

# Quantum backreactions in slow-roll and de Sitter spacetimes

by

Bojan Losic

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M.Sc., UBC, 2001

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(Signature) \_\_\_\_\_

Department of Physics and Astronomy

The University Of British Columbia  
Vancouver, Canada

Date \_\_\_\_\_

# ABSTRACT

This thesis is comprised of three projects. In the first, I consider fluctuations in a perfect irrotational fluid coupled to gravity in an Einstein static universe background. I show that a linearization instability occurs in Einstein static spacetimes despite the presence of matter, and that this instability can only be avoided by inducing spatially homogeneous perturbations of the spacetime. Since the first order homogeneous perturbations in this case are well known to be exponentially (dynamically) unstable, the tactic of neglecting these modes to create a long-lived, perturbed Einstein static universe does not work, even if all higher order ( $L > 1$ ) modes are dynamically stable. The main conclusion is that Einstein static is unconditionally unstable at first order in perturbation theory despite the presence of a large class of neutrally stable, inhomogeneous, modes.

In the second, I examine the importance of second order corrections to linearized cosmological perturbation theory in an inflationary background, taken to be a spatially flat FRW spacetime. The full second order problem is solved in the sense that I evaluate the effect of the superhorizon second order corrections on the inhomogeneous and homogeneous modes of the linearized fluctuations. In order to quantify their physical significance I study their effective equation of state by looking at the perturbed energy density and isotropic pressure to second order. I define the energy density (isotropic pressure) in terms of the (averaged) eigenvalues associated with timelike (spacelike) eigenvectors of a total stress energy for the metric and matter fluctuations, and find that the second order contributions to the dispersion of these eigenvalues becomes of the same order or exceeds that of the linear contributions. This occurs for a wide range of initial conditions for slow-roll inflation and results in a constraint on the small slow-roll parameter of that model. The main conclusion is that the linearized approximation of a slowly rolling spacetime may, under reasonable circumstances, be intrinsically sick since higher order contributions are comparable to, or substantially larger than, the linear contributions.

In the third and final project, backreactions are considered in a pure de Sitter space whose cosmological constant is generated by the potential of a scalar field. The leading order effect of matter backreactions on the gravitational field is considered. The initial value problem for the perturbed Einstein equations is proven to generically possess linearization instabilities. I furthermore show that these linearization instabilities can be avoided by assuming strict de Sitter invariance of the quantum states of the linearized fluctuations. This invariance constraint applies to the entire spectrum of states, from the vacuum to the excited states, and is in that sense much stronger than the usual Poincaré invariance of the Minkowski vacuum. Some sketches are presented on how to construct de Sitter invariant states. The main conclusion is that to leading order in their effect on the gravitational field, the quantum states of the matter and metric fluctuations must be de Sitter invariant.

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# CHAPTER 1

## INTRODUCTION

The gradual decline of the political and intellectual authority of the medieval church in Italy and northern Europe ushered in a new era of Western thinking about the origin and nature of the large-scale Universe, and by extension our place in it. New standards for, and indeed the very emergence of, critical thought sharpened the formulation of these fundamental questions and slowly led to the establishment of what is today called the science of cosmology.

Many ancient cosmogonies competed with various different views on the shape of the universe and its early history, but in most cases there seemed to be a consensus that the world was not in a high state of symmetry. In particular, the condition of man was terrible and the world was running amuck and askew from an early history of bliss whose rational elements could be described in what today would be called mathematically elegant terms (such as, say, the three Gorgons in elucidating galactic motions or a three-brane in some current cosmological models). Current observations, however, hold that the large scale structure of the universe is remarkably spatially symmetric while theory is inadequate to comment on the precise nature of the initial state. Whatever one's view, remarkable technological developments in observational techniques have recently firmly established cosmology as a precision science wherein competing theories can actually lead to testably different predictions.

In this thesis I examine some techniques used to study a large class of cosmological models, and even though the treatment is largely mathematical the physical aspects of these studies are emphasized throughout. In the following brief introduction we will broadly motivate this work at a basic level and outline its accomplishments, using the opportunity to introduce some terminology and set some notation. In Section 1.1 general relativity is introduced and in Section 1.2 the early universe scenario known as inflation is introduced. In Section 1.3 some introductory and basic comments are made about what cosmological perturbation theory is, and Section 1.4 ends the chapter with a technical summary of the results of this thesis. In this first chapter it will prove impossible to specifically cite all of the many people who have contributed to our current understanding of cosmology, without which this thesis would be nothing more than my idle speculation on matters way over my head.

