

where

$$\overline{\mathcal{H}}_{\vec{n}_b} = (\{\vec{m} : 0 \leq m_i \leq n_i, i = 1,2,3\} \cap \{\vec{m} : 0 \leq m_i \leq n_i - 2, i \in G\}) \setminus \mathcal{H}_{\vec{n}_b} \tag{6.87}$$

and

$$\mathcal{H}_{\vec{n}_b} = \{\vec{m} : 0 \leq m_i \leq \frac{n_i}{3} \; i = 1,2,3\} \cap \{\vec{m} : m_i \leq n_i - 2, i \in G\}. \tag{6.88}$$

As in Lemma 4.3.4, $|\phi_{\mathcal{N}}^{\pi}(\vec{y})|$ is bounded by

$$\begin{aligned} &C \left| \sum_{\vec{m} \in \overline{\mathcal{H}}_{\vec{n}_b}} \sum_{\vec{u}} \pi_{\vec{m}}(\vec{u}) \sum_{\vec{y}} \prod_{i=1}^{3} \sum_{v_i} D(v_i - u_i) t_{\mathcal{N}_i^-}(\vec{y}_{v_i}) \right| \\ &\leq C \sum_{N=1}^{\infty} \sum_{\vec{m} \in \overline{\mathcal{H}}_{\vec{n}_b}} \sum_{\vec{u}} \pi_{\vec{m}}^{N}(\vec{u}) \sum_{\vec{y}} \prod_{i=1}^{3} \sum_{v_i} D(v_i - u_i) t_{\mathcal{N}_i^-}(\vec{y}_{v_i}), \end{aligned} \tag{6.89}$$

where $\mathcal{N}_i^- = (\mathcal{N} \setminus \mathcal{S}_{\vec{m}}^{\Delta})$, and $\vec{y}_{v_i}$ denotes the vector of displacements associated to the branches of $\mathcal{N}_i^-$ (determined by $\vec{v}$, $\vec{y}$, and the labelling of the branches of $\mathcal{N}$).

Summing over the $v_i$ and $\vec{y}$ and using Lemma 4.4.1 this is bounded by

$$\begin{aligned} &C \sum_{N=1}^{\infty} \sum_{\vec{m} \in \overline{\mathcal{H}}_{\vec{n}_b}} \sum_{\vec{u}} \pi_{\vec{m}}^{N}(\vec{u}) \prod_{i=1}^{3} K^{\#\mathcal{N}_i^-} \\ =&C \sum_{N=1}^{\infty} \sum_{\vec{m} \in \overline{\mathcal{H}}_{\vec{n}_b}} \sum_{\vec{u}} \pi_{\vec{m}}^{N}(\vec{u}) \\ \leq& \sum_{N=1}^{\infty} N^3 \sum_{j=1}^{3} \sum_{\vec{m}: m_j \geq \frac{n_j}{3}} B_N(\vec{m}) \leq \sum_{j=1}^{3} \frac{C\beta^{2-\frac{8\nu}{d}}}{n_j^{\frac{d-8}{2}}}, \end{aligned} \tag{6.90}$$

applying Proposition 4.3.2 and Lemma 4.3.3 in the last line. This verifies the third bound of Lemma 4.2.1. ☐

### 6.6.3 Proof of the second bound of Lemma 4.2.1

Recall the definition of $\phi^b_{\mathcal{N}}(\vec{y})$ in (4.20). In this section we prove that

$$\sum_{\vec{y}} |\phi^b_{\mathcal{N}}(\vec{y})| \leq C \sum_{j=1}^{3} \frac{1}{n_j^{\frac{d-8}{2}}}. \tag{6.91}$$

It follows immediately from the definition of $\phi^b_{\mathcal{N}}(\vec{y})$ that

$$|\phi^b_{\mathcal{N}}(\vec{y})| = \sum_{\omega \in \Omega_{\mathcal{N}}(\vec{y})} W(\omega) \prod_{s \in \mathcal{N}} \sum_{R_s \in \mathcal{T}(\omega(s))} W(R_s) \left| \sum_{\Gamma \in \mathcal{E}^b_{\mathcal{N}}} \prod_{\flat \in \Gamma} U_\flat \right|, \tag{6.92}$$

where $\mathcal{E}^b_{\mathcal{N}}$ is defined in Definition 2.1.1 and is only nonempty if $\mathcal{N}$ contains more than 1 branch point ($r \geq 4$). In particular recall that graphs in $\mathcal{E}^b_{\mathcal{N}}$ contain no bonds in $\mathcal{R}$. We use an approach similar to that of [22] to analyse $\phi^b_{\mathcal{N}}(\vec{y})$.

Let $G(\mathcal{N}) \subset \{2, 3\}$ be the set of labels of branches of $\mathcal{N}$ incident to $b$ and another branch point of $\mathcal{N}$. For $F \subset G$ and $e \in F$, let $b_e$ be the other branch point in $\mathcal{N}$ incident to branch $\mathcal{N}_e$. Let

$$\mathcal{E}^b_{F,\mathcal{N}} = \{\Gamma \in \mathcal{E}^b_{\mathcal{N}} : \text{ for every } e \in F, \ \mathcal{A}_b(\Gamma) \text{ contains a nearest neighbour of } b_e\}. \tag{6.93}$$

Then,

$$\sum_{\Gamma \in \mathcal{E}^b_{\mathcal{N}}} \prod_{st \in \Gamma} U_{st} = \sum_{\Gamma \in \mathcal{E}^{(b)}_{\{2\},\mathcal{N}}} \prod_{st \in \Gamma} U_{st} + \sum_{\Gamma \in \mathcal{E}^b_{\{3\},\mathcal{N}}} \prod_{st \in \Gamma} U_{st} - \sum_{\Gamma \in \mathcal{E}^b_{\{2,3\},\mathcal{N}}} \prod_{st \in \Gamma} U_{st}, \tag{6.94}$$

where some of these sums could be empty if $G \neq \{2, 3\}$. Thus,

$$\left| \sum_{\Gamma \in \mathcal{E}^b_{\mathcal{N}}} \prod_{st \in \Gamma} U_{st} \right| \leq \sum_{\substack{F \subset G(\mathcal{N}) \\ F \neq \emptyset}} \left| \sum_{\Gamma \in \mathcal{E}^b_{F,\mathcal{N}}} \prod_{st \in \Gamma} U_{st} \right|. \tag{6.95}$$

Note that if $r = 4$ then one of $\mathcal{E}^b_{\{2\},\mathcal{N}}$ or $\mathcal{E}^b_{\{3\},\mathcal{N}}$ is empty and $\mathcal{E}^b_{\{2,3\},\mathcal{N}}$ is empty. This may also be true for $r > 4$, depending on the shape $\alpha$.

Define $\Gamma_F \subset \Gamma$ to be the set of bonds $st \in \Gamma$ such that

- $st$ is the bond in $\Gamma$ associated to $e$ at $b$ for some $e \in F$, or

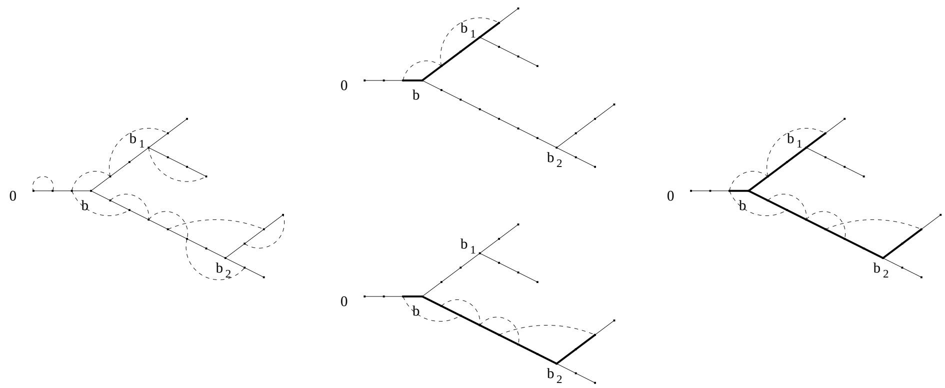


Figure 6.16: An illustration of the construction of a lace from a graph on some $\mathcal{N}$ in the case $b_1, b_2 \in \mathcal{A}_{\mathcal{N}}(\Gamma)$. The first figure shows a graph $\Gamma$ on a network $\mathcal{N}$. The remaining figures highlight the subnetworks $\mathcal{S}_F(\Gamma)$ for $F = \{2\}, \{3\}, \{2,3\}$.

- $st$ is the bond in $\Gamma$ associated to $e$ at $b_e$ for some $e \in F$ and $b_e \in \mathcal{A}_b(\Gamma)$, or
- $s, t \in \mathcal{N}_e$ for some $e \in F$.

Let $\mathcal{S}_F(\Gamma)$ be the largest subnetwork of $\mathcal{N}$ covered by $\Gamma_F \subset \Gamma$. Clearly $\Gamma|_{\mathcal{S}_F(\Gamma)} = \Gamma_F$ is a connected graph on $\mathcal{S}_F(\Gamma)$.

For each $e \in F$, $\mathcal{S}_F(\Gamma)$ by definition contains a nearest neighbour of $b_e$ in $\mathcal{N}$, and may contain $b_e$ itself. Since $\Gamma_F$ contains at most one bond that covers $b_e$, if $b_e \in \mathcal{S}_F(\Gamma)$ then it is not a branch point of $\mathcal{S}_F(\Gamma)$. Moreover if $F = \{2\}$ or $F = \{3\}$ then $b$ is also not a branch point of $\mathcal{S}_F(\Gamma)$, and hence $\mathcal{S}_F(\Gamma)$ is a network with no branch point (of course it contains at least one branch point of $\mathcal{N}$, namely $b$). If $F = \{2,3\}$ then $\mathcal{S}_F(\Gamma)$ may be a star-shaped network of degree 3.

Fix $\mathcal{N}, F$. Write $\mathcal{S} \sqsubset_F \mathcal{N}$, if $\mathcal{S} \subset \mathcal{N}$ is a star-shaped network with the following properties:

(a) for every $e \in F$, $\mathcal{S}$ contains a vertex $v$ that is adjacent to a branch point $b_e$ of $\mathcal{N}$, and

(b) $\mathcal{S}$ contains no branch points of $\mathcal{N}$ other than $b$ and $b_e$, $e \in F$.

Such star-shaped networks are exactly those for which there exists $\Gamma \in \mathcal{G}_{\mathcal{N}}^{-\mathcal{R}}$ such that $\mathcal{S} = \mathcal{S}_F(\Gamma)$. Define $\mathcal{L}_{\mathcal{S}}^*$ to be the set of laces $L$ on $\mathcal{S}$ such that

1. For each $e$ in $F$, if $b_e \in \mathcal{S}$ then there is exactly one bond $s^e t^e \in L$ covering branch point (of $\mathcal{N}$) $b_e \neq b$, and that bond has $s$ or $t$ strictly on branch $\mathcal{N}_e$.

2. If $F=\{2\}$ or $F=\{3\}$ then there is exactly one bond in $\mathcal{L}^*_{\mathcal{S}}$ covering $b$, while if $F=\{2,3\}$ there are at most 2 bonds in $\mathcal{L}^*_{\mathcal{S}}$ covering $b$.

3. $L$ contains no elements of $\mathcal{R}$ (i.e. no bonds which cover $\geq 2$ branch points of $\mathcal{N}$).

Then recalling the definition of $\mathbf{L}_\Gamma$ from Definition 2.1.4 we have

$$\begin{aligned}
\sum_{\Gamma\in\mathcal{E}^b_{F,\mathcal{N}}}\prod_{st\in\Gamma}U_{st} &= \sum_{\mathcal{S}\subset\mathcal{N}}\sum_{\substack{\Gamma\in\mathcal{E}^b_{F,\mathcal{N}}:\\ \mathcal{S}_F(\Gamma)=\mathcal{S}}}\prod_{st\in\Gamma}U_{st}\\
&= \sum_{\mathcal{S}\sqsubset_F\mathcal{N}}\sum_{L\in\mathcal{L}^*_{\mathcal{S}}}\left[\prod_{st\in L}U_{st}\right]\left[\sum_{\substack{\Gamma\in\mathcal{E}^b_{F,\mathcal{N}}:\\ \mathcal{S}_F(\Gamma)=\mathcal{S},\mathbf{L}_{\Gamma_F}=L}}\prod_{s't'\in\Gamma\setminus L}U_{s't'}\right]\\
&= \sum_{\mathcal{S}\sqsubset_F\mathcal{N}}\sum_{L\in\mathcal{L}^*_{\mathcal{S}}}\left[\prod_{st\in L}U_{st}\right]\left[\sum_{\substack{\Gamma\in\mathcal{G}^{-\mathcal{R},con}_{\mathcal{S}}:\\ \mathbf{L}_\Gamma=L}}\prod_{st\in\Gamma\setminus L}U_{st}\right]\\
&\quad\times\left[\sum_{\Gamma'\in\mathcal{G}^{-\mathcal{R}}_{\mathcal{N}\setminus\mathcal{S}}}\prod_{st\in\Gamma'}U_{st}\right]\sum_{\substack{\Gamma^*\in\mathcal{G}^{-\mathcal{R}}_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}:\\ \mathcal{S}_F(L\cup\Gamma^*)=\mathcal{S}}}\prod_{st\in\Gamma^*}U_{st},
\end{aligned}\tag{6.96}$$

where

$$\mathcal{G}^{-\mathcal{R}}_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}=\{\Gamma\in\mathcal{G}^{-\mathcal{R}}:\text{ for every } st\in\Gamma,\ [s\in\mathcal{S},t\in\mathcal{N}\setminus\mathcal{S}]\text{ or }[t\in\mathcal{S},s\in\mathcal{N}\setminus\mathcal{S}]\}.\tag{6.97}$$

Now note that for any set of sets of bonds $\mathcal{H}$ with the property that there exists some $N\in\mathbb{N}$ and $\{s_it_i\}\in\mathcal{H}$, $i=1,\dots N$ such that every element of $\mathcal{H}$ is a subset of $\{s_1t_1,\dots,s_Nt_N\}$, we have $\sum_{\Gamma\in\mathcal{H}}\prod_{st\in\Gamma}U_{st}=\prod_{\{st\}\in\mathcal{H}}[1+U_{st}]$. Let $\mathcal{L}^{N,*}_{\mathcal{S}}$ be the set of laces in $\mathcal{L}^*_{\mathcal{S}}$ consisting of exactly $N$ bonds. Then (6.96) is equal to

$$\sum_{N=1}^{\infty}(-1)^N\sum_{\mathcal{S}\sqsubset_F\mathcal{N}}\sum_{L\in\mathcal{L}^{N*}_{\mathcal{S}}}\left[\prod_{st\in L}-U_{st}\right]\left[\prod_{st\in\mathcal{C}(L)}[1+U_{st}]\right]\left[\prod_{\substack{st\in\mathbf{E}_{\mathcal{N}\setminus\mathcal{S}}\\ st\notin\mathcal{R}}}[1+U_{st}]\right]\times$$
$$\left[\prod_{\substack{s\in\mathcal{S},t\in\mathcal{N}\setminus\mathcal{S}:\\ \mathcal{S}_F(L\cup st)=\mathcal{S},st\notin\mathcal{R}}}[1+U_{st}]\right].\tag{6.98}$$

If $U_{st} \in \{-1, 0\}$ for each $st$, then each quantity involving $U_{st}$ in (6.98) is nonnegative and we have

$$\left|\sum_{\Gamma\in\mathcal{E}_{\mathcal{N}}^{F}}\prod_{st\in\Gamma}U_{st}\right| \leq \sum_{N=1}^{\infty}\sum_{\mathcal{S}\sqsubset_{F}\mathcal{N}}\sum_{L\in\mathcal{L}_{\mathcal{S}}^{N*}}\left[\prod_{st\in L}-U_{st}\right]\left[\prod_{st\in\mathcal{C}(L)}[1+U_{st}]\right]\times$$
$$\prod_{i=1}^{\Delta_{\mathcal{N}\setminus\mathcal{S}}}\left[\prod_{\substack{st\in\mathbf{E}_{(\mathcal{N}\setminus\mathcal{S})^{i}}\\ st\notin\mathcal{R}}}[1+U_{st}]\right], \tag{6.99}$$

where $\Delta_{\mathcal{N}\setminus\mathcal{S}}$ is the number of disjoint components $(\mathcal{N}\setminus\mathcal{S})_i$ of $\mathcal{N}\setminus\mathcal{S}$. This quantity is bounded above by the sum of four terms (corresponding to the 4 possible branches incident to $b_2$ and $b_3$ if $F = \{2, 3\}$) each of the form

$$\sum_{N=1}^{\infty}\left(\prod_{e\in F}\sum_{m_e=n_e-1}^{n_e+(n_{e'}-1)}\right)\sum_{m_1=0}^{n_1}\left(\prod_{e\in\{2,3\}\setminus F}\sum_{m_e=0}^{n_e-1}\right)\times$$
$$\sum_{L\in\mathcal{L}_{\mathcal{S}_{\vec{m}}^{\Delta}}^{N}}\left[\prod_{st\in L}-U_{st}\right]\left[\prod_{st\in\mathcal{C}(L)}[1+U_{st}]\right]$$
$$\times\prod_{i=1}^{\Delta_{\mathcal{N}\setminus\mathcal{S}_{\vec{m}}^{\Delta}}}\prod_{\overline{e}\in(\mathcal{N}\setminus\mathcal{S}_{\vec{m}}^{\Delta})'^{i}}\left[\prod_{\substack{s,t\in((\mathcal{N}\setminus\mathcal{S}_{\vec{m}}^{\Delta})'^{i})_{\overline{e}}:\\ 0<s<t<n_{\overline{e}}(\vec{m})'^{i}}}[1+U_{st}]\right], \tag{6.100}$$

where $e'$ denotes one of the two branches (other than $e$) incident to $b_e$, $\mathcal{S}_{\vec{m}}^{\Delta}$ is the star-shaped network defined by (2.12), and $(\mathcal{N}\setminus\mathcal{S}_{\vec{m}}^{\Delta})'^{i}$ denotes the fact that part of branch $\mathcal{N}_{e'}$ is being removed if $m_e \geq n_e$. In addition $n_{\overline{e}}(\vec{m})'^{i}$ is the length of branch $\overline{e}$ of $(\mathcal{N}\setminus\mathcal{S}_{\vec{m}}^{\Delta})'^{i}$. Since the analysis does not depend on the $e'$, we ignore the fact that there are 4 such terms from this point on.