### 1.1 General relativity

The convincing success of Newtonian mechanics as applied to inter-planetary scales in our own solar system is truly impressive, and it seemed plausible in the mid 20th century that larger scale systems like galaxies and even the universe would similarly be well-described. However in a dramatic interplay between pure thought and some rudimentary measurements indicating small

experimental discrepancies from Newtonian theory, an entirely new framework of viewing the large scale universe (called General Relativity) was nearly singlehandedly produced by Albert Einstein some 90 years ago. The full implications of this new framework for cosmology are still actively being explored today, surely in part because the extent of the departure from Newtonian thought, and complexity of equations, can hardly be overemphasized.

One of the hallmarks of Newtonian cosmological ideas, which pointedly only came under scrutiny around the time GR was developed, was the notion that the distant past and distant future of the universe as a whole were in a one to one correspondence. In other words, a particular early history in Newtonian physics *always* corresponds to a particular future. The notion that this might not be so in a deterministic theory, that for example the classically evolving universe might not even *exist* in the distant past of a given future or vice-versa, is now known to be one of the most dazzling consequences of the general theory of relativity (GR). In this theory the apparently inert spacetime which merely labels events with one time and three spatial coordinates (so that it is four-dimensional in total) assumes a *dynamical* character and can interact not only with matter, but also with itself. Such a dynamic and nonlinear system<sup>1</sup> can easily admit situations where e.g. the past may not exist for a given future, depending on the 'shape' or geometry of the four-dimensional spacetime. Within GR, the geometry of the universe acquires a much deeper and more general physical meaning than before, and in that way completely alters how we even formulate a potential cosmology (i.e. what kind of questions we can ask). It thus becomes important to extract the central message GR has for cosmology, and it is in this effort we slowly become aware of the stunning convergence of modern theory. Namely, that apparently disjoint bodies of knowledge, even applied to completely different physical systems, can actually overlap and be useful in their respective domains. With this in mind, we turn to pure geometry.

Using one of the great advances in mathematics, Gauss's *Theorema Egregium*, it is possible to completely characterize the geometry of a space in an *intrinsic* way, without referring to any embeddings in a higher 'background' space as one would for example with a 2-dimensional (2-d) sphere or cylinder embedded in three dimensions. This effectively means that any being that lived on a given surface and had no knowledge of how that surface was embedded in some complicated higher space could still completely determine the geometry of that surface. Thus, Gauss's advance was to provide the most general possible characterization of the intrinsic geometry of a surface, and in so doing show that the extrinsic features of its geometry, those related to its embedding, were not essential in this characterization. A very important and related idea, which Einstein spent a great deal of time thinking about, was that the particular coordinatization of the surface is not important. The intrinsic features of the surface are unique, which the details of a particular choice of e.g. 'time' or 'space' coordinates can only obscure or illuminate. Therefore it is natural for us to apply Gauss's framework to space-time itself using the GR intuition that space-time is a surface with some nontrivial geometry. In fact, much of the technical and intuitive complication of GR is related to the usage of this intrinsic language (largely developed by Riemann in the late 19th century), and indeed to the very notion of not having an absolute, 'background', space-time as in Newtonian physics.

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<sup>1</sup>Often viewed by analogy to a ball on a rubber tarp, which isn't a very good analogy since it implies only *spatial* curvature occurs and that the tarp is always flat when there is no ball, both of which are not always true in GR.

A rather key feature of this space-time geometry is its (intrinsic) curvature<sup>2</sup>. For example, the 2-d sphere and cylinder mentioned above are distinguished by the fact that the former is *intrinsically* curved while the latter is not, even though both appear curved when *embedded* in three dimensions. In GR, once again, this abstract idea of intrinsic curvature is applied to an entire space-time. In a great leap, the theory describes how to straightforwardly relate the intrinsic curvature of spacetime to a generalized concept of the gravitational field which, in an appropriate limit, reduces to the Newtonian gravitational field. Therefore, assembling all of the above ideas, the key message of general relativity is: *the intrinsic curvature of a spacetime is directly related to a generalized concept of a gravitational field*. The link between the intrinsic curvature of spacetime and a generalized notion of the gravitational field is one of the most phenomenal examples of the so-called modern convergence of theory<sup>3</sup>.