Combining (6.95), (6.99) and (6.100) we have that $\left|\sum_{\Gamma\in\mathcal{E}_{\mathcal{N}}^{b}}\prod_{st\in\Gamma}U_{st}\right|$ is bounded by a constant times

$$\sum_{\substack{F\subset G(\mathcal{N})\\ F\neq\emptyset}} \sum_{N=1}^{\infty} \left(\prod_{e\in F}\sum_{m_e=n_e-1}^{n_e+(n_{e'}-1)}\right) \sum_{m_1=0}^{n_1} \left(\prod_{e\in\{2,3\}\setminus F}\sum_{m_e=0}^{n_e-1}\right) \times$$
$$\sum_{L\in\mathcal{L}^N_{\mathcal{S}^{\Delta}_{\vec{m}}}} \left[\prod_{st\in L} -U_{st}\right]\left[\prod_{st\in\mathcal{C}(L)}[1+U_{st}]\right] \tag{6.101}$$
$$\times \prod_{i=1}^{\Delta_{\mathcal{N}\setminus\mathcal{S}^{\Delta}_{\vec{m}}}} \prod_{\overline{e}\in(\mathcal{N}\setminus\mathcal{S}^{\Delta}_{\vec{m}})'^i} \left[\prod_{\substack{s,t\in((\mathcal{N}\setminus\mathcal{S}^{\Delta}_{\vec{m}})'^i)_{\overline{e}}:\\ 0<s<t<n_{\overline{e}}(\vec{m})'^i)}} [1+U_{st}]\right].$$

Putting this back into (6.92), the sum over laces on the star-shaped network gives rise to the quantity $\pi_{\vec{M}}(\bullet)$ and the final product gives rise to $Ch_{n_{\overline{e}}(\vec{m})'^i}(\bullet)$, with displacements summed over. On the latter on which we use (5.6) with $l=1$ to bound $\|h_{n_{\overline{e}}(\vec{m})'^i}\|_1$ by a constant and we obtain an upper bound on (6.92) of a constant times

$$\sum_{\substack{F\subset G(\mathcal{N})\\ F\neq\emptyset}} \sum_{N=1}^{\infty} \left(\prod_{e\in F}\sum_{m_e=n_e-1}^{n_e+(n_{e'}-1)}\right) \sum_{m_1=0}^{n_1} \left(\prod_{e\in\{2,3\}\setminus F}\sum_{m_e=0}^{n_e-1}\right) \sum_{\vec{u}} \pi^N_{\vec{m}}(\vec{u})$$
$$\times \prod_{i=1}^{\Delta_{\mathcal{N}\setminus\mathcal{S}^{\Delta}_{\vec{m}}}} \prod_{\overline{e}\in(\mathcal{N}\setminus\mathcal{S}^{\Delta}_{\vec{m}})'^i} K. \tag{6.102}$$

By Proposition 4.3.2 this is bounded above by

$$C \sum_{\substack{F\subset G(\mathcal{N})\\ F\neq\emptyset}} \sum_{N=1}^{\infty} N^3 \left(\prod_{e\in F}\sum_{m_e=n_e-1}^{n_e+(n_{e'}-1)}\right) \sum_{m_1=0}^{n_1} \left(\prod_{e\in\{2,3\}\setminus F}\sum_{m_e=0}^{n_e-1}\right) B_N(\vec{m})$$
$$\leq C \sum_{\substack{F\subset G(\mathcal{N})\\ F\neq\emptyset}} \sum_{e\in F} \frac{C\beta^{2-\frac{8\nu}{d}}}{n_e^{\frac{d-8}{2}}}. \tag{6.103}$$

Since the remaining sums are finite, this establishes the second bound of Lemma 4.2.1. □

This completes the proof of Lemma 4.2.1.

# Chapter 7

# Convergence to CSBM.

In this chapter we relate the convergence of the $r$-point functions (as in Theorems 1.4.3 and 1.4.5) to convergence of $\mu_n \in M_F(D(M_F(\mathbb{R}^d)))$ (defined by (1.17)) to the canonical measure of super-Brownian motion (CSBM).

The fact that CSBM is the weak limit of certain branching random walk models is a standard result in the theory of measure-valued processes, and we take such a result as our definition of CSBM in Section 7.1. In Section 7.2 we restate Theorem 1.3.1 and briefly discuss some related results. In Section 7.3 we prove a general result (Proposition 7.3.3) that relates convergence of finite-dimensional distributions to convergence of certain functionals and the existence of certain exponential moments for the limiting measures. We conclude in Section 7.4 by proving Theorem 1.3.1 and noting that Theorems 1.4.3 and 1.4.5, together with the convergence of the *survival probability* (which in general need not be a probability measure) implies convergence of the finite-dimensional distributions of our model to those of CSBM.

## 7.1 The canonical measure of super-Brownian Motion

In this section we indirectly define *the canonical measure of super-Brownian motion* as the weak limit of a branching random walk model with critical branching.

### 7.1.1 Branching Random Walk

We describe a particle model (branching random walk) where we label particles by multi-indicies as in [27] and some of the references therein. The construction we describe here is somewhat nonstandard but is done to resemble the construction of our lattice tree model. A particle is described by

$$\alpha \in I = \cup_{m=0}^{\infty} \mathbb{N}^{\{0,1,\ldots,m\}} = \{(\alpha_0, \ldots, \alpha_m) : \alpha_i \in \mathbb{N}, m \in \mathbb{Z}_+\}. \tag{7.1}$$

We start with a single particle, and set $\alpha_0 = 1$ for all $\alpha$. Let $|(\alpha_0, \ldots, \alpha_m)| = m$ be the generation of $\alpha$ and write $\bar{\alpha} < \alpha \iff \bar{\alpha} = (\alpha_0, \ldots, \alpha_i)$ for some $i < |\alpha|$, i.e. if $\bar{\alpha}$ is an ancestor of $\alpha$. If $\alpha = (\alpha_0, \ldots, \alpha_{m-1}, \alpha_m)$ and $\bar{\alpha} = (\alpha_0, \ldots, \alpha_{m-1})$ then we say that $\bar{\alpha}$ is the parent of $\alpha$ and $\alpha$ is the $\alpha_m^{th}$ child of $\bar{\alpha}$. Fix $M \in \mathbb{Z}_+$ and let $Y$ be a random variable with $Y(\omega) \in \{0, 1, \ldots, M\}$, $E[Y] = 1$, $0 < E\left[(Y - E[Y])^2\right] = \gamma < \infty$.

Let $\{Y_\alpha : \alpha \in I\}$ be i.i.d $\sim Y$ random variables. Let $G_0 = \{(1)\}$ and for each $m \in \mathbb{N}$ we define $G_m$ recursively as follows. At time $m^-$, each particle $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in G_{m-1}$ gives birth to $Y_\alpha$ children

$$(\alpha_0, \ldots, \alpha_{m-1}, 1), \ldots, (\alpha_0, \ldots, \alpha_{m-1}, Y_\alpha), \tag{7.2}$$

and immediately dies. We let $G_m$ be the set of particles alive at time $m$. Note that each particle $\alpha \in G_m$ satisfies $|\alpha| = m$ and has a unique parent $\bar{\alpha} \in G_{m-1}$ by definition. Clearly if $G_m = \emptyset$ then $G_n = \emptyset$ for every $n \geq m$. A well known result due to Kolmogorov (see [27] Theorem II.1.1.(a)) states that

$$m\gamma P(G_m \neq \emptyset) \to 2, \text{ as } m \to \infty, \tag{7.3}$$

and therefore $P(\cap_{m=0}^{\infty}\{G_m \neq \emptyset\}) = 0$ so that $G = \cup_{m=0}^{\infty} G_m$ is almost surely a finite set. We call a set $G$ that can be constructed in this way a *geneology.* Let $\mathcal{G}$ denote the set of possible geneologies. Since the number of children of a particle is bounded above, the number of possible $G_m$'s is finite for each $m$. Therefore $\mathcal{G}$ is a countable set.

Given a function $D(x)$ defined by Definition 1.2.4, and a set of particles (multindicies) $G$ we choose a random embedding $B$ of $G$ into $\mathbb{R}^d$ as follows. Let $\Omega_G = \{B : G \mapsto \mathbb{Z}^d, B((1)) = 0\}$ be the set of maps from $G$ to $\mathbb{Z}^d$ that map the initial particle (1) to the origin. Then we define a probability measure $P' \in M_1(\Omega)$ on the set $\Omega = \{(G, B) : G \in \mathcal{G}, B \in \Omega_G\}$ of embedded geneologies by

$$P'\left((G, B) = (G^*, B^*)\right) = P(G = G^*) \prod_{(\bar{\alpha}, \alpha) \in G^*} D(B^*(\alpha) - B^*(\bar{\alpha})) I_{\{B^* \in \Omega_{G^*}\}}, \tag{7.4}$$

where the product is over all (parent, child) pairs $(\bar{\alpha}, \alpha)$ in $G^*$.

Now given $(G, B) \in \Omega$, $n \in \mathbb{N}$, we define measures $X_{\frac{i}{n}}^{n,(G,B)} \in M_F(\mathbb{R}^d)$, $i \in \mathbb{N}$ by

$$X_{\frac{i}{n}}^{n,(G,B)} = \frac{C_1'}{n} \sum_{x : \sqrt{C_2' n}\, x \in B(G_i)} \delta_x. \tag{7.5}$$

We extend this to all $t \in \mathbb{R}_+$ by $X_t^{n,(G,B)} = X_{\frac{\lfloor nt \rfloor}{n}}^{n,(G,B)}$. Finally we define $\mu_n' \in M_F(D(M_F(\mathbb{R}^d)))$ by

$$\mu_n'(H) = nC_3' P'\left((G, B) : \{X_t^{n,(G,B)}\}_{t \in \mathbb{R}_+} \in H\}\right), \qquad H \in \mathcal{B}(D(M_F(\mathbb{R}^d))). \tag{7.6}$$

Here, the $C_i'$ are some fixed, known constants that may depend on the function $D$ (recall that $D$ depends on $L$) and $\gamma$.

### 7.1.2 BRW converges to CSBM

In this section (much of which is taken from [27] chapter II.7., but with different notation) we define the precise way in which branching random walk converges to super-Brownian motion.

The *survival time* (often called the *extinction time*) $S : D([0,\infty), M_F(\mathbb{R}^d)) \to [0,\infty]$ is defined by

$$S\left(\{X_t\}_{t\in\mathbb{R}_+}\right) = \inf\{s > 0 : X_s = 0_M\}, \tag{7.7}$$

where $0_M$ is the zero measure on $\mathbb{R}^d$ satisfying $0_M(\mathbb{R}^d) = 0$. Let

$$\begin{aligned} D^*([M_F(\mathbb{R}^d)) &= \{\{X_t\} \in D([0,\infty), M_F(\mathbb{R}^d)) : \ \ S(\{X_t\}) > 0, \ \ X_t = 0_M \ \ \forall t \geq S\}, \\ C_0^*(M_F(\mathbb{R}^d)) &= \{\{X_t\} \in D^* : X_0 = 0_M, X_\bullet \text{ is continuous}\}, \end{aligned} \tag{7.8}$$

with the topologies inherited from $D([0,\infty), M_F(\mathbb{R}^d)), C([0,\infty), M_F(\mathbb{R}^d))$ (the topology for $C$ is the topology of uniform convergence on compact sets). Note that

$$\begin{aligned} \mu_n'(S < \epsilon) &\geq nC_3'P'\left(X^{n,(G,B)}_{\frac{\lfloor n\epsilon\rfloor}{n}} = 0_M\right) = nC_3'P\left(G_{\lfloor n\epsilon\rfloor} = \emptyset\right) \\ &\geq nC_3'P\left(G_1 = \emptyset\right), \qquad \text{for } n \geq \frac{1}{\epsilon} \\ &\to \infty \text{ as } n \to \infty. \end{aligned} \tag{7.9}$$

Let us now also define the finite dimensional distributions of $\nu \in M_F(D(M_F(\mathbb{R}^d)))$. Let $R \geq 1$, and $\vec{t} = \{t_1, \ldots, t_R\} \in [0,\infty)^R$. Let $h_{\vec{t}} : D(M_F(\mathbb{R}^d)) \to M_F(\mathbb{R}^d)^R$ denote the projection map satisfying $h_{\vec{t}}(\{X_\bullet\}) = (X_{t_1}, \ldots X_{t_R})$. Then the *finite dimensional distributions* of $\nu$ are the measures $\nu h_{\vec{t}}^{-1} \in M_F(M_F(\mathbb{R}^d)^R)$ given by

$$\nu h_{\vec{t}}^{-1}(H) = \nu\left(\{X_\bullet\} : h_{\vec{t}}(\{X_\bullet\}) \in H\right), \qquad H \in \mathcal{B}(M_F(\mathbb{R}^d)^R). \tag{7.10}$$

**Definition 7.1.1 (Convergence in $D^*$).** *Suppose $\{\nu_n : n \in \mathbb{N} \cup \infty\} \subset M_\sigma(D^*)$, the set of $\sigma$-finite measures on $D^*$. We write $\nu_n \stackrel{w}{\Longrightarrow} \nu_\infty$ on $D^*$ if for every $\epsilon > 0$,*

$$\begin{aligned} \nu_n(S > \epsilon) &< \infty, \qquad \forall n \in \mathbb{N} \cup \infty, \ \textit{and} \\ \nu_n(\bullet, S > \epsilon) &\stackrel{w}{\Longrightarrow} \nu_\infty(\bullet, S > \epsilon), \qquad \textit{as } n \to \infty, \end{aligned} \tag{7.11}$$

*where the weak convergence in the second condition is convergence in* $M_F(D(M_F(\mathbb{R}^d)))$. *We write* $\nu_n \overset{f.d.d}{\Longrightarrow} \nu_\infty$ *on* $D^*$ *if for every* $\epsilon > 0$, $m \in \mathbb{N}$ *and* $\vec{t} \in (\epsilon, \infty)^m$,

$$\begin{aligned} \nu_n(S > \epsilon) &< \infty, && \forall n \in \mathbb{N} \cup \infty, \text{ and} \\ \nu_n h_{\vec{t}}^{-1}(\bullet, S > \epsilon) &\overset{w}{\Longrightarrow} \nu_\infty h_{\vec{t}}^{-1}(\bullet, S > \epsilon), && \text{as } n \to \infty, \end{aligned} \tag{7.12}$$

*where the weak convergence in the second condition is convergence in* $M_F(M_F(\mathbb{R}^d)^m)$.

The following Theorem, which states that branching random walk converges weakly to the canonical measure of super-Brownian motion is fairly well known, and a version of it is proved in [27] (see [27] Theorem II.7.3.).

**Theorem 7.1.2.** *Let* $\mu'_n$ *be defined by (7.6) for the branching random walk model defined in section 7.1.1. There exist constants* $C'_1, C'_2, C'_3$ *such that*

*(a) for every* $s > 0$ *there exists* $R_s \in M_F(M_F(\mathbb{R}^d) \setminus \{0_M\})$ *such that for every* $s > 0$,

$$\mu'_n(X_s \in \bullet, X_s \neq 0_M) \overset{w}{\Longrightarrow} R_s(\bullet) \text{ on } M_F(\mathbb{R}^d), \quad \text{and } R_s(M_F(\mathbb{R}^d) \setminus \{0_M\}) = \frac{2}{\gamma s}. \tag{7.13}$$

*(b) There exists a* $\sigma$*-finite measure* $\mathbb{N}_0$ *on* $C_0^*(M_F(\mathbb{R}^d))$ *such that* $\mu'_n \overset{w}{\Longrightarrow} \mathbb{N}_0$ *on* $D^*(M_F(\mathbb{R}^d))$, *and for every* $s > 0$

1. $\mathbb{N}_0(X_s \in \bullet, S > s) = R_s(\bullet)$,
2. $P'\left(\{X_t^{n,(G,B)}\} \in \bullet | X_s^{n,(G,B)} \neq 0_M\right) \overset{w}{\Longrightarrow} \mathbb{N}_0(\bullet | S > s)$ *on* $D(M_F(\mathbb{R}^d))$.

**Definition 7.1.3.** *The* $\sigma$*-finite measure* $\mathbb{N}_0$ *on* $C^*([0,\infty), M_F(\mathbb{R}^d))$ *defined by part (b) of Theorem 7.1.2 is called the* canonical measure of super-Brownian motion.

## 7.2 Lattice trees

In this section we recall the setup of our measure-valued process and briefly discuss the context of our results.

Recall from Sections 1.3 and 1.4 the following definitions:

- $X_t^{n,T} \in M_F(\mathbb{R}^d)$ therefore $\{X_t^{n,T}\}_{t \in \mathbb{R}_+} \in D(M_F(\mathbb{R}^d))$ defined by

$$X_{\frac{i}{n}}^{n,T} = \frac{1}{VA^2 n} \sum_{x : \sqrt{\sigma^2 v n} x \in T_i} \delta_x, \text{ and } X_t^{n,T} = X_{\frac{\lfloor nt \rfloor}{n}}^{n,T}. \tag{7.14}$$

- $\mathbb{P} \in M_1(\mathcal{T}_0)$ defined by
$$\mathbb{P}(\{T\}) = \frac{W(T)}{\rho(0)}. \tag{7.15}$$

- $\mu_n \in M_F(D(M_R(\mathbb{R}^d)))$ defined by
$$\mu_n(H) = nVA\rho(0)\mathbb{P}\left(\{T : \{X_t^{n,T}\}_{t\in\mathbb{R}} \in H\}\right). \tag{7.16}$$

Note that for $\epsilon > 0$ and $n \geq \frac{1}{\epsilon}$,

$$\begin{aligned}
\mu_n(S \leq \varepsilon) &= \mu_n(X_t = 0_M \text{ for all } t > \varepsilon) \\
&= nVA\rho(0)P(T : X^{n,T}_{\frac{\lfloor nt \rfloor}{n}} = 0_M \text{ for all } t > \varepsilon) \\
&\geq nVA\rho(0)P(T : X^{n,T}_{\frac{\lfloor n\epsilon \rfloor}{n}} = 0_M) \\
&\geq nVA\rho(0)P(T = \{0\}) = nVA\rho(0),
\end{aligned} \tag{7.17}$$

i.e. $\mu_n(S \leq \varepsilon) \nearrow \infty = \mathbb{N}_0(S \leq \varepsilon)$. Recall however from the previous section that our statements about convergence to CSBM include the condition that $S > \epsilon$. Another way of removing the contribution from processes that have arbitrarily small lifetime is to include the total mass at time $\epsilon$ in the expectation as in Theorem 1.3.1, which we restate with our new notation.

**Theorem (1.3.1).** *There exists $L_0 \gg 1$ such that for every $L \geq L_0$, with $\mu_n$ defined by (1.17) the following holds: For every $\epsilon, \lambda > 0$, $m \in \mathbb{N}$, $\vec{t} \in (\epsilon, \infty)^m$ and every $F : \left(M_F(\mathbb{R}^d)^m\right) \to \mathbb{R}$ bounded by a polynomial and such that $\mathbb{N}_0 h_{\vec{t}}^{-1}(\mathcal{D}_F) = 0$,*

*1.*
$$E_{\mu_n h_{\vec{t}}^{-1}}\left[X_\epsilon(1)F(\vec{X})\right] \to E_{\mathbb{N}_0 h_{\vec{t}}^{-1}}\left[X_\epsilon(1)F(\vec{X})\right], \tag{7.18}$$

*and*

*2.*
$$E_{\mu_n h_{\vec{t}}^{-1}}\left[F(\vec{X})I_{\{X_\epsilon(1)>\lambda\}}\right] \to E_{\mathbb{N}_0 h_{\vec{t}}^{-1}}\left[F(\vec{X})I_{\{X_\epsilon(1)>\lambda\}}\right]. \tag{7.19}$$

We show in Section 7.4 that the convergence of the survival probability (together with our results) would be sufficient to prove the following conjecture.

**Conjecture 7.2.1.** *Let $\mu_n$ be defined by (7.16) for the $L, D$ lattice tree model defined in section 1.3, and let $d > 8$. There exists $L_0(d) \gg 1$ such that for every $L \geq L_0$,*

$$\mu_n \overset{f.d.d}{\Longrightarrow} \mathbb{N}_0, \qquad \textit{on } D^*. \tag{7.20}$$

As in [3], convergence as a stochastic process follows from convergence of the finite-dimensional distributions (Conjecture 7.2.1) and tightness. Tightness for this model is also an open problem and is less well understood at present.

### 7.2.1 ISE

Derbez and Slade [7] proved results closely related to Theorems 1.4.3 and 1.4.5. They showed using generating function methods that the scaling limit of lattice trees (sufficiently spread out for $d > 8$, or nearest neighbour model for $d \gg 8$) is *integrated super-Brownian excursion* (ISE). We describe their results in the form that is most most relevant to this paper.

Let $\mathcal{T}_N$ denote the set of lattice trees containing exactly $N$ vertices, one of which is 0.

**Remark 7.2.2.** *Roughly speaking, a lattice tree that survives until time $n$ has, on average, order $n$ particles alive at that time. We infer from this that the total size of such a tree is $N \approx n^2$. Thus scaling space by $n^{-\frac{1}{2}}$ should be equivalent (in terms of the leading asymptotics) to scaling space by $N^{-\frac{1}{4}}$.*

For fixed $N \in \mathbb{Z}_+$ and $T \in \mathcal{T}_N$, define

$$X^{N,T} = \frac{1}{N} \sum_{x: D_1 N^{\frac{1}{4}} x \in T} \delta_x, \tag{7.21}$$

where $D_1$ is a constant defined in [7]. Since $T$ contains exactly $N$ vertices, $X^{N,T}$ is a probability measure on $\mathbb{R}^d$. Keeping $N$ fixed, choose a random tree according to

$$\mu^N(\{T\}) = \frac{W_p(T)}{\sum_{T \in \mathcal{T}_N} W_p(T)}. \tag{7.22}$$

Then $X^{N,T}$ is a random probability measure described by $\mu^N$.

Define $\mathcal{I}_N \in M_1\left(M_1(\mathbb{R}^d)\right)$ by

$$\mathcal{I}_N(A) = \mu^N(\{T : X^{N,T} \in A\}), \qquad A \in \mathcal{B}(M_1(\mathbb{R}^d)). \tag{7.23}$$

with $\mathcal{B}$ denoting the Borel sets and $M_1(E)$ the space of probability measures on $E$ with the weak topology.

Slade [28] shows that the results of Derbez and Slade [7] imply $\mathcal{I}_N \stackrel{w}{\Longrightarrow} \mathcal{I}$ as $N \to \infty$, where the probability measure $\mathcal{I} \in M_1\left(M_1(\mathbb{R}^d)\right)$ is called *integrated super-Brownian excursion.* This is a statement that for all $f \in C_b(M_1(\mathbb{R}^d))$ (i.e. $f$ bounded continuous on $M_1(\mathbb{R}^d)$),

$$\int f d\mathcal{I}_N \to \int f d\mathcal{I}. \tag{7.24}$$

Derbez and Slade [7] prove (7.24) for functions of the form

$$f_{\vec{k}}(\nu) = \int e^{i\vec{k}\cdot\vec{x}} \nu(d\vec{x}), \tag{7.25}$$

and Slade [28] shows that this is sufficient to prove weak convergence.

To prove their results, Derbez and Slade [7] define for $\zeta_i, p \in \mathbb{C}$, $r \geq 2$, $\alpha \in \Sigma_r$, and $\vec{y} \in \mathbb{Z}^{d(2r-3)}$, the set $\mathcal{T}^{\alpha}_{\vec{n}}(\vec{y})$ of trees of skeleton shape $\alpha$ with skeleton displacements $y_i$ and the generating functions

$$G^{r,\alpha}_{p,\vec{\zeta}}(\vec{y}) = \sum_{\vec{n} \in \mathbb{Z}_+^{2r-3}} \prod_{j=1}^{2r-3} \zeta_j^{n_j} \sum_{T \in \mathcal{T}_{\mathcal{N}(\vec{n},\alpha)}(0,\vec{y})} W_p(T). \tag{7.26}$$

They then write

$$\widehat{G}^{r,\alpha}_{p,\vec{\zeta}}(\vec{\kappa}) = V^{r-2} \prod_{j=1}^{2r-3} \frac{1}{C_1 \kappa_j^2 + C_2(1-\frac{p}{p_c})^{\frac{1}{2}} + C_3(1-\zeta_j)} + \widehat{E}^{r,\alpha}_{p,\vec{\zeta}}(\vec{\kappa}), \tag{7.27}$$

for specific constants $C_1, C_2, C_3$. They show that $\widehat{E}^{r,\alpha}_{p,\vec{\zeta}}(\vec{\kappa})$ is an error term when:

- $r = 2, 3$ for all $p < p_c$ and $\|\vec{\zeta}\|_\infty \leq 1$, and when
- $r \geq 2$ for all $p < p_c$ and $\vec{\zeta} = \vec{1}$.

Essentially in [7] backbones were very well understood for $r = 2, 3$ but less so for $r \geq 4$. Since $\vec{n}$ is summed over in the definition of $G$ in 7.26, setting $\vec{\zeta} = \vec{1}$ removes all time (backbone length) information from the results of [7] for $r \geq 4$. Thus we do not expect Theorem 4.1.8 to follow from the analysis of [7] for $r \geq 4$ and at least for $r \geq 4$, Theorem 4.1.8 is an entirely new result. The following non-rigorous argument suggests that Theorems 1.4.3 and 1.4.5 for $r = 2, 3$ may follow from the analysis of [7] without too much difficulty (perhaps with less sharp error bounds).

When $p = p_c$, (7.26) implies that the coefficient of $\prod_{j=1}^{2r-3} \zeta_j^{n_j}$ in $\widehat{G}^{r,\alpha}_{p_c,\vec{\zeta}}(\frac{\vec{\kappa}}{\sqrt{n}})$ is

$$\sum_{\vec{y}} e^{i\vec{\kappa}\cdot\vec{y}n^{-\frac{1}{2}}} \sum_{T \in \mathcal{T}_{\mathcal{N}(\vec{n},\alpha)}(0,\vec{y})} W(T) = \hat{t}_{\mathcal{N}(\vec{n},\alpha)}\left(\frac{\vec{\kappa}}{\sqrt{n}}\right). \tag{7.28}$$

Using the fact that for $x < a$, $\frac{1}{a-x} = \frac{1}{a}\sum_{n=0}^{\infty}(\frac{x}{a})^n$, and assuming that $\widehat{E}$ is an error term, (7.27) implies that this same coefficient is approximately

$$\frac{V^{r-2}}{C_3^{2r-3}} \prod_{j=1}^{2r-3} \frac{1}{(1+\frac{C_1\kappa_j^2}{C_3 n})^{n_j+1}} \approx \frac{V^{r-2}}{C_3^{2r-3}} \prod_{j=1}^{2r-3} e^{-\frac{C_1}{C_3}\kappa_j^2\frac{n_j}{n}}, \quad \text{for large } n. \tag{7.29}$$

This is of the form of Theorem 4.1.8 (resp. Theorem 1.4.3 for $r = 2$), which was the main ingredient in the proof of Theorem 1.4.5. It is likely that one could adapt this rough argument to get a rigorous proof of a version of Theorem 1.4.3 and Theorem 1.4.5 with $r = 3$.

We describe the connection between ISE and CSBM as follows. Let $\{X_t\} \in C^*([0,\infty), M_F(\mathbb{R}^d))$. By definition of the survival time, $S$, we have that $X_t = 0_M$ for every $s \geq S$, and $X_t \neq 0_M$ for all $t \in (0, S)$.

If $S$ is finite (note that under $\mathbb{N}_0$, $S$ is indeed finite almost everywhere) then by continuity on $[0, S]$, $\sup_t X_t(\mathbb{R}^d) \leq K$ for some $0 < K < \infty$. Define a measure $Y(\bullet) = Y_{\{X_t\}}(\bullet)$ on $\mathbb{R}^d$ by

$$Y(\bullet) = \int_0^\infty X_t(\bullet)dt. \tag{7.30}$$

Then by the above discussion, $Y(\mathbb{R}^d) \leq \int_0^S K dt < \infty$, and we may define a probability measure $\mathcal{P}$ on $\mathbb{R}^d$ by

$$\mathcal{P}(\bullet) = \frac{Y(\frac{\bullet}{Y(\mathbb{R}^d)^{1/4}})}{Y(\mathbb{R}^d)}, \tag{7.31}$$

where for $A \in \mathcal{B}(\mathbb{R}^d)$ and $C > 0$, $\frac{A}{C} = \{x \in \mathbb{R}^d : Cx \in A\}$. Now if we choose $X_t$ randomly according to $\mathbb{N}_0\left(\bullet | Y(\mathbb{R}^d) = 1\right)$ (which is a probability measure on $D(M_F(\mathbb{R}^d)))$ then $\mathcal{P}$ has law $\mathcal{I}$.

## 7.3 Finite dimensional distributions

In this section we prove some results within the general theory of measure-valued processes. The main result of this section is Proposition 7.3.3. The motivation for proving Proposition 7.3.3 is to obtain a statement about convergence of the measures $\mu_n$ of (1.17) to $\mathbb{N}_0$ from convergence of the $r$-point functions (Theorems 1.4.3 and 1.4.5). We use Proposition 7.3.3 in Section 7.4 to prove Theorem 1.3.1 as a consequence of convergence of the $r$-point functions and in Section 7.5 to show that convergence of finite dimensional distributions of our model follows from convergence of the $r$-point functions and the survival probability.

The applications of Proposition 7.3.3 carried out in Sections 7.4 and 7.5 are also implicitly being used in [20] for oriented percolation and in [17] for the contact process in connecting convergence of the $r$-point functions to convergence of finite-dimensional distributions.

**Definition 7.3.1 (Tightness for finite measures).** *A set of finite measures $F \subset M_F(E)$ on the Borel $\sigma$-algebra of a metric space $E$ is* spatially tight *if for every $\eta > 0$ there exists $K \subset E$ compact such that $\sup_{\mu \in F} \mu(K^c) < \eta$. A set $F \subset M_F(E)$ is* tight *if it is spatially tight and $\sup_{\mu\in F} \mu(E) < \infty$.*

**Lemma 7.3.2.** *If $F \subset M_F(E)$ is tight, then every sequence in $F$ has a further subsequence which converges in $M_F(E)$ (weak convergence).*

*Proof.* Let $\{\mu_n\} \subset F$. If there exists a subsequence $\mu_{n_l}$ such that $\mu_{n_l}(E) \to 0$ then we have $\mu_{n_l} \to 0_M$ by definition (for every bounded continuous $f$....) and we are done. So without loss of generality there exists $\eta_0 > 0$ such that $\inf_n \mu_n(E) = \eta_0$. Therefore

$$P_n(\bullet) = \frac{\mu_n(\bullet)}{\mu_n(E)} \tag{7.32}$$

are probability measures. Let $\eta > 0$. Since the $\mu_n$ are (spatially) tight there exists $K \subset E$ compact such that $\sup_n \mu_n(K^c) < \eta\eta_0$. Therefore

$$\sup_n P_n(K^c) = \sup_n \frac{\mu_n(K^c)}{\mu_n(E)} < \frac{\eta\eta_0}{\eta_0} = \eta, \tag{7.33}$$

so $\{P_n\}$ is tight as a set of probability measures. Therefore there exists a subsequence $P_{n_k} \to P_\infty$.

Since $\{\mu_{n_k}\}$ is tight, $\{\mu_{n_k}(E)\}$ is a bounded, real-valued sequence, and therefore has a convergent subsequence $\mu_{n_k^*}(E) \to C \geq \eta_0$. So $\frac{\mu_{n_k^*}}{\mu_{n_k^*}(E)} \to P_\infty$ and $\mu_{n_k^*}(E) \to C > 0$ and therefore $\mu_{n_k^*} \to CP_\infty \in M_F(E)$ as required. □

That the full statement of tightness is necessary (i.e. spatial tightness is not sufficient) for the conclusion of Lemma 7.3.2 is illustrated in Example 7.3.4.

Let $\mathcal{F}$ denote a $M_F(\mathbb{R}^d)$ convergence determining class of bounded continuous functions $\phi : \mathbb{R}^d \to \mathbb{C}$ (i.e. $\nu_n \to \nu$ in $M_F(\mathbb{R}^d)$ if and only if $\nu_n(\phi) \to \nu(\phi)$ for every $\phi \in \mathcal{F}$), that contains a constant function, $\phi(x) = C_{\mathcal{F}} \neq 0$. The main result of this section is the following proposition.

**Proposition 7.3.3.** *Let $\varepsilon > 0$ and $\mu_n, \mu \in M_F\left(D(M_F(\mathbb{R}^d))\right)$. Suppose that for every $l \in \mathbb{Z}_+$ and every $\vec{t} \in (\varepsilon, \infty)^l$, $\vec{m} \in \mathbb{Z}_+^l$ we have*

1. *there exists a $\delta = \delta(\vec{t}) > 0$ such that for all $\theta_i < \delta$, $E_{\mu h_{\vec{t}}^{-1}}[e^{\sum_{j=1}^{l} \theta_i X_i(\mathbb{R}^d)}] < \infty$, and*
2. *for every $\vec{\phi} = \{\phi_{11}, \dots, \phi_{lm_l}\} \in \mathcal{F}^{\sum_{i=1}^{l} m_i}$,*

$$E_{\mu_n h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \to E_{\mu h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] < \infty, \tag{7.34}$$

*where an empty product is 1 by definition.*

*Then for every $m \in \mathbb{N}$ and every $\vec{t} \in (\varepsilon, \infty)^m$, $\mu_n h_{\vec{t}}^{-1} \to \mu h_{\vec{t}}^{-1}$ in $M_F\left((M_F(\mathbb{R}^d))^m\right)$.*

The importance of the $l = 0$ case in the Proposition is evident from the following example.