What does this mean for cosmology? For one it implies that the history and spatial properties of the large-scale universe are best described in terms of an intrinsic description of spacetime which, as a whole, does not evolve with respect to anything and yet dynamically participated in physical processes. This is revolutionary. Furthermore, it implies that the matter content of the universe can significantly influence how it evolves, as it interacts with spacetime, and that the very spacetime interacts with itself (colloquially, 'gravity gravitates') in processes which can be described as *gravitational* interactions. In view of all of this, the programme of cosmology turns to characterizing the spacetime geometry according to the dynamical laws which govern its interaction with itself and the matter content of the cosmos. Thus in this thesis we have replaced the philosophical prejudice that spacetime is an inert arena which absolutely labels physical processes with the prejudice that spacetime dynamically participates in such processes, in the precise way spelled out by GR.

In this thesis we shall consider techniques which approximate these spacetime dynamics, generally called perturbation theory, or in the cosmological context, 'cosmological perturbation theory'. The reason the full spacetime dynamics are approximated at all is that they are expressed in terms of coupled, nonlinear, partial differential equations whose only known exact solutions assume many simplifying assumptions which, for reasons spelled out later, are not acceptable for many detailed studies of current cosmological models. Despite their complexity, the common expression of these equations is deceptively simple, namely

$$\underbrace{F(R)}_{\text{Linear function of curvature}} = \underbrace{T}_{\text{Matter stress-energy}}, \quad (1.1)$$

where  $R$  is some measure of the spacetime curvature. In four dimensions, these are ten coupled equations whose solutions give the values of the generalized gravitational field for a given spacetime and matter configuration. To make analytic headway, perturbation theory in general relativity inherently considers only perturbations from a given exact solution and in that way attempts to

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<sup>2</sup>Roughly, if one transports a tangent vector around a closed circuit on a given curved surface, the initial vector and final vector do not coincide. The angle between them is linearly proportional to the 'intrinsic curvature' of a surface.

<sup>3</sup>It is no accident that this link makes no use of any particular kind of symmetry (such as the Poincare group), or the notion of 'particles'. GR is a background independent (i.e., wholly intrinsic) field theory which only admits a usable particle interpretation in special cases.

approximate an exact solution <sup>4</sup> to (1.1) near the highly symmetric known solution using a power series (assuming these perturbations are ‘small’ in some reasonable measure). The first order correction is called linear, the higher order corrections are generally referred to as backreactions.

In the next section I shall describe which solutions to (1.1) are postulated to be extremely relevant for the early history of the universe.

## 1.2 Inflationary scenario

Equations (1.1) suddenly make possible giving the universe dynamics as a whole. It can shrink and grow or even pinch off as suggested above, and space itself can be created and destroyed. We can use the equations to study the consequences of one of the most phenomenal observational discoveries about our universe, namely that it is incredibly smooth and homogeneous on the largest scales. Assuming such a uniform distribution of matter and energy in the universe, this rate of creation or destruction is driven in a straightforward way, by equations (1.1), by the energy density and pressure of the stress energy of the matter. If one insists on positive energy densities and pressures and only looks at the case of space being created (expanding universe), then it is easy to see this implies the energy density must have been larger in the past. This in turn implies that, by equations (1.1), the rate of space creation/expansion was larger in the past. Furthermore if the pressure is always bigger than zero, the energy density must have been infinite at some *finite* time in the past and one can only presume that at this finite time the universe came into being. However, light travelling from this finite beginning time could have only travelled a distance which is *much, much* smaller than the scales of the universe we observe today. We know this from looking at the well-known Cosmic Microwave Background (CMB) radiation and furthermore assume, as I do in this thesis, that no physical processes can propagate faster than the speed of light. Clearly equations (1.1) are a mixed blessing, since they have replaced the mystery of what dynamics the universe can have as a whole with the problem of providing a causal explanation for the large-scale uniformity of the observed universe.

There are a variety of ways one can try and solve this conundrum. By fiat, one can invoke some kind of being that created the universe as we see it and end the inquiry as such right there<sup>5</sup>. However, if we can find some way around the conclusion that in the finite time since the beginning of the universe light could only travel a small portion of the visible universe, then perhaps there is hope for actually acquiring some explanatory power. As usual, this requires crushing a naive assumption we have made, and in the late '70's and early '80's this move was made almost simultaneously in the USSR and USA. The idea was to create enough time for the entire universe to be in causal contact with itself by *slowing down the rate at which space was created in the past*. This will require some kind of mechanism to slow down the increase of energy density as the universe grows smaller in the past. This is achieved by simply allowing the pressure to do work on the universe as it collapses, i.e. allow a *negative pressure* to develop, and in fact if the pressure is large enough the

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<sup>4</sup>The only people who are brave enough to even contemplate trying to solve equations (1.1) without as many special simplifying assumptions are the numerical relativists. The fruits of their effort are the only insight (aside from the Hawking-Penrose singularity theorems) we have about the analytic structure of general solutions to the field equations, which appears to be extremely rich.