**Example 7.3.4.** *Let $\mu_n' \in M_F\left(D(M_F(\mathbb{R}^d))\right)$ be the measure that puts all its mass (n) on the measure-valued process $X_t = \frac{\delta_0}{n^2}$ for all $t \geq 0$, and $\mu$ be the measure that puts all its mass (1) on the measure-valued process $X_t = \delta_1$ for all $t \geq 0$. Next let $\mu_n = \mu_n' + \mu$ i.e.*

$$\mu_n(\bullet) = n\delta_{\frac{\delta_0}{n^2}} + \delta_{\delta_1}, \quad \mu = \delta_{\delta_1}. \tag{7.35}$$

*Then $E_\mu\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right] = \prod_{i=1}^l \prod_{j=1}^{m_i} \delta_1(\phi_{ij}) = \prod_{i=1}^l \prod_{j=1}^{m_i} \phi_{ij}(1)$, and*

$$E_{\mu_n}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right] = n\prod_{i=1}^{l}\prod_{j=1}^{m_i}\frac{\delta_0(\phi_{ij})}{n^2} + \prod_{i=1}^{l}\prod_{j=1}^{m_i}\delta_1(\phi_{ij}) \to 0 + \prod_{i=1}^{l}\prod_{j=1}^{m_i}\phi_{ij}(1). \tag{7.36}$$

*Thus we have*

$$E_{\mu_n}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right] \to E_{\mu}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right] \tag{7.37}$$

*for every $l \geq 1$, $\vec{m}$, $\vec{t}$, and $E_\mu\left[e^{\sum \theta_i X_{t_i}(\phi_i)}\right] = e^{\sum \theta_i} < \infty$. However $\mu_n(S > \epsilon) = n+1$ (resp. $\mu_n\left(D(M_F(\mathbb{R}^d))\right) = n+1$) and $\mu(S > \epsilon) = 1$ (resp. $\mu\left(D(M_F(\mathbb{R}^d))\right) = 1$), so that none of the finite-dimensional distributions can converge, and no subsequence of $\mu_n$ can converge in $M_F\left(D(M_F(\mathbb{R}^d))\right)$. Note that $\{\mu_n\}_{n\in\mathbb{N}}$ is spatially tight but not tight.*

We prove Proposition 7.3.3 in the form of 5 lemmas. The first, Lemma 7.3.5 establishes tightness of the $\{\mu_n h_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ for each fixed $l, \vec{t}$. Thus every subsequence of the $\mu_n h_{\vec{t}}^{-1}$ has a further subsequence that converges. The second, Lemma 7.3.6 states that any limit point of the $\{\mu_n h_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ must have the same moments (7.34) as $\mu h_{\vec{t}}^{-1}$. The third, Lemma 7.3.7 states that if a certain moment condition holds for every $\phi_i \in \mathcal{F}$, we also have that result for all continuous $0 \leq \phi_i \leq 1$. The fourth, Lemma 7.3.8 says that each subsequential limit point is uniquely determined by certain class of functionals $E_\bullet[e^{-\sum_{i=1}^m X_i(\phi_i)}]$, $\phi_i \geq 0$ bounded, continuous. Finally, Lemma 7.3.9 says that these functionals are uniquely determined by certain moments of the form (7.34). Taken together they show that since all subsequential limit points have the same moments (7.34), the limit points all coincide, and thus the whole sequence converges to that limit point.

**Lemma 7.3.5.** *Let $\mu_n, \mu \in M_F\left(D(M_F(\mathbb{R}^d))\right)$. Suppose that for every $t \in (0, \infty)$, and every $\phi \in \mathcal{F}$,*

$$E_{\mu_n h_t^{-1}}[X(\phi)] \to E_{\mu h_t^{-1}}[X(\phi)] < \infty. \tag{7.38}$$

*Then for each $m \in \mathbb{N}$ and every $\vec{t} \in (0,\infty)^m$, the set of measures $\{\mu_n h_{\vec{t}}^{-1} : n \in \mathbb{N}\}$ is tight.*

*Proof.* Taking $m = 0$ gives $\mu_n\left(D(M_F(\mathbb{R}^d))\right) \to \mu\left(D(M_F(\mathbb{R}^d))\right) < \infty$ so it remains to prove spatial tightness.

We first prove the $m = 1$ case. Let $\epsilon > 0$. Define $\nu_n = E_{\mu_n h_t^{-1}}[X]$, and $\nu = E_{\mu h_t^{-1}}[X]$. Then $\nu(\mathbb{R}^d) = L_0 < \infty$ and applying Fubini to (7.38) we have $\int \phi_i(x)\nu_n(dx) \to \int \phi_i(x)\nu(dx)$ for every $\phi_i \in \mathcal{F}$ hence $\nu_n \to \nu$. Therefore there exists $n_0$ such that for every $n \geq n_0$, $\nu_n(\mathbb{R}^d) \leq L_0 + 1$. Since the $\nu_n$ are finite, there exists $L_1$ such that $\nu_n(\mathbb{R}^d) \leq L_1$ for all $n \leq n_0$.

Let $L = (L_0 + 1) \wedge L_1$ and choose $M$ such that $\frac{L}{M} < \frac{\epsilon}{2}$. Then

$$\begin{aligned} L \geq &\sup_n E_{\mu_n h_t^{-1}}[X(\mathbb{R}^d)] \geq \sup_n E_{\mu_n h_t^{-1}}[M I_{X(\mathbb{R}^d)>M}] \\ =&M \sup_n \mu_n h_t^{-1}(X(\mathbb{R}^d) > M). \end{aligned} \tag{7.39}$$

Dividing through by $M$, we get that

$$\sup_n \mu_n h_t^{-1}(X(\mathbb{R}^d) > M) \leq \frac{L}{M} < \frac{\epsilon}{2}. \tag{7.40}$$

Fix $\eta > 0$. There exists $K_{-1} \subset \mathbb{R}^d$ compact such that $\nu(K_{-1}^c) < \frac{\eta}{3}$. Furthermore there exists $K_0 \subset \mathbb{R}^d$ compact such that $\nu(\overline{K_0^c}) \leq \nu(K_{-1}^c)$ (e.g. the set $K_0 = \{x : d(x, K_{-1}) \leq 1\}$). Since $\nu_n \to \nu$ in $M_F(\mathbb{R}^d)$ and $\overline{K_0^c}$ is closed,

$$\limsup_n \nu_n(\overline{K_0^c}) \leq \nu(\overline{K_0^c}) < \frac{\eta}{3}. \tag{7.41}$$

Therefore there exists $n_0 \geq 0$ such that for all $n \geq n_0$, $\nu_n(K_0^c) < \eta$. Also since $\nu_1, \ldots, \nu_{n_0-1}$ are finite measures there exist $K_i \subset \mathbb{R}^d$ compact such that $\nu_i(K_i^c) < \eta$ Then $K = \bigcup_{i=0}^{n_0-1} K_i$ is compact and

$$\sup_n \nu_n(K^c) < \eta. \tag{7.42}$$

Now

$$\begin{aligned} \sup_n \eta^{\frac{1}{4}} \mu_n h_t^{-1}\left(X(K^c) > \eta^{\frac{1}{4}}\right) &\leq \sup_n E_{\mu_n h_t^{-1}}\left[X(K^c) I_{X(K^c)>\eta^{\frac{1}{4}}}\right] \leq \sup_n E_{\mu_n h_t^{-1}}[X(K^c)] \\ &= \sup_n \nu_n(K^c) < \eta. \end{aligned} \tag{7.43}$$

Dividing through by $\eta^{\frac{1}{4}}$ we get that

$$\sup_n \mu_n h_t^{-1}\left(X(K^c) > \eta^{\frac{1}{4}}\right) < \eta^{\frac{3}{4}}. \tag{7.44}$$

Choose $\eta^{\frac{1}{4}} = \frac{1}{2^j}$. Then there exists $K_j \subset \mathbb{R}^d$ compact such that

$$\sup_n \mu_n h_t^{-1}\left(X(K_j^c) > \frac{1}{2^j}\right) \leq \frac{1}{2^{3j}}. \tag{7.45}$$

Choose $m > \frac{\log \frac{2}{\epsilon}}{\log 8} + 1$ so that $\frac{1}{8^{m-1}} < \frac{\epsilon}{2}$ and let

$$\mathbf{K} = \bigcap_{j \geq m} \left\{X : X(K_j^c) \leq \frac{1}{2^j}\right\} \bigcap \{X : X(\mathbb{R}^d) \leq M\}. \tag{7.46}$$

Now $\mathbf{K}$ is (sequentially) compact (see for example in the proof of Theorem II.4.1 of [27]), and

$$\mathbf{K}^c = \bigcup_{j \geq m} \left\{X : X(K_j^c) > \frac{1}{2^j}\right\} \bigcup \left\{X : X(\mathbb{R}^d) > M\right\}. \tag{7.47}$$

Thus,

$$\begin{aligned} \sup_n \mu_n h_t^{-1}(\mathbf{K}^c) \leq& \sup_n \mu_n h_t^{-1} \left( \bigcup_{j \geq m} \left\{X : X(K_j^c) > \frac{1}{2^j}\right\}\right) \\ &+ \sup_n \mu_n h_t^{-1}\left(\{X : X(\mathbb{R}^d) > M\}\right) \\ \leq& \sup_n \sum_{j=m}^{\infty} \mu_n h_t^{-1}\left(\left\{X : X(K_j^c) > \frac{1}{2^j}\right\}\right) + \frac{\epsilon}{2} \\ \leq& \sum_{j=m}^{\infty} \frac{1}{2^{3j}} + \frac{\epsilon}{2} \leq \frac{1}{8^{m-1}} + \frac{\epsilon}{2} < \epsilon, \end{aligned} \tag{7.48}$$

which verifies that the $\mu_n h_t^{-1}$ are spatially tight, for $m = 1$.

For $m > 1$, and $\vec{t} \in (0,\infty)^m$, We have from (7.48) that for each $i \in \{1,\ldots,m\}$ there exists $\mathbf{K_i} \subset M_F(\mathbb{R}^d)$ compact such that $\sup_n \mu_n h_{t_i}^{-1}(\mathbf{K_i}^c) < \frac{\epsilon}{m}$. Let $\mathbf{K} = \mathbf{K_1} \times \mathbf{K_2} \times \cdots \times \mathbf{K_m}$. Then $\mathbf{K} \subset (M_F(\mathbb{R}^d))^m$ is compact and

$$\begin{aligned} \sup_n \mu_n h_{\vec{t}}^{-1}\left(\{\vec{X} : \vec{X} \in \mathbf{K}^c\}\right) &= \sup_n \mu_n h_{\vec{t}}^{-1}\left(\bigcup_{i=1}^m \{\vec{X} : X_i \in \mathbf{K_i}^c\}\right) \\ &\leq \sup_n \sum_{i=1}^m \mu_n h_{\vec{t}}^{-1}\left(\{\vec{X} : X_i \in \mathbf{K_i}^c\}\right) \\ &= \sup_n \sum_{i=1}^m \mu_n h_{t_i}^{-1}(\mathbf{K_i}^c) \leq \sum_{i=1}^m \sup_n \mu_n h_{t_i}^{-1}(\mathbf{K_i}^c) < \epsilon, \end{aligned} \tag{7.49}$$

which gives the result.

**Lemma 7.3.6.** *If the hypotheses of Proposition 7.3.3 hold for $\mu_n, \mu \in M_F(D(M_F(\mathbb{R}^d)))$, and if for fixed $\vec{t} \in (0,\infty)^l$, $\mu_{n_k} h_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \nu$ in $M_F\left((M_F(\mathbb{R}^d))^l\right)$, then for each $\vec{m} \in \mathbb{Z}_+^l$*

$$E_\nu \left[ \prod_{i=1}^{l} \prod_{j=1}^{m_i} X_i(\phi_{ij}) \right] = E_{\mu h_{\vec{t}}^{-1}} \left[ \prod_{i=1}^{l} \prod_{j=1}^{m_i} X_i(\phi_{ij}) \right]. \tag{7.50}$$

*Proof.* Let $\mu_{n_k} h_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \nu$. Then in particular we have $\mu_{n_k} h_{\vec{t}}^{-1}(1) \to \nu(1)$. Assume $\nu(1) \neq 0$. Then there exists $k_0$ such that for every $k \geq k_0$, $\frac{1}{2}\nu(1) \leq \mu_{n_k} h_{\vec{t}}^{-1}(1) \leq 2\nu(1)$ and we define for $k \geq k_0$ the probability measures,

$$P_{n_k}(\bullet) = \frac{\mu_{n_k} h_{\vec{t}}^{-1}(\bullet)}{\mu_{n_k} h_{\vec{t}}^{-1}(1)}, \quad P(\bullet) = \frac{\nu(\bullet)}{\nu(1)}. \tag{7.51}$$

Then we have that $P_{n_k} \overset{w}{\Longrightarrow} P$ as probability measures. Let $\vec{X}_{n_k} \sim P_{n_k}$ and $\vec{X} \sim P$. Then $\vec{X}_{n_k} \overset{\mathcal{D}}{\longrightarrow} \vec{X}$ and since $(M_F(\mathbb{R}^d))^l$ is separable, we may assume that $\vec{X}_{n_k}$ and $\vec{X}$ are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By Corollary 1 of Theorem 5.1 of [3], $F(\vec{X}^n) \overset{\mathcal{D}}{\longrightarrow} F(\vec{X})$ for every $F$ such that $\mathbb{P}(\vec{X} \in \mathcal{D}_F) = 0$, i.e. such that $P(\mathcal{D}_F) = 0$, where $\mathcal{D}_F$ denotes the set of discontinuities of $F$. We apply this to the continuous function $F_{\vec{\phi}} : \vec{X} \to \prod_{i=1}^{l} \prod_{j=1}^{m_i} X_i(\phi_{ij})$.

We now show that $E_{\mathbb{P}}\left[F_{\vec{\phi}}(\vec{X}_{n_k})\right] \to E_{\mathbb{P}}\left[F_{\vec{\phi}}(\vec{X})\right]$. By Example 7.10(15) of [10], it is enough to show that $\sup_k E_{\mathbb{P}}\left[\left(F_{\vec{\phi}}(\vec{X}_{n_k})\right)^2\right] < \infty$. But,

$$\begin{aligned} \sup_k E_{\mathbb{P}}\left[\left(F_{\vec{\phi}}(\vec{X}_{n_k})\right)^2\right] &= \sup_k E_{P_{n_k}}\left[\left(F_{\vec{\phi}}(\vec{X})\right)^2\right] \\ &= \sup_k \frac{E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[\left(\prod_{i=1}^{l} \prod_{j=1}^{m_i} X_i(\phi_{ij})\right)^2\right]}{\mu_{n_k} h_{\vec{t}}^{-1}(1)} < \infty, \end{aligned} \tag{7.52}$$

since $(\prod_{i=1}^{l} \prod_{j=1}^{m_i} X_i(\phi_{ij}))^2$ is also a polynomial of the form appearing in (7.34).

Thus we have

$$E_{\mathbb{P}}\left[F_{\vec{\phi}}(\vec{X}_{n_k})\right] \to E_{\mathbb{P}}\left[F_{\vec{\phi}}(\vec{X})\right], \tag{7.53}$$

which implies that

$$E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[F_{\vec{\phi}}(\vec{X})\right] \to E_\nu\left[F_{\vec{\phi}}(\vec{X})\right]. \tag{7.54}$$

Since we also have $E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[F_{\vec{\phi}}(\vec{X})\right] \to E_{\mu h_{\vec{t}}^{-1}}\left[F_{\vec{\phi}}(\vec{X})\right]$, we have verified the claim in the case $\nu(1) \neq 0$.

Consider now the case that $\nu(1) = 0$. Then $\nu$ is the zero measure and we have $E_\nu\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = 0$. By Cauchy-Schwarz,

$$\left(E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right]\right)^2 \le E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[1^2\right] E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[\left(\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right)^2\right]. \tag{7.55}$$

Since 1 is a bounded continuous function and $\mu_{n_k} h_{\vec{t}}^{-1} \to 0_M$ we have that the first expectation on the right converges to 0. Since $\sup_k E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[\left(\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right)^2\right] < \infty$ we obtain

$$E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \to 0. \tag{7.56}$$

Since also

$$E_{\mu_{n_k} h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \to E_{\mu h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right], \tag{7.57}$$

we have that $E_{\mu h_{\vec{t}}^{-1}}\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = 0 = E_\nu\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right]$ which verifies the result. □

**Lemma 7.3.7.** *Suppose $\mu, \mu' \in M_F\left((M_F(\mathbb{R}^d))^l\right)$. If*

$$E_\mu\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = E_{\mu'}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \tag{7.58}$$

*holds (and both quantities are finite) for every $\vec{\phi} \in \mathcal{F}^{\sum m_i}$ then (7.58) holds for every $\vec{\phi}$ such that for each $i, j$, $0 \le \phi_{ij} \le 1$ is continuous.*

*Proof.* Applying Fubini to (7.58), using the facts that $E_\mu[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(1)] < \infty$ (by choosing $\phi_i = C_{\mathcal{F}} \neq 0$, the constant function in $\mathcal{F}$) and the $\phi_i$ are bounded we have

$$\int \cdots \int \prod_{i=1}^{l}\prod_{j=1}^{m_i} \phi_{ij} E_\mu\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(dx_{ij})\right] = \int \cdots \int \prod_{i=1}^{l}\prod_{j=1}^{m_i} \phi_{ij} E_{\mu'}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(dx_{ij})\right]. \tag{7.59}$$

Since $\mathcal{F}$ is a determining class for $M_F(\mathbb{R}^d)$ one can verify that $\mathcal{F}^{\sum m_i}$ is a determining class for $M_F(\mathbb{R}^{d\sum m_i})$ (using the fact that this class of functions determines the conditional distribution of the $n$th coordinate given the first $n-1$ and proceeding by induction).

Now

$$\left\{\prod_{i=1}^{l}\prod_{j=1}^{m_i}\phi_{ij} : \phi_{ij}\in\mathcal{F}\right\} = \left\{\prod_{k=1}^{\sum m_i}\phi_k : \phi_k\in\mathcal{F}\right\} = \mathcal{F}^{\sum m_i}, \tag{7.60}$$

so the products of $\phi_{ij}$ in (7.59) uniquely determine the measure defined by $\nu(d\vec{x}) = E_\mu[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(dx_{ij})]$. Therefore (7.59) holds for all $\phi_i$ bounded, continuous, so in particular for all continuous $0 \le \phi_i \le 1$. Applying Fubini again we get the result. □

**Lemma 7.3.8.** *Suppose $\mu, \mu' \in M_F\left((M_F(\mathbb{R}^d))^m\right)$ and assume $D_0 \subset (\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+))^m$ satisfies $\overline{D_0}^{bp} = (\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+))^m$. If for all $\vec{\phi} \in D_0$*

$$E_\mu\left[e^{-\sum_{j=1}^m X_i(\phi_j)}\right] = E_{\mu'}\left[e^{-\sum_{j=1}^m X_i(\phi_j)}\right], \tag{7.61}$$

*then $\mu = \mu'$.*

*Proof.* We follow the proof of Lemma II.5.9 of [27].
**(a) (7.61) holds for every $\vec{\phi} \in (C_b(\mathbb{R}^d, \mathbb{R}_+))^m$.** We verify the stronger result that the class $\mathcal{L}$ of $\vec{\phi}$ for which (7.61) holds contains $(\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+))^m$.

Let $\vec{\phi}_n \in \mathcal{L}$ be such that $\vec{\phi}_n \overset{bp}{\to} \vec{\phi}$. Now by dominated convergence (using the fact that $\mu$ is a finite measure and dominating by $e^0 = 1$),

$$\begin{aligned} E_\mu\left[e^{-\sum_{j=1}^m X_i(\phi_j)}\right] &= E_\mu\left[\lim_{n\to\infty} e^{-\sum_{j=1}^m X_i(\phi_{j,n})}\right] \\ &= \lim_{n\to\infty} E_\mu\left[e^{-\sum_{j=1}^m X_i(\phi_{j,n})}\right] \\ &= \lim_{n\to\infty} E_{\mu'}\left[e^{-\sum_{j=1}^m X_i(\phi_{j,n})}\right] \\ &= E_{\mu'}\left[e^{-\sum_{j=1}^m X_i(\phi_j)}\right]. \end{aligned} \tag{7.62}$$

Thus $\mathcal{L}$ is closed under bounded pointwise convergence. Since $D_0 \subset \mathcal{L}$ by hypothesis this shows that $(\mathcal{B}_b(\mathbb{R}^d, \mathbb{R}_+))^m \subset \mathcal{L}$ as required.

Define $e_{\vec{\phi}} : (M_F(\mathbb{R}^d))^m \to \mathbb{R}_+$ by $e_{\vec{\phi}}(\vec{\nu}) = e^{-\sum_{j=1}^m \nu_j(\phi_j)}$. Now let

$$\mathcal{H} = \{\Phi \in \mathcal{B}_b((M_F(\mathbb{R}^d))^m, \mathbb{R}) : E_\mu[\Phi(\vec{X})] = E_{\mu'}[\Phi(\vec{X})]\} \tag{7.63}$$

and

$$\mathcal{H}_0 = \{e_{\vec{\phi}} : \vec{\phi} \in (C_b(\mathbb{R}^d, \mathbb{R}_+))^m\}. \tag{7.64}$$

**(b) $\mathcal{H}$ contains all bounded $\sigma(\mathcal{H}_0)$ measurable functions**. We show that $\mathcal{H}$ is a linear class containing 1, closed under $\overset{bp}{\to}$, and that $\mathcal{H}_0 \subset \mathcal{H}$ is closed under products. Once we achieve this, we have by Lemma II.5.2 of [27] that $\mathcal{H}$ contains all bounded $\sigma(\mathcal{H}_0)$-measurable functions.

1) that $\mathcal{H}$ is a linear class is immediate by linearity of the integral.

2) $1 \in \mathcal{H}$ by taking $\vec{\phi} = \vec{0}$ and using part (a).

3) Let $\Phi_n \in \mathcal{H}, \Phi_n \overset{bp}{\to} \Phi$. Then $\Phi \in \mathcal{H}$ by dominated convergence since $\mu, \mu'$ are finite measures.

4) Let $f_1, f_2 \in \mathcal{H}_0$. Then $f_i = e_{\phi_i}$ and

$$f_1 f_2 = e^{-\sum_{j=1}^m X_j(\phi_{j,1})} e^{-\sum_{j=1}^m X_j(\phi_{j,2})} = e^{-\sum_{j=1}^m X_j(\phi_{j,1}+\phi_{j,2})} = e_{\phi_1+\phi_2} \in \mathcal{H}_0. \tag{7.65}$$

5) $\mathcal{H}_0 \subset \mathcal{H}$ was verified in part (a).

**(c) There exists a countable convergence determining set for** $(M_F(\mathbb{R}^d))^m$. We use the construction of Proposition 3.4.4 of [8] to obtain a countable set $V \subset (C_b(\mathbb{R}^d, \mathbb{R}_+))^m$ such that $\vec{\nu}_n \to \vec{\nu}$ in $(M_F(\mathbb{R}^d))^m$ if and only if $\vec{\nu}_n(\vec{\phi}) \to \vec{\nu}(\vec{\phi})$ for every $\vec{\phi} \in V$). Let $\{\vec{q}_1, \vec{q}_2, \dots\}$ be an enumeration of $\mathbb{Q}^d$, a dense subset of $\mathbb{R}^d$. For each $(i,j) \in \mathbb{N}^2$ define

$$f_{i,j}(\vec{x}) = 2\,(1 - j|\vec{x} - \vec{q}_i|) \vee 0, \tag{7.66}$$

and for $A \subset \mathbb{N}^2$ define

$$g_A^m(\vec{x}) = \left( \sum_{\substack{i,j \leq m \\ (i,j) \in A}} f_{i,j} \right) \wedge 1. \tag{7.67}$$

It is an exercise left for the reader to verify that

$$V_0 = \{g_A^m : m \in \mathbb{N}, A \subset \{1, \dots, m\}^2\} \subset C_b(\mathbb{R}^d), \tag{7.68}$$

is a countable convergence determining set for $M_F(\mathbb{R}^d)$. It follows that $V = \{(\phi_1, \dots, \phi_m) : \phi_i \in V_0 \cup \{0\}\}$ is a countable convergence determining set for $(M_F(\mathbb{R}^d))^m$.

Define

$$\mathcal{G} = \sigma(e_{\vec{\phi}} : \vec{\phi} \in V). \tag{7.69}$$

**(d)** $\mathcal{B}((M_F(\mathbb{R}^d))^m) \subset \mathcal{G} \subset \sigma(\mathcal{H}_0)$**, where** $\mathcal{G} = \sigma(e_{\vec{\phi}} : \vec{\phi} \in V)$. The second inclusion is trivial since $V \subset (C_b(\mathbb{R}^d, \mathbb{R}))^m$. We claim that $\mathcal{G}$ contains all the open sets in

$(M_F(\mathbb{R}^d))^m$ and hence contains $\mathcal{B}((M_F(\mathbb{R}^d))^m)$. Define the metric

$$\varrho'(\vec{\mu},\vec{\nu}) = \sum_{j=1}^{m}\sum_{n=1}^{\infty}\frac{|\int \phi_n d\mu_i - \int \phi_n d\nu_i|}{2^n}, \tag{7.70}$$

where $\{\phi_1, \phi_2, \dots\}$ is some fixed enumeration of $V_0 \cup \{0\}$. It is a standard result that $\varrho'$ induces the topology of weak convergence.

Let $U$ be an open set in the topology of weak convergence. Then $U$ is also open in $((M_F(\mathbb{R}^d))^m, \varrho')$. Now $M_F(\mathbb{R}^d)$ is separable so every open set is a countable union of balls $B_{\varrho'}(\vec{\nu}, r)$ and therefore to show that $U \in \mathcal{G}$, it is enough to show that $B_{\varrho'}(\vec{\nu}, r) \in \mathcal{G}$. But

$$B_{\varrho'}(\vec{\nu}, r) = \left\{\vec{\mu} : \sum_{j=1}^{m}\sum_{n=1}^{\infty}\frac{|\int \phi_n d\mu_j - \int \phi_n d\nu_j|}{2^n} < r\right\} \in \mathcal{G} \tag{7.71}$$

since an infinite series of measurable functions is measurable. We have now verified that $\mathcal{G}$ contains all the open sets of $(M_F(\mathbb{R}^d))^m$ and therefore contains $\mathcal{B}((M_F(\mathbb{R}^d))^m)$.
**(e) Conclusion**. We have now verified that $\mathcal{B}((M_F(\mathbb{R}^d))^m) \subset \mathcal{G} \subset \sigma(\mathcal{H}_0)$. Therefore every bounded continuous function is measurable with respect to $\sigma(\mathcal{H}_0)$. Furthermore we have that $\mathcal{H}$ contains all $\sigma(\mathcal{H}_0)$-measurable functions (and in particular all the bounded continuous functions). Since $\mu = \mu'$ if and only if $\int f d\mu = \int f d\mu'$ for all bounded continuous $f : (M_F(\mathbb{R}^d))^m \to \mathbb{R}$, we have proved the result.

**Lemma 7.3.9.** *Let $\mu \in M_F\left((M_F(\mathbb{R}^d))^m\right)$. Suppose there exists a $\delta > 0$ such that for all $\theta_i < \delta$,*

$$E_\mu[e^{\sum_{i=1}^m \theta_i X_i(\mathbb{R}^d)}] < \infty. \tag{7.72}$$

*Then for every $0 \le \varphi_i$ bounded, continuous, the quantity*

$$E_\mu\left[e^{-\sum_{i=1}^m X_i(\varphi_i)}\right] \tag{7.73}$$

*is uniquely determined by the mixed moments (7.76) of $\vec{X}(\vec{\phi})$, $0 \le \phi_i \le 1$ continuous, $i = 1, \dots, m$.*

*Proof.* Fix $\vec{\phi} = (\phi_1, \dots, \phi_m)$ and let $\vec{x} = \vec{X}(\vec{\phi})$. Then for all $\vec{z} \in \mathbb{C}^m$,

$$E_\mu\left[e^{\vec{X}(\vec{\phi})\cdot\vec{z}}\right] = E_\mu\left[\sum_{l=0}^{\infty}\frac{(\vec{z}\cdot\vec{x})^l}{l!}\right]. \tag{7.74}$$

By dominated convergence with (7.72), and using the Multinomial Theorem, we have for $\|\vec{z}\|_\infty < \delta$ that the right hand side is equal to

$$\sum_{l=0}^{\infty} \frac{1}{l!} E_\mu \left[ (\vec{z}\cdot\vec{x})^l \right] = \sum_{l=0}^{\infty} \frac{1}{l!} E_\mu \left[ \sum_{\substack{\vec{n}\in\mathbb{Z}_+^m: \\ \sum n_i = l}} \frac{l!}{\prod_{i=1}^m n_i!} \prod_{i=1}^m (z_i x_i)^{n_i} \right], \tag{7.75}$$

which is a multivariable power series in $z_1, \ldots z_m$ with coefficients that are linear combinations of quantities of the form

$$E_\mu \left[ \prod_{i=1}^m x_i^{n_i} \right] = E_\mu \left[ \prod_{i=1}^m X_i(\phi_i)^{n_i} \right]. \tag{7.76}$$

Now let $\|\vec{z}\|_\infty < \delta$ and note that $0 \le x_i = X_i(\phi_i) \le X_i(\mathbb{R}^d)$ for $0 \le \phi_i \le 1$. Then

$$\begin{aligned} &\lim_{\Delta z_i \to 0} \left| \int \frac{e^{(\vec{z}+\Delta z_i)\cdot\vec{x}} - e^{\vec{z}\cdot\vec{x}}}{\Delta z_i} d\mu - \int x_i e^{\vec{z}\cdot\vec{x}} d\mu \right| \\ &= \lim_{\Delta z_i \to 0} \left| \int e^{\vec{z}\cdot\vec{x}} \left[ \frac{e^{\Delta z_i \cdot x_i} - 1}{\Delta z_i} - x_i \right] d\mu \right| \\ &= \lim_{\Delta z_i \to 0} \left| \int e^{\vec{z}\cdot\vec{x}} \left[ \Delta z_i \sum_{l=2}^{\infty} \frac{(\Delta z_i)^{l-2} x_i^l}{l!} \right] d\mu \right| \\ &\le \lim_{\Delta z_i \to 0} |\Delta z_i| \int e^{\sum_{j=1}^m Re(z_j) x_j} x_i^2 e^{|\Delta z_i| x_i} d\mu \\ &= \lim_{\Delta z_i \to 0} |\Delta z_i| \int e^{(Re(z_i)+\epsilon+|\Delta z_i|)x_i + \sum_{j\ne i}^m Re(z_j) x_j} x_i^2 e^{-\epsilon x_i} d\mu. \end{aligned} \tag{7.77}$$

Now $x_i^2 e^{-\epsilon x_i} \le C_\epsilon$ so the integral converges for all $\vec{z}$ such that $Re(z_i) + \epsilon + |\Delta z_i| < \delta$ and $Re(z_j) < \delta$ for all $j \ne i$. Thus the limit in the above is zero.

Choosing $i = 1$ we get that for fixed $\vec{z}_{-1} = (z_2, \ldots z_m)$ with $\|\vec{z}_{-1}\|_\infty < \delta$, $\varphi^1(\vec{z}) = \int e^{\vec{z}\cdot\vec{x}} d\mu$ is analytic in $z_1$ such that $Re(z_1) + \epsilon < \delta$ for every $\epsilon > 0$, and thus in particular for $z_1$ such that $Re(z_1) < 0$. In particular, $\varphi^1(\vec{z})$ is the analytic continuation (in $z_1$) of $\int e^{\vec{z}\cdot\vec{x}} d\mu$ for $\|\vec{z}\|_\infty < \delta$ and as such $\varphi^1(\vec{z})$ is uniquely determined by the moments (7.76).

For $1 < i \le m$, and fixed $z_j$ such that $Re(z_j) < 0$ for $j < i$ and $|z_j| < \delta$ for $j > i$ suppose we have $\varphi^{i-1}(\vec{z})$ is analytic in each $z_j$ in the regions $Re(z_j) < 0$ for $j < i$ and $|z_j| < \delta$ for $j \ge i$. Then we define $\varphi^i(\vec{z})$ as follows.

As in the last line of (7.77), and using the fact that $z_j x_j \le 0$ for $j < i$, we have

$$\lim_{\Delta z_i \to 0} \left| \int \frac{e^{(\vec{z}+\Delta z_i)\cdot \vec{x}} - e^{\vec{z}\cdot\vec{x}}}{\Delta z_i} d\mu - \int x_i e^{\vec{z}\cdot\vec{x}} d\mu \right| \\ \leq \lim_{\Delta z_i \to 0} |\Delta z_i| \int e^{(Re(z_i)+\epsilon+|\Delta z_i|)x_i + \sum_{j>i}^m Re(z_j)x_j} x_i^2 e^{-\epsilon x_i} d\mu. \tag{7.78}$$

This integral converges for all $\vec{z}$ such that $Re(z_i) + \epsilon + |\Delta z_i| < \delta$ and $Re(z_j) < \delta$ for $j > i$. Thus for $\vec{z}$ such that $Re(z_1) < 0, \ldots, Re(z_{i-1}) < 0$ and fixed $|z_j| < \delta$ for $j > i$ the function

$$\varphi^i(\vec{z}) = \int e^{\vec{z}\cdot\vec{x}} d\mu \tag{7.79}$$

is analytic in each $z_j$ in the region $Re(z_j) < 0$ for $j \leq i$ and $|z_j| < \delta$ for $j > i$, and is the analytic continuation of $\varphi^{i-1}(\vec{z})$, as a function of $z_i$. As such, $\varphi^{i-1}$ is uniquely determined by the moments (7.76).

Therefore we have $\varphi^m(\vec{z}) = \int e^{\vec{z}\cdot\vec{x}} d\mu$ is analytic in each $z_j$ in the region $Re(z_j) < 0$, and is uniquely determined by the moments (7.76).