<sup>5</sup>However, despite everything, we push on: Qui tacet, consentire videtur.

energy density will be a constant independent of the size of the universe. The first investigations demonstrated that an era of such negative pressure could lead to a slow enough rate of expansion in the past, or in other words a sufficiently long enough time for light to traverse the fledgling universe, and allow a causal explanation for the large scale homogeneity we see.

Generally the above scenario is known as *inflation* and the universe started out in a hot, dense, event called the *Big Bang* (as coined, ironically, by the principal advocate of the competing Steady State models, Fred Hoyle)<sup>6</sup>. The inflationary scenario provides a way to view why the Big Bang should lead to a homogeneous universe, i.e. it allows us to understand the spatially homogeneous solutions to (1.1) in terms of a *dynamical explanation* of the large scale homogeneity of the universe *if* negative pressures can be somehow arranged. It turns out that this is relatively simple to do if the dominant stress energy of the universe is 'potential dominated' (as opposed to the 'rest mass', 'kinetic', dominated stress energy we mostly observe in our every day lives) and thus not terribly affected by the universe as it expands or contracts. The inflationary scenario manages to create an era like this by postulating such a special stress-energy in the early universe, and in this way it is usually stated that inflation solves the major causality problems of the Big Bang scenario<sup>7</sup>.

Another compelling success of inflation is that it miraculously solves the question of local inhomogeneities by postulating that quantum fluctuations in the early, negative pressure dominated, matter were amplified to a classical level during the transition from the inflationary era to a more 'matter dominated' era. The amplitude of the quantum fluctuations is directly related to the rate at which the (classical) negative pressure matter transforms to positive pressure matter, and in fact the very first models of inflation had a very tiny rate of conversion so the fluctuations were too large and created a very inhomogeneous universe. The miracle of inflation is in embedding this mechanism for explaining the inhomogeneity of the universe within the potential dominated era which explains its homogeneity. Though extremely compelling, there is still widespread agreement in the community (with notable exceptions) that inflation can still only claim the status of a scenario, rather than a full fledged theory. In this thesis we ignore these tough foundational issues and only directly address the part of the inflationary scenario which deals with the generation and amplification of quantum fluctuations.

### 1.3 Cosmological perturbation theory

The theory of linearized fluctuations, or linearized gravitational fluctuations in the language of GR, in an expanding universe has become one of the cornerstones of modern cosmology. The quantum fluctuations invoked in the inflationary scenario are necessary to explain how nonlinear structures on the scale of galaxies and clusters came to be. With perturbation theory we can describe these fluctuations and study the growth of structure in the universe, including the predicted fluctuations in the CMB, using the essential intuition that the perturbations will grow in time as a result of

<sup>6</sup>It is perhaps worth emphasizing that the Big Bang should be distinguished from a moment of time in which the universe began, which is a common misconception. In GR, one may think of the Big Bang as a point in parameter space, called a singularity, in which the spacetime of this model ceases to exist (which is considerably worse).

<sup>7</sup>Inflation does *not* solve the problem of the initial singularity of the universe because it actually requires relatively special initial conditions to work. Vexingly, this actually leads to successful arguments that it does not generically solve the homogeneity problems either.

gravitational instability. In a static (nonexpanding) universe, Newtonian intuition suggests that this growth will be exponential whilst in an expanding universe one's intuition might be that the growth slows down to a power law dependence (which is correct).

Just as (1.1) is compactly written down and explained, the above words are easy to say or write down. Expanding equations (1.1) order by order and solving them order by order about one's favourite exact cosmological solution in one or more small parameters (depending on one's approximations), however, is fraught with technical difficulty which will be described in some detail in later chapters. Key issues include filtering out fluctuations in coordinates as opposed to real fluctuations in spacetime, and interpreting these fluctuations in a way which does not depend on how one coordinatizes the background solution<sup>8</sup>. Here and henceforth, 'background solution' or 'background' will refer to the exact solution of (1.1) about which the perturbations are defined. In this thesis we will also pay special attention to issues which are peculiar to cosmological perturbation theory (and other theories whose field equations are nonlinear) whenever the background solution is sufficiently symmetric.