Thus for each $\vec{z}$ with $Re(z_j) < 0$ for each $j$, and every $\vec{\phi}$ with $0 \leq \phi_j \leq 1$ continuous for each $j$, we have that

$$\int e^{\vec{z}\cdot\vec{X}(\vec{\phi})} d\mu = \int e^{-\sum_{j=1}^m X_j(-z_j \phi_j)} d\mu \tag{7.80}$$

is uniquely determined by the moments (7.76). Therefore for each $\vec{\phi}'$ such that $\phi'_j$ is bounded, nonnegative, and continuous we have

$$\int e^{-\sum_{j=1}^m X_j(\phi'_j)} d\mu \tag{7.81}$$

is uniquely determined by the moments (7.76).

## 7.4 Proof of Theorem 1.3.1

In this section we use Proposition 7.3.3 together with the convergence of the $r$-point fucntions (Theorems 1.4.3 and 1.4.5) to prove Theorem 1.3.1.

The first hypothesis of Proposition 7.3.3 is the existence of an exponential moment for the limiting measure. The following Lemma will be used to verify this hypothesis in the proof of Proposition 7.3.3.

**Lemma 7.4.1.** *For every $\epsilon > 0$ the following hold.*

1. *For every $\lambda > 0$, $\mathbb{N}_0^\epsilon(X_\epsilon(1) = \lambda) = 0$.*

*2. For every $\vec{t} \in (\epsilon, \infty)^m$ there exists a $\delta(\vec{t}) > 0$ such that for all $\|\vec{\theta}\|_\infty < \delta$,*

$$E_{\mathbb{N}_0}\left[X_\epsilon(1)e^{\sum_{i=1}^m \theta_i X_{t_i}(1)}\right] < \infty. \tag{7.82}$$

*Proof.* By Theorem II.7.2(iii) of [27] we have

$$\mathbb{N}_0\left(X_\epsilon(1) \in A\right) = \frac{c}{\epsilon}\int_A e^{cx}dx, \tag{7.83}$$

so the first assertion is trivial.

The second assertion of Lemma 7.4.1 is also a standard result and can be proved using the Markov property of the *local time* of the Brownian excursion under Ito's excursion measure, or the fact that $\mathbb{N}_0$ is an entrance law for SBM. We choose to give a direct and elementary calculation relying on the representation of SBM as a Poisson Point Process of excursions with intensity $\mathbb{N}_0$ (see (7.84)).

Let $\vec{t} = (t_1, \ldots, t_m) \in (\epsilon, \infty)^m$ and set $t_0 = \epsilon$. Then Theorem II.7.3(c) of [27] implies that for $\theta_i \geq 0$,

$$E_{\delta_0}\left[e^{\sum_{i=0}^m \theta_i Y_{t_i}(1)}\right] = \exp\left\{\int e^{\sum_{i=0}^m \theta_i \nu_{t_i}(1)} - 1 \, d\mathbb{N}_0(\nu)\right\}, \tag{7.84}$$

where $\{Y_t\}_{t\geq 0}$ is a super-Brownian motion starting at $\delta_0$ (i.e. with initial law $\delta_{\delta_0}$). Lemma III.3.6 of [27] with Cauchy-Schwarz and with $f_i = \theta_i$ (constant functions) implies that the expressions in (7.84) are finite provided $\|\vec{\theta}\|_\infty \leq \frac{2c_0}{\|\vec{t}\|_\infty}$ where $c_0$ is some constant depending on $m$. Therefore for $\|\vec{\theta}\|_\infty \leq \frac{c_0}{\|\vec{t}\|_\infty}$ a standard application of the Dominated Convergence Theorem allows us to take differentiation through the integral on the left side of (7.84) and obtain

$$\frac{d}{d\theta_0^+}E_{\delta_0}\left[e^{\sum_{i=0}^m \theta_i Y_{t_i}(1)}\right]\Bigg|_{\theta_0=0} = E_{\delta_0}\left[Y_\epsilon(1)e^{\sum_{i=1}^m \theta_i Y_{t_i}(1)}\right]. \tag{7.85}$$

That this quantity is finite follows easily from the fact that (7.84) is finite. The derivative of the right side of (7.84) is

$$\exp\left\{\int e^{\sum_{i=0}^m \theta_i \nu_{t_i}(1)} - 1 \, d\mathbb{N}_0(\nu)\right\}\frac{d}{d\theta_0^+}\int e^{\sum_{i=0}^m \theta_i \nu_{t_i}(1)} - 1 \, d\mathbb{N}_0(\nu)\Bigg|_{\theta_0=0}. \tag{7.86}$$

Therefore we have

$$\begin{aligned}\frac{d}{d\theta_0^+}\int e^{\sum_{i=0}^m \theta_i \nu_{t_i}(1)} - 1 \, d\mathbb{N}_0(\nu)\Bigg|_{\theta_0=0} &= \frac{E_{\delta_0}\left[Y_\epsilon(1)e^{\sum_{i=1}^m \theta_i Y_{t_i}(1)}\right]}{E_{\delta_0}\left[e^{\sum_{i=0}^m \theta_i Y_{t_i}(1)}\right]} \\ &\equiv H(\epsilon, \vec{t}, \vec{\theta}) < \infty.\end{aligned} \tag{7.87}$$

By Fatou's Lemma we have

$$\begin{aligned} H(\epsilon,\vec{t},\vec{\theta}) &= \lim_{\theta_0\searrow 0}\int e^{\sum_{i=1}^m \theta_i\nu_{t_i}(1)}\left[\frac{e^{\theta_0\nu_\epsilon(1)}-1}{\theta_0}\right]\,d\mathbb{N}_0(\nu) \\ &\geq \int e^{\sum_{i=1}^m \theta_i\nu_{t_i}(1)}\liminf_{\theta_0\searrow 0}\left[\frac{e^{\theta_0\nu_\epsilon(1)}-1}{\theta_0}\right]\,d\mathbb{N}_0(\nu) \\ &= \int e^{\sum_{i=1}^m \theta_i\nu_{t_i}(1)}\nu_\epsilon(1)\,d\mathbb{N}_0(\nu). \end{aligned} \tag{7.88}$$

Thus for $\|\theta\|_\infty \leq \frac{c_0}{\|\vec{t}\|_\infty}$ we have

$$E_{\mathbb{N}_0}\left[X_\epsilon(1)e^{\sum_{i=1}^m \theta_i X_{t_i}(1)}\right] \leq H(\epsilon,\vec{t},\vec{\theta}) < \infty, \tag{7.89}$$

as required. $\square$

Recall the statement of Theorem 1.3.1, where $\mathcal{D}_F$ is the discontinuity set of $F$. In Section 7.2 we restated this theorem using the notation of this chapter as follows.

**Theorem (1.3.1).** *There exists $L_0 \gg 1$ such that for every $L \geq L_0$, with $\mu_n$ defined by (1.17) the following holds: For every $\epsilon,\lambda > 0$, $m \in \mathbb{N}$, $\vec{t} \in (\epsilon,\infty)^m$ and every $F : \left(M_F(\mathbb{R}^d)^m\right) \to \mathbb{R}$ bounded by a polynomial and such that $\mathbb{N}_0 h_{\vec{t}}^{-1}(\mathcal{D}_F) = 0$,*

*1.*

$$E_{\mu_n h_{\vec{t}}^{-1}}\left[X_\epsilon(1)F(\vec{X})\right] \to E_{\mathbb{N}_0 h_{\vec{t}}^{-1}}\left[X_\epsilon(1)F(\vec{X})\right], \tag{7.90}$$

*and*

*2.*

$$E_{\mu_n h_{\vec{t}}^{-1}}\left[F(\vec{X})I_{\{X_\epsilon(1)>\lambda\}}\right] \to E_{\mathbb{N}_0 h_{\vec{t}}^{-1}}\left[F(\vec{X})I_{\{X_\epsilon(1)>\lambda\}}\right]. \tag{7.91}$$

*Proof.* Define $\mu_n^\epsilon, \mathbb{N}_0^\epsilon \in M_F\left(D(M_F(\mathbb{R}^d))\right)$ by

$$\mu_n^\epsilon(A) = \int_A X_\epsilon(\mathbb{R}^d)d\mu_n, \quad \mathbb{N}_0^\epsilon(A) = \int_A X_\epsilon(\mathbb{R}^d)d\mathbb{N}_0. \tag{7.92}$$

That these measures are finite (in fact uniformly bounded) follows from the fact that

$$\mu_n^\epsilon\left(D(M_F(\mathbb{R}^d))\right) = E_{\mu_n}\left[X_\epsilon(1)\right] \to E_{\mathbb{N}_0}\left[X_\epsilon(1)\right] < \infty. \tag{7.93}$$

We take $\mathcal{F} = \{e^{ik\cdot x} : k \in \mathbb{R}^d\}$ which is a convergence determining class of bounded $\mathbb{C}$-valued functions containing the constant function $C_{\mathcal{F}} = 1$. Now for all

$l \geq 0$, $\vec{m} \in \mathbb{Z}_+^l$,

$$E_{\mu_n^\epsilon h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = E_{\mu_n^\varepsilon}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right] = E_{\mu_n}\left[X_\epsilon(1)\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right]$$
$$\to E_{\mathbb{N}_0}\left[X_\epsilon(1)\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_{t_i}(\phi_{ij})\right]$$
$$= E_{\mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}}\left[\prod_{i=1}^{l}\prod_{j=1}^{m_i} X_i(\phi_{ij})\right], \tag{7.94}$$

where even in the $l = 0$ case, the presence of the factor $X_\varepsilon(1)$ ensures that the convergence in (7.94) is a statement of convergence of $r$-point functions. By Lemma 7.3.5 the measures $\{\mu_n^\epsilon h_{\vec{t}}^{-1}\}$ are spatially tight. Since they are also uniformly bounded by (7.93), $\{\mu_n^\epsilon h_{\vec{t}}^{-1}\}$ are in fact tight.

By Lemma 7.4.1 we have that

$$E_{\mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}}\left[e^{\sum_{i=1}^m \theta_i X_i(1)}\right] < \infty, \tag{7.95}$$

for all $\vec{\theta}$ sufficiently small depending on $\vec{t}$. In view of (7.93), (7.94) and (7.95) we may apply Proposition 7.3.3 to the measures $\mu_n^\epsilon, \mathbb{N}_0^\epsilon$ to get

$$\mu_n^\epsilon h_{\vec{t}}^{-1} \stackrel{w}{\Longrightarrow} \mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}. \tag{7.96}$$

Thus $\mu_n^\epsilon h_{\vec{t}}^{-1}(1) \to \mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}(1)$. In particular, there exists $n_0(\epsilon, \vec{t})$ such that for $n \geq n_0$,

$$\frac{\mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}(1)}{2} \leq \mu_n^\epsilon h_{\vec{t}}^{-1}(1) \leq 2\mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}(1). \tag{7.97}$$

Therefore for $n \geq n_0$ we may define $P_{n,\vec{t}}^\epsilon \in M_1\left((M_F(\mathbb{R}^d))^m\right)$ by

$$P_{n,\vec{t}}^\epsilon(\bullet) = \frac{\mu_n^\epsilon h_{\vec{t}}^{-1}(\bullet)}{\mu_n^\epsilon h_{\vec{t}}^{-1}(1)}. \tag{7.98}$$

Now $\mu_n^\epsilon h_{\vec{t}}^{-1} \stackrel{w}{\Longrightarrow} \mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}$ implies that $P_{n,\vec{t}}^\epsilon \stackrel{w}{\Longrightarrow} P_{\vec{t}}^\epsilon$ as probability measures, where $P_{\vec{t}}^\epsilon(\bullet) = \frac{\mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}(\bullet)}{\mathbb{N}_0^\epsilon h_{\vec{t}}^{-1}(1)}$. Let $\vec{X}^n \sim P_{n,\vec{t}}^\epsilon$ and $\vec{X} \sim P_{\vec{t}}^\epsilon$. Then we have $\vec{X}^n \xrightarrow{\mathcal{D}} \vec{X}$, and since $(M_F(\mathbb{R}^d))^m$ is separable, we may assume that $\vec{X}^n$ and $\vec{X}$ are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By Corollary 1 of Theorem 5.1 of Billingsley $F(\vec{X}^n) \xrightarrow{\mathcal{D}} F(\vec{X})$ for every $F$ such that $\mathbb{P}(\vec{X} \in \mathcal{D}_F) = 0$, i.e. such that $P_{\vec{t}}^\epsilon(\mathcal{D}_F) = 0$.

We now show that if $F$ is also bounded by a polynomial, then $E_{\mathbb{P}}\left[F(\vec{X}^n)\right] \to E_{\mathbb{P}}\left[F(\vec{X})\right]$. By Example 7.10(15) of Grimmett and Stirzaker, it is enough to show that $\sup_n E_{\mathbb{P}}\left[(F(\vec{X}^n))^2\right] < \infty$. But,

$$E_{\mathbb{P}}\left[(F(\vec{X}^n))^2\right] = E_{P^{\epsilon}_{n,\vec{t}}}\left[(F(\vec{X}))^2\right] = \frac{E_{\mu^{\epsilon}_n h^{-1}_{\vec{t}}}\left[(F(\vec{X}))^2\right]}{\mu^{\epsilon}_n h^{-1}_{\vec{t}}(1)} \leq \frac{E_{\mu^{\epsilon}_n h^{-1}_{\vec{t}}}\left[(Q(\vec{X}))^2\right]}{\mu^{\epsilon}_n h^{-1}_{\vec{t}}(1)}, \tag{7.99}$$

where $Q$ is a polynomial such that $|F| \leq Q$. Since $\sup_n E_{\mu^{\epsilon}_n h^{-1}_{\vec{t}}}\left[(Q(\vec{X}))^2\right] < \infty$ we have the result.

Thus we have that

$$E_{\mathbb{P}}\left[F(\vec{X}^n)\right] \to E_{\mathbb{P}}\left[F(\vec{X})\right], \tag{7.100}$$

which implies that

$$E_{\mu^{\epsilon}_n h^{-1}_{\vec{t}}}\left[F(\vec{X})\right] \to E_{\mathbb{N}_0}\left[F(\vec{X})\right]. \tag{7.101}$$

Therefore for every function $F$ that is bounded by a polynomial and that satisfies $\mathbb{N}^{\epsilon}_0(\mathcal{D}_F) = 0$,

$$E_{\mu_n h^{-1}_{\vec{t}}}\left[X_{\epsilon}(1)F(\vec{X})\right] \to E_{\mu_n h^{-1}_{\vec{t}}}\left[X_{\epsilon}(1)F(\vec{X})\right]. \tag{7.102}$$

Define

$$F_1 \equiv \begin{cases} 0 & , \text{ if } X_{\epsilon}(1) < \lambda \\ \frac{I_{\{X_{\epsilon}(1)>\lambda\}}}{X_{\epsilon}(1)} & , \text{ otherwise.} \end{cases} \tag{7.103}$$

Then $F_1$ continuous except at $X_{\epsilon}(1) = \lambda$, and is bounded above by $\frac{1}{\lambda}$, Lemma 7.4.1 and (7.102) show that

$$\begin{aligned} E_{\mu_n h^{-1}_{\vec{t}}}\left[X_{\epsilon}(1)F_1 F\right] &\to E_{\mu_n h^{-1}_{\vec{t}}}\left[X_{\epsilon}(1)F_1 F\right], \quad \text{i.e.} \\ E_{\mu_n h^{-1}_{\vec{t}}}\left[I_{\{X_{\epsilon}(1)>\lambda\}}F\right] &\to E_{\mu_n h^{-1}_{\vec{t}}}\left[I_{\{X_{\epsilon}(1)>\lambda\}}F\right]. \end{aligned} \tag{7.104}$$

□

## 7.5 A note on convergence of finite dimensional distributions

Recall the definitions of $\mu_n$ (depending on $L$) and $\mathbb{N}_0$ in (1.17) and Definition 7.1.3 respectively. In this section give a brief discussion about how Conjecture 7.2.1 follows from the following conjecture.

**Conjecture 7.5.1.** *There exists an $L_0 \gg 1$ such that for every $L \geq L_0$, and for every $\varepsilon > 0$,*

$$\mu_n(S > \varepsilon) \to \mathbb{N}_0(S > \varepsilon). \tag{7.105}$$

Fix $\varepsilon > 0$ and define $\mu_n^\varepsilon, \mathbb{N}_0^\varepsilon \in M_F\left(D(M_F(\mathbb{R}^d))\right)$ by

$$\mu_n^\varepsilon(\bullet) = \mu_n(\bullet, S > \varepsilon), \qquad \mathbb{N}_0^\varepsilon(\bullet) = \mathbb{N}_0(\bullet, S > \varepsilon). \tag{7.106}$$

That these are finite measures follows from Theorem 7.1.2 and the fact that $\mu_n$ is a finite measure for each $n$. We wish to apply Proposition 7.3.3 to the finite measures $\mu_n^\varepsilon, \mathbb{N}_0^\varepsilon$.

Using the representation of SBM as Poisson point process with intensity $\mathbb{N}_0$, one can show that (for the $\delta(\vec{t})$ of Lemma 7.4.1) for all $\|\vec{\theta}\|_\infty < \delta$,

$$E_{\mathbb{N}_0^\varepsilon h_{\vec{t}}^{-1}}\left[e^{\sum_{i=1}^m \theta_i X_i(1)}\right] \equiv E_{\mathbb{N}_0}\left[e^{\sum_{i=1}^m \theta_i X_{t_i}(1)} I_{S>\epsilon}\right] < \infty. \tag{7.107}$$

This verifies condition 1 of Proposition 7.3.3.

The $l = 0$ ($\vec{m} = \vec{0}$) case of the second condition of Proposition 7.3.3 is provided by (7.105) since,

$$\mu_n^\varepsilon(1) = \mu_n(S > \varepsilon) \to \mathbb{N}_0(S > \varepsilon) = \mathbb{N}_0^\varepsilon(1) < \infty. \tag{7.108}$$

By definition of the finite dimensional distributions, and the fact that if $X_\varepsilon(1) = 0$ then $X_t(1) = 0$ for all $t \geq \varepsilon$, we have for $t > \varepsilon$,

$$E_{\mu_n^\varepsilon h_{\vec{t}}^{-1}}\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = E_{\mu_n h_{\vec{t}}^{-1}}\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right], \tag{7.109}$$

and

$$E_{\mathbb{N}_0^\varepsilon h_{\vec{t}}^{-1}}\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right] = E_{\mathbb{N}_0 h_{\vec{t}}^{-1}}\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right]. \tag{7.110}$$

Now $\mathcal{F} = \{e^{ik\cdot x} : k \in [-\pi, \pi]^d\}$ is a convergence determining class for $M_F(\mathbb{R}^d)$ containing 1. In view of (7.109)–(7.110), Theorems 1.4.3 and 1.4.5 show that for every $l \geq 1$, $\vec{t} \in (0,\infty)^l$, $\vec{m} \in \mathbb{Z}_+^l \setminus \vec{0}$ and every $\vec{\phi} \in \mathcal{F}^{\sum m_i}$,

$$E_{\mu_n^\varepsilon h_{\vec{t}}^{-1}}\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right] \to E_{\mathbb{N}_0^\varepsilon h_{\vec{t}}^{-1}}\left[\prod_{i=1}^l \prod_{j=1}^{m_i} X_i(\phi_{ij})\right]. \tag{7.111}$$

Applying Proposition 7.3.3 to the measures $\mu_n^\varepsilon, \mathbb{N}_0^\varepsilon$ shows that for every $m \in \mathbb{N}$ and every $\vec{t} \in (\varepsilon, \infty)^m$,

$$\mu_n^\varepsilon h_{\vec{t}}^{-1} \overset{w}{\Longrightarrow} \mathbb{N}_0^\varepsilon h_{\vec{t}}^{-1}, \tag{7.112}$$

which is exactly the statement that $\mu_n \overset{\text{f.d.d}}{\Longrightarrow} \mathbb{N}_0$ (Conjecture 7.2.1). $\square$

# Appendix A
# Extending the inductive approach

## A.1 Motivation

We have already noted in Chapter 1 why we expect a Gaussian scaling limit for our lattice trees model in dimensions $d > 8$. We have also discussed results of Derbez and Slade [7] in Chapter 7, and in particular how their analysis might be used to verify Gaussian behaviour of the 2-point and 3-point functions. An alternative method is to attempt to analyse the 2-point function by extending the inductive approach of van der Hofstad and Slade [19].

Suppose we have for every $z \in [0, 2]$ say, $\sup_{n,k} |f_n(k; z)| \leq K$ with $f_0 = 1$, and

$$f_{n+1}(k; z) = \sum_{m=1}^{n+1} g_m(k; z) f_{n+1-m}(k; z) + e_{n+1}(k; z), \quad (n \geq 0). \tag{A.1}$$

For the following nonrigorous argument we also suppose that $g_1(k; 1) = \widehat{D}(k) \approx \widehat{D}(0) - \frac{k^2\sigma^2}{2d}$, where $D$ is defined in 1.2.4, and that $e_m, g_{m+1} \approx 0$ for $m \geq 1$. Then we have $f_{n+1} \approx g_1 f_n$ and so $f_n(k) \approx g_1(k)^n \approx \left(1 - \frac{k^2\sigma^2}{2d}\right)^n$. Thus

$$f_n\left(\frac{k}{\sqrt{n\sigma^2}}\right) \approx \left(1 - \frac{k^2}{2dn}\right)^n \to e^{-\frac{k^2}{2d}}. \tag{A.2}$$

The inductive method of [19], which followed on from previous work of van der Hofstad, den Hollander and Slade [14] proves an important result detailing specific bounds on the quantities appearing in the recursion equation (A.1), that ensures that there exists a critical $z_c \approx 1$ at which $f_n\left(\frac{k}{\sqrt{n\sigma^2}}; z_c\right) \to e^{-\frac{k^2}{2d}}$. The result of [19] is applied to sufficiently spread out models of self-avoiding walk [19], oriented percolation [20] and the contact process [16], each of which is believed to have critical dimension $d_c = 4$. In each case the lace expansion is used to derive a recursion relation of the form (A.1) and the required bounds on the quantities in the recursion equation are shown to hold (provided $d > 4$) by estimating Feynman diagrams. The required bounds are typically of the form $|h_m(k, z)| \leq \frac{C}{m^{\frac{d}{2}-b}}$, for some functions $h_m$ and power $b \geq 0$ that varies from bound to bound. What turns out to be important in the analysis is that $\frac{d}{2} = \frac{d}{2} = 2 + \frac{d-4}{2}$ is greater than 2 when $d > 4$.

In unpublished work [18] the authors note that the analysis of [19] should be robust enough to permit extension to certain other models where the lace expansion is applicable, above $d_c \neq 4$. In particular they outline how [19] might be adapted to analyse lattice trees in dimensions $d > 8$. While deviating somewhat in the details, our analysis in this chapter (and its application to lattice trees) is based on the ideas of [18].

In our analysis we introduce two new parameters $\theta(d), p^*$ and a set $B$. We will discuss the significance of $p^*$ and $B$ when they appear shortly. The most important parameter, $\theta(d)$, is taking the place of $\frac{d}{2}$ in exponents appearing in various bounds. As in [19] we require that $\theta > 2$, and we apply the results of this chapter to lattice trees model with the choice $\theta = 2 + \frac{d-8}{2}$. We also expect the result to be applicable to other models where the lace expansion is used in the analysis above a critical dimension $d_c$. In such cases the lace expansion for $d > d_c$ suggests setting $\theta = 2 + \frac{d-d_c}{2}$. In particular for percolation ($d_c = 6$) we would expect $\theta = 2 + \frac{d-6}{2}$. Note that in the case $d_c = 4$, $2 + \frac{d-d_c}{2} = \frac{d}{2}$, which is that appearing in [19].

There is an unpublished version [15] of this chapter consisting of full proofs of the material in [19], adapted to our more general setting, with generally only cosmetic changes (e.g. $\frac{d}{2} \mapsto \theta$) required. In this thesis we will state the assumptions and results explicitly, but for the sake of brevity we will present only significant changes in the proof and leave the reader to refer to [19] when the changes are only cosmetic.

Therefore the chapter is organised as follows. In Section A.2 we state the assumptions $S, D, E_\theta$, and $G_\theta$ on the quantities appearing in the recursion equation, and the "$\theta$-theorem" to be proved. In Section A.3, we introduce the induction hypotheses on $f_n$ that will be used to prove the $\theta$-theorem. We advancement of the induction hypotheses is highly technical and our extension does not require significant alterations from the analysis of [19]. We therefore briefly discuss the role of $\theta$ in this section and direct the interested reader to [19] for the analysis. Once the induction hypotheses have been advanced the $\theta$-theorem follows without difficulty.

## A.2 Assumptions on the Recursion Relation

Suppose that for $z > 0$ and $k \in [-\pi, \pi]^d$, we have $f_0(k; z) = 1$ and

$$f_{n+1}(k; z) = \sum_{m=1}^{n+1} g_m(k; z) f_{n+1-m}(k; z) + e_{n+1}(k; z), \qquad (n \geq 0), \tag{A.3}$$

where the functions $g_m$ and $e_m$ are to be regarded as given. The goal is to understand the behaviour of the solution $f_n(k; z)$ of (A.3).

### A.2.1 Assumptions S,D,$\mathbf{E}_\theta$,$\mathbf{G}_\theta$

The first assumption, Assumption S, requires that the functions appearing in the recursion equation (A.3) respect the lattice symmetries of reflection and rotation, and that $f_n$ remains bounded in a weak sense. This assumption remains unchanged from [19].

**Assumption S**. For every $n \in \mathbb{N}$ and $z > 0$, the mapping $k \mapsto f_n(k;z)$ is symmetric under replacement of any component $k_i$ of $k$ by $-k_i$, and under permutations of the components of $k$. The same holds for $e_n(\cdot;z)$ and $g_n(\cdot;z)$. In addition, for each $n$, $|f_n(k;z)|$ is bounded uniformly in $k \in [-\pi,\pi]^d$ and $z$ in a neighbourhood of 1 (which may depend on $n$).

The next assumption, Assumption D, introduces a function $D = D_L$ which defines the underlying random walk model and involves a non-negative parameter $L$ which will typically be $\gg 1$. This serves to spread out the steps of the random walk over a large set. An example of a family of $D$'s obeying the assumption was given in Definition 1.2.4 and the remarks following it. In particular Assumption D implies that $D$ has a finite second moment, and we define

$$\sigma^2 \equiv -\nabla^2 \hat{D}(0) = -\left[\sum_{j=1}^{d} \frac{\partial^2}{\partial k_j^2} \sum_x e^{ik\cdot x} D(x)\right]_{k=0} = \sum_x |x|^2 D(x). \tag{A.4}$$

Let

$$a(k) = 1 - \hat{D}(k). \tag{A.5}$$

**Assumption D**. We assume that

$$f_1(k;z) = z\hat{D}(k), \quad e_1(k;z) = 0. \tag{A.6}$$

In particular, this implies that $g_1(k;z) = z\hat{D}(k)$. As part of Assumption D, we also assume:

(i) $D$ is normalised so that $\hat{D}(0) = 1$, and has $2+2\epsilon$ moments for some $0 < \epsilon < \theta - 2$, i.e.,

$$\sum_{x\in\mathbb{Z}^d} |x|^{2+2\epsilon} D(x) < \infty. \tag{A.7}$$

(ii) There is a constant $C$ such that, for all $L \geq 1$,

$$\|D\|_\infty \leq CL^{-d}, \qquad \sigma^2 = \sigma^2 \leq CL^2, \tag{A.8}$$

(iii) There exist constants $\eta, c_1, c_2 > 0$ such that

$$c_1 L^2 k^2 \le a(k) \le c_2 L^2 k^2 \quad (\|k\|_\infty \le L^{-1}), \tag{A.9}$$

$$a(k) > \eta \quad (\|k\|_\infty \ge L^{-1}), \tag{A.10}$$

$$a(k) < 2 - \eta \quad (k \in [-\pi, \pi]^d). \tag{A.11}$$

Assumptions E and G of [19] are now adapted to general $\theta > 2$ as follows. The relevant bounds on $f_m$, which *a priori* may or may not be satisfied, are that for some $p^* \ge 1$, some nonempty $B \subset [1, p^*]$ and

$$\beta = \beta(p^*) = L^{-\frac{d}{p^*}} \tag{A.12}$$

we have for every $p \in B$,

$$\|\hat{D}^2 f_m(\cdot; z)\|_p \le \frac{K}{L^{\frac{d}{p}} m^{\frac{d}{2p} \wedge \theta}}, \quad |f_m(0; z)| \le K, \quad |\nabla^2 f_m(0; z)| \le K\sigma^2 m, \tag{A.13}$$

for some positive constant $K$. The full generality in which this has been presented is not required for our application to lattice trees where we have $p^* = 2$ and $B = \{2\}$. This is because we require only the $p = 2$ case in (A.13) to estimate the diagrams arising from the lace expansion for lattice trees and verify the assumptions $\mathbf{E}_\theta$, $\mathbf{G}_\theta$ which follow. In other applications it may be that a larger collection of $\| \bullet \|_p$ norms are required to verify the assumptions and the set $B$ is allowing for this possibility. The parameter $p^*$ serves to make this set bounded so that $\beta(p^*)$ is small for large $L$.

The bounds in (A.13) are identical to the ones in [19](1.27), except for the first bound, which only appears in [19] with $p = 1$ and $\theta = \frac{d}{2}$.

**Assumption $\mathbf{E}_\theta$.** There is an $L_0$, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0,1)$, and a function $K \mapsto C_e(K)$, such that if (A.13) holds for some $K > 1$, $L \ge L_0$, $z \in I$ and for all $1 \le m \le n$, then for that $L$ and $z$, and for all $k \in [-\pi, \pi]^d$ and $2 \le m \le n+1$, the following bounds hold:

$$|e_m(k; z)| \le C_e(K)\beta m^{-\theta}, \quad |e_m(k; z) - e_m(0; z)| \le C_e(K) a(k) \beta m^{-\theta+1}. \tag{A.14}$$

**Assumption $\mathbf{G}_\theta$.** There is an $L_0$, an interval $I \subset [1-\alpha, 1+\alpha]$ with $\alpha \in (0,1)$, and a function $K \mapsto C_g(K)$, such that if (A.13) holds for some $K > 1$, $L \ge L_0$, $z \in I$ and for all $1 \le m \le n$, then for that $L$ and $z$, and for all $k \in [-\pi, \pi]^d$ and $2 \le m \le n+1$, the following bounds hold:

$$|g_m(k; z)| \le C_g(K)\beta m^{-\theta}, \quad |\nabla^2 g_m(0; z)| \le C_g(K)\sigma^2 \beta m^{-\theta+1}, \tag{A.15}$$

$$|\partial_z g_m(0;z)| \le C_g(K)\beta m^{-\theta+1}, \tag{A.16}$$

$$|g_m(k;z) - g_m(0;z) - a(k)\sigma^{-2}\nabla^2 g_m(0;z)| \le C_g(K)\beta a(k)^{1+\epsilon'} m^{-\theta+(1+\epsilon')}, \tag{A.17}$$

with the last bound valid for any $\epsilon' \in [0,\epsilon]$, with $0 < \epsilon < \theta - 2$ given by (A.7).

**Theorem A.2.1.** *Let $d > d_c$ and $\theta(d) > 2$, and assume that Assumptions $S$, $D$, $E_\theta$ and $G_\theta$ all hold. There exist positive $L_0 = L_0(d,\epsilon)$, $z_c = z_c(d,L)$, $A = A(d,L)$, and $v = v(d,L)$, such that for $L \ge L_0$, the following statements hold.*
*(a) Fix $\gamma \in (0, 1\wedge\epsilon)$ and $\delta \in (0, (1\wedge\epsilon) - \gamma)$. Then*

$$f_n\Big(\frac{k}{\sqrt{v\sigma^2 n}}; z_c\Big) = Ae^{-\frac{k^2}{2d}}[1 + \mathcal{O}(k^2 n^{-\delta}) + \mathcal{O}(n^{-\theta+2})], \tag{A.18}$$

*with the error estimate uniform in $\{k \in \mathbb{R}^d : a(k/\sqrt{v\sigma^2 n}) \le \gamma n^{-1}\log n\}$.*
*(b)*

$$-\frac{\nabla^2 f_n(0;z_c)}{f_n(0;z_c)} = v\sigma^2 n[1 + \mathcal{O}(\beta n^{-\delta})]. \tag{A.19}$$

*(c) For all $p \ge 1$,*

$$\|\hat{D}^2 f_n(\cdot;z_c)\|_p \le \frac{C}{L^{\frac{d}{p}} n^{\frac{d}{2p}\wedge\theta}}. \tag{A.20}$$

*(d) The constants $z_c$, $A$ and $v$ obey*

$$\begin{aligned} 1 &= \sum_{m=1}^{\infty} g_m(0;z_c), \\ A &= \frac{1 + \sum_{m=1}^{\infty} e_m(0;z_c)}{\sum_{m=1}^{\infty} m g_m(0;z_c)}, \\ v &= -\frac{\sum_{m=1}^{\infty} \nabla^2 g_m(0;z_c)}{\sigma^2 \sum_{m=1}^{\infty} m g_m(0;z_c)}. \end{aligned} \tag{A.21}$$

It follows immediately from Theorem A.2.1(d) and the bounds of Assumptions E and $G$ that

$$z_c = 1 + \mathcal{O}(\beta), \quad A = 1 + \mathcal{O}(\beta), \quad v = 1 + \mathcal{O}(\beta). \tag{A.22}$$

## A.3 Induction hypotheses

The recursion relation (A.3) is analysed using induction on $n$, as done in [19].

The induction hypotheses involve a sequence $v_n$, which is defined exactly as in [19] as follows. We set $v_0 = b_0 = 1$, and for $n \ge 1$ we define

$$b_n = -\frac{1}{\sigma^2}\sum_{m=1}^{n} \nabla^2 g_m(0;z), \quad c_n = \sum_{m=1}^{n}(m-1)g_m(0;z), \quad v_n = \frac{b_n}{1+c_n}. \tag{A.23}$$

The induction hypotheses also involve several constants. Let $\theta > 2$, and recall that $\epsilon$ was specified in (A.7). We fix $\gamma, \delta > 0$ and $\lambda > 2$ according to

$$\begin{aligned} 0 < \gamma &< 1 \wedge (\theta - 2) \wedge \epsilon \\ 0 < \delta &< (1 \wedge (\theta - 2) \wedge \epsilon) - \gamma \\ \theta - \gamma < \lambda &< \theta. \end{aligned} \tag{A.24}$$

Here $\lambda$ replaces $\rho + 2$ from [19] simply to avoid confusion with $\rho(0)$ from other chapters in this thesis.
We also introduce constants $K_1, \dots, K_5$, which are independent of $\beta$. We define

$$K_4' = \max\{C_e(cK_4), C_g(cK_4), K_4\}, \tag{A.25}$$

where $c$ is a constant determined in Lemma A.3.6 below. To advance the induction, we will need to assume that

$$K_3 \gg K_1 > K_4' \geq K_4 \gg 1, \quad K_2 \geq K_1, 3K_4', \quad K_5 \gg K_4. \tag{A.26}$$

Here $a \gg b$ denotes the statement that $a/b$ is sufficiently large. The amount by which, for instance, $K_3$ must exceed $K_1$ is independent of $\beta$, but may depend on $p^*$, and will be determined during the course of the advancement of the induction in Section A.4.

Let $z_0 = z_1 = 1$, and define $z_n$ recursively by

$$z_{n+1} = 1 - \sum_{m=2}^{n+1} g_m(0; z_n), \qquad n \geq 1. \tag{A.27}$$

For $n \geq 1$, we define intervals

$$I_n = [z_n - K_1 \beta n^{-\theta+1}, z_n + K_1 \beta n^{-\theta+1}]. \tag{A.28}$$

In particular this gives $I_1 = [1 - K_1\beta, 1 + K_1\beta]$.

Recall the definition $a(k) = 1 - \hat{D}(k)$ from (A.5). Our induction hypotheses are that the following four statements hold for all $z \in I_n$ and all $1 \leq j \leq n$.

**(H1)** $|z_j - z_{j-1}| \leq K_1 \beta j^{-\theta}$.

**(H2)** $|v_j - v_{j-1}| \leq K_2 \beta j^{-\theta+1}$.

**(H3)** For $k$ such that $a(k) \leq \gamma j^{-1} \log j$, $f_j(k; z)$ can be written in the form

$$f_j(k; z) = \prod_{i=1}^{j} \left[1 - v_i a(k) + r_i(k)\right],$$

with $r_i(k) = r_i(k; z)$ obeying

$$|r_i(0)| \leq K_3 \beta i^{-\theta+1}, \quad |r_i(k) - r_i(0)| \leq K_3 \beta a(k) i^{-\delta}.$$

**(H4)** For $k$ such that $a(k) > \gamma j^{-1} \log j$, $f_j(k;z)$ obeys the bounds

$$|f_j(k;z)| \leq K_4 a(k)^{-\lambda} j^{-\theta}, \quad |f_j(k;z) - f_{j-1}(k;z)| \leq K_5 a(k)^{-\lambda+1} j^{-\theta}.$$

Note that these four statements are those of [19] with the replacement

$$\rho + 2 \mapsto \lambda \tag{A.29}$$

in **(H4)** and the global replacement

$$\frac{d}{2} \mapsto \theta, \tag{A.30}$$

By global replacement we also mean that $\frac{d-2}{2} \mapsto \theta - 1$, $\frac{d-4}{2} \mapsto \theta - 2$, etc. whenever such quantities appear in exponents.

### A.3.1 Initialisation of the induction

The verification that the induction hypotheses hold for $n = 0$ remains unchanged from the $p = 1$ case, up to the replacements (A.29-A.30).

### A.3.2 Consequences of induction hypotheses

The key result of this section is that the induction hypotheses imply (A.13) for all $1 \leq m \leq n$, from which the bounds of Assumptions $E_\theta$ and $G_\theta$ then follow, for $2 \leq m \leq n+1$.

As in [19] throughout this chapter:

- $C$ denotes a strictly positive constant that may depend on $d, \gamma, \delta, \lambda$, but *not* on the $K_i$, *not* on $k$, *not* on $n$, and *not* on $\beta$ (provided $\beta$ is sufficiently small, possibly depending on the $K_i$). The value of $C$ may change from line to line.

- We frequently assume $\beta \ll 1$ without explicit comment.

Lemmas A.3.1 and A.3.3 are proved in [19] and the proof in our context requires only the global change (A.30).

**Lemma A.3.1.** *Assume (H1) for* $1 \leq j \leq n$. *Then* $I_1 \supset I_2 \supset \cdots \supset I_n$.

**Remark A.3.2.** *We were unable to verify [19](2.19) as stated. Instead of [19](2.19) we use*

$$|s_i(k)| \leq K_3(2 + C(K_2 + K_3)\beta)\beta a(k) i^{-\delta}, \tag{A.31}$$

*the only difference being the constant 2 appears here instead of a constant 1 in [19](2.19). This does not affect the proof. To verify (A.31) we use the fact that* $\frac{1}{1-x} \le 1 + 2x$ *for* $0 \le x \le \frac{1}{2}$ *to write for small enough* $\beta$*,*

$$\begin{aligned} |s_i(k)| &\le [1 + 2K_3\beta]\,[(1 + |v_i - 1|)a(k)r_i(0) + |r_i(k) - r_i(0)|] \\ &\le [1 + 2K_3\beta]\left[(1 + CK_2\beta)a(k)\frac{K_3\beta}{i^{\theta-1}} + \frac{K_3\beta a(k)}{i^{\delta}}\right] \\ &\le \frac{K_3\beta a(k)}{i^{\delta}}[1 + 2K_3\beta][2 + CK_2\beta] \le \frac{K_3\beta a(k)}{i^{\delta}}[2 + C(K_2 + K_3)\beta]. \end{aligned} \tag{A.32}$$

*Here we have used the bounds of (H3) as well as the fact that* $\theta - 1 > \delta$*.*

**Lemma A.3.3.** *Let* $z \in I_n$ *and assume (H2–H3) for* $1 \le j \le n$*. Then for* $k$ *with* $a(k) \le \gamma j^{-1}\log j$*,*

$$|f_j(k;z)| \le e^{CK_3\beta} e^{-(1-C(K_2+K_3)\beta)ja(k)}. \tag{A.33}$$

The middle bound of (A.13) follows, for $1 \le m \le n$ and $z \in I_m$, directly from Lemma A.3.3. We next state two lemmas which provide the other two bounds of (A.13). The first concerns the $\| \bullet \|_p$ norms and contains the most significant changes from [19]. As such we present the full proof of this lemma.

**Lemma A.3.4.** *Let* $z \in I_n$ *and assume (H2), (H3) and (H4). Then for all* $1 \le j \le n$*, and* $p \ge 1$*,*

$$\|\hat{D}^2 f_j(\cdot;z)\|_p \le \frac{C(1+K_4)}{L^{\frac{d}{p}} j^{\frac{d}{2p}\wedge\theta}}, \tag{A.34}$$

*where the constant* $C$ *may depend on* $p, d$*.*

*Proof.* We show that

$$\|\hat{D}^2 f_j(\cdot;z)\|_p^p \le \frac{C(1+K_4)^p}{L^d j^{\frac{d}{2}\wedge\theta p}}. \tag{A.35}$$

For $j = 1$ the result holds since $|f_1(k)| = |z\widehat{D}(k)| \le z \le 2$ and by using (A.8) and the fact that $p \ge 1$. We may therefore assume that $j \ge 2$ where needed in what follows, so that in particular $\log j \ge \log 2$.

Fix $z \in I_n$ and $1 \le j \le n$, and define

$$\begin{aligned} R_1 &= \{k \in [-\pi,\pi]^d : a(k) \le \gamma j^{-1}\log j,\ \|k\|_\infty \le L^{-1}\}, \\ R_2 &= \{k \in [-\pi,\pi]^d : a(k) \le \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\}, \\ R_3 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty \le L^{-1}\}, \\ R_4 &= \{k \in [-\pi,\pi]^d : a(k) > \gamma j^{-1}\log j,\ \|k\|_\infty > L^{-1}\}. \end{aligned}$$

The set $R_2$ is empty if $j$ is sufficiently large. Then

$$\|\hat{D}^2 f_j\|_p^p = \sum_{i=1}^{4} \int_{R_i} \left(\hat{D}(k)^2 |f_j(k)|\right)^p \frac{d^d k}{(2\pi)^d}. \tag{A.36}$$

We will treat each of the four terms on the right side separately.

On $R_1$, we use (A.9) in conjunction with Lemma A.3.3 and the fact that $\hat{D}^2 \le 1$, to obtain for all $p > 0$,

$$\begin{aligned} \int_{R_1} \left(\hat{D}(k)^2\right)^p |f_j(k)|^p \frac{d^d k}{(2\pi)^d} &\le \int_{R_1} C e^{-cpj(Lk)^2} \frac{d^d k}{(2\pi)^d} \\ &\le \prod_{i=1}^{d} \int_{-\frac{1}{L}}^{\frac{1}{L}} C e^{-cpj(Lk_i)^2} dk_i \le \frac{C}{L^d (pj)^{d/2}} \\ &\le \frac{C}{L^d j^{d/2}}. \end{aligned} \tag{A.37}$$

Here we have used the substitution $k_i' = Lk_i\sqrt{pj}$. On $R_2$, we use Lemma A.3.3 and (A.10) to conclude that for all $p > 0$, there is an $\alpha(p) > 1$ such that

$$\int_{R_2} \left(\hat{D}(k)^2 |f_j(k)|\right)^p \frac{d^d k}{(2\pi)^d} \le C \int_{R_2} \alpha^{-j} \frac{d^d k}{(2\pi)^d} = C\alpha^{-j} |R_2|, \tag{A.38}$$

where $|R_2|$ denotes the volume of $R_2$. This volume is maximal when $j = 3$, so that

$$\begin{aligned} |R_2| &\le |\{k : a(k) \le \tfrac{\gamma \log 3}{3}\}| \le |\{k : \hat{D}(k) \ge 1 - \tfrac{\gamma \log 3}{3}\}| \\ &\le (\tfrac{1}{1 - \frac{\gamma \log 3}{3}})^2 \|\hat{D}^2\|_1 \le (\tfrac{1}{1 - \frac{\gamma \log 3}{3}})^2 C L^{-d}, \end{aligned} \tag{A.39}$$

using (A.8) in the last step. Therefore $\alpha^{-j}|R_2| \le CL^{-d} j^{-d/2}$ since $\frac{j^{\frac{d}{2}}}{\alpha^j} \le C$ for every $j$ , and

$$\int_{R_2} \left(\hat{D}(k)^2 |f_j(k)|\right)^p \frac{d^d k}{(2\pi)^d} \le CL^{-d} j^{-d/2}. \tag{A.40}$$

On $R_3$ and $R_4$, we use (H4). As a result, the contribution from these two regions is bounded above by

$$\left(\frac{K_4}{j^\theta}\right)^p \sum_{i=3}^{4} \int_{R_i} \frac{\hat{D}(k)^{2p}}{a(k)^{\lambda p}} \frac{d^d k}{(2\pi)^d}. \tag{A.41}$$

On $R_3$, we use $\hat{D}(k)^2 \le 1$ and (A.9). From (A.9), $k \in R_3$ implies that $L^2|k|^2 > Cj^{-1} \log j$ so that

$$\frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \int_{R_3} \frac{1}{|k|^{2\lambda p}} d^d k \le \frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \int_{\sqrt{\frac{C \log j}{L^2 j}}}^{\frac{C}{L}} r^{d-1-2\lambda p} dr. \tag{A.42}$$

Since $\log 1 = 0$, this integral will not be finite if both $j = 1$ and $p \geq \frac{d}{2\lambda}$, but recall that we can restrict our attention to $j \geq 2$.

For $d > 2\lambda p$, we have an upper bound on (A.42) of

$$\frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \int_0^{\frac{C}{L}} r^{d-1-2\lambda p} dr \leq \frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \left(\frac{C}{L}\right)^{d-2\lambda p} \leq \frac{CK_4^p}{j^{\theta p} L^d}. \tag{A.43}$$

For $d = 2\lambda p$, (A.42) is

$$\frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \int_{\sqrt{\frac{C \log j}{L^2 j}}}^{\frac{C}{L}} \frac{1}{r} dr \leq \frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \log\left(\frac{C\sqrt{L^2 j}}{L\sqrt{\log j}}\right) = \frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \log\left(\frac{Cj}{\log j}\right). \tag{A.44}$$

Now since $d = 2\lambda p$, we have that $\theta p = \frac{\theta d}{2\lambda} > \frac{d}{2}$ using the fact that $\lambda < \theta$. This gives an upper bound on this term of $\frac{CK_4^p}{j^{\frac{d}{2}} L^d}$.

Lastly for $d < 2\lambda p$, since $\lambda < \theta$, (A.42) is bounded by

$$\frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \int_{\sqrt{\frac{C \log j}{CL^2 j}}}^{\infty} r^{d-1-2\lambda p} dr \leq \frac{CK_4^p}{j^{\theta p} L^{2\lambda p}} \left(\frac{CL^2 j}{\log j}\right)^{\frac{2\lambda p - d}{2}} \leq \frac{CK_4^p}{j^{\frac{d}{2}} L^d}, \tag{A.45}$$

as required.

On $R_4$, we use (A.8) and (A.10) to obtain the bound

$$\frac{CK_4^p}{j^{\theta p}} \int_{[-\pi,\pi]^d} \hat{D}(k)^{2p} \frac{d^d k}{(2\pi)^d} \leq \frac{CK_4^p}{j^{\theta p}} \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \frac{d^d k}{(2\pi)^d} \leq \frac{CK_4^p}{j^{\theta p} L^d}, \tag{A.46}$$

where we have used the fact that $p \geq 1$ and $|\widehat{D}| \leq 1$. Since $K_4^p \leq (1 + K_4)^p$, this completes the proof. $\square$

**Lemma A.3.5.** *Let $z \in I_n$ and assume (H2) and (H3). Then, for $1 \leq j \leq n$,*

$$|\nabla^2 f_j(0; z)| \leq (1 + C(K_2 + K_3)\beta)\sigma^2 j. \tag{A.47}$$

The proof is identical to [19]. We merely point out one small correction to the first line of [19](2.35), where a constant 2 is missing. It should read

$$|\nabla^2 s_i(0)| = 2\Big|\sum_{l=1}^{d} \lim_{t\to 0} \frac{s_i(te_l) - s_i(0)}{t^2}\Big|, \tag{A.48}$$

however once again this does not affect the proof.

The next lemma, whose proof is the same as in [19], is the key to advancing the induction, as it provides bounds for $e_{n+1}$ and $g_{n+1}$.

**Lemma A.3.6.** *Let $z \in I_n$, and assume (H2), (H3) and (H4). For $k \in [-\pi, \pi]^d$, $2 \le j \le n+1$, and $\epsilon' \in [0, \epsilon]$, the following hold:*
*(i)* $|g_j(k;z)| \le K_4' \beta j^{-\theta}$,
*(ii)* $|\nabla^2 g_j(0;z)| \le K_4' \sigma^2 \beta j^{-\theta+1}$,
*(iii)* $|\partial_z g_j(0;z)| \le K_4' \beta j^{-\theta+1}$,
*(iv)* $|g_j(k;z) - g_j(0;z) - a(k)\sigma^{-2}\nabla^2 g_j(0;z)| \le K_4' \beta a(k)^{1+\epsilon'} j^{-\theta+1+\epsilon'}$,
*(v)* $|e_j(k;z)| \le K_4' \beta j^{-\theta}$,
*(vi)* $|e_j(k;z) - e_j(0;z)| \le K_4' a(k) \beta j^{-\theta+1}$.

## A.4 The induction advanced

In this section we advance the induction hypotheses (H1)–(H4) from $n$ to $n+1$. For (H1)-(H2) the proofs are identical to those in [19] up to the global replacement (A.30) due to the following observations. Since $\theta > 2$ and $\epsilon' < \epsilon \le \theta - 2$ we have that

$$\sum_{m=2}^{\infty} \frac{1}{m^{\theta-1}} \le \sum_{m=2}^{\infty} \frac{1}{m^{\theta-1-\epsilon'}} < \infty, \qquad \sum_{j=n+2-m}^{n} \frac{1}{m^{\theta-1}} \le \frac{C}{(n+2-m)^{\theta-2}}. \tag{A.49}$$

Similarly, convolution bounds used in [19] to verify (H1)–(H3) remain applicable under the global replacement (A.30).

The above bounds are also used to advance (H3)-(H4). In addition, in (H3) we require that there exists a $q > 1$ but sufficiently close to 1 so that

$$(n+1)^{\gamma q - 1} \log(n+1) \times \begin{cases} (n+1)^{0 \vee (3-\theta)}, & (\theta \neq 3) \\ \log(n+1), & (\theta = 3), \end{cases} \tag{A.50}$$

is bounded by $(n+1)^{-\delta}$. This holds since $\delta + \gamma < 1 \wedge (\theta - 2)$ by (A.24). This corresponds to [19](3.43). Other similar bounds required to verify (H3) (corresponding to [19](3.50)-(3.51) and [19](3.58) for example) also follow from $\delta + \gamma < 1 \wedge (\theta - 2)$.

To advance (H4) we make the additional global replacement (A.29). Then using the fact that $\gamma + \lambda - \theta > 0$ we have that there exists $q'$ close to 1 so that for $a(k) \le \gamma n^{-1} \log n$,

$$\frac{C}{n^\theta} \frac{n^\lambda}{n^{q'\gamma + \lambda - \theta}} \le \frac{C}{n^\theta a(k)^\lambda}. \tag{A.51}$$

This corresponds to [19](3.62), and is used to advance the first and second bounds of (H4). In addition we use the fact that $\lambda > 2$ so that $a(k)^{\lambda-2} \le C$ (recall that $a(k) \le 2$ from (A.11)) to get $\frac{1}{a(k)} \le \frac{C}{a(k)^{\lambda-1}}$.

The proof of Theorem A.2.1 now proceeds as in [19] with the global replacement (A.30).

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