The vast majority of previous work in cosmological perturbation theory has been on the linearized sector. Back of the envelope calculations easily reveal that the rms amplitude of the anisotropies in the CMB roughly give the amplitude of the linearized fluctuations, which is roughly  $10^{-5}$ , and so naively the second order fluctuations should be extremely subdominant. However, the relative importance of the second order contributions is subtle and can sometimes affect the linearized perturbations and the background spacetime in a certain cumulative sense which defies naive estimates. This process is commonly referred to as 'backreaction' and has been investigated seriously in the past by many authors, often in near heroic calculations. However there is still considerable debate on how to assess the importance of these higher order fluctuations, primarily because of the technical complications required to do even simple calculations in second order cosmological perturbation theory but also because the leading order nonlinearities of GR bring up fundamental issues in the full theory which have not been resolved<sup>9</sup>. Despite this, interest is growing in these sorts of calculations as people realize that its results may have important things to add to a whole host of pressing issues in theoretical cosmology, and even in the effort to reconcile the well-known incompatibility between GR and quantum field theory. Indeed, a better and more definitive understanding of what the leading order nonlinearities in standard relativistic (quantum) cosmological perturbation theory might be telling us is, in my opinion, at the very least as valuable as pursuing modifications to the foundations of GR itself.

## 1.4 What is new in this thesis?

In this thesis I make a modest step in this direction by considering the relative importance of a certain class of backreactions on a certain class of linearized fluctuations, all within a general class of inflationary models. I also answer questions concerning their effect on the background spacetime and details of their quantum states. This step is taken in three studies, which to my knowledge

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<sup>8</sup>Not all perturbed gravitational fields correspond to perturbed spacetimes.

<sup>9</sup>Such as the question of observables. The only known observables in classical GR are constants.

are original.

- In the first study I examine a peculiarity (called a linearization instability) in relativistic perturbation theory which allows for the possibility that some solutions of the linearized field equations are not in fact linearizations of solutions to the full field equations. I extend the existing theory by explicit construction, showing that such situations still exist even when matter is present (which, in a sense to be spelled out later, is counterintuitive). We show an example of how the exclusion of all spurious solutions to the linearized equations actually leads to critical dynamical instabilities in a class of proposed cosmological models. These dynamical instabilities effectively rule out these cosmological models, and in particular rule out their candidates for a quantum initial state of the universe. Noteworthy aspects include the extension of the known results in vacuum to cases involving matter and the first known example that linearization instabilities can alter the physical conclusions of linearized perturbation theory. This is peer-reviewed work [1].
- In the second study the importance of long-wavelength backreactions on linearized cosmological fluctuations about an inflating spacetime is assessed. Particular attention is paid to the limit where the inflating spacetime tends to a well-known vacuum solution<sup>10</sup> and two key results are obtained. The first is that the effect of these backreactions can be described as an additional negative cosmological constant in the background spacetime. The second is that there is a wide class of inflationary initial conditions under which some completely reasonable assumptions about the background matter<sup>11</sup> will lead the relative amplitude of the backreactions to dominate over that of the linearized fluctuations. Noteworthy features of this analysis are its treatment of higher order gauge ambiguities, solution of the backreaction equations of motion, its novel construction of observables and reformulation of the backreaction programme, and its suggestion that the linearized perturbative approximation of a wide class of inflationary spacetimes fails. This is peer-reviewed work [2].
- In the third and final study, I examine the classical and quantum backreaction problem in a special vacuum spacetime with a positive cosmological constant (de Sitter). This is done in order to gain insight into another aspect of the second study, and ironically the techniques of the first study are crucial in achieving some of its results. In the classical problem we follow a novel gauge-reduction technique, which in the quantum case we show does not contain any anomalies. I then proceed to formulate the theory in Hamiltonian form, and find that linearization instabilities can exist. I find that the quantum mechanical statement that all solutions to the quantized linearized equations of motion be legitimate is equivalent to requiring that the states of these solutions be deSitter invariant. This requirement seems to unduly restrict the possible symmetry of the linearized states and forces one to directly confront the problem of constructing observables in de Sitter. I examine several ideas on how to do this and show some sketches on how to achieve nontrivial deSitter invariant states. Noteworthy aspects include the imposition of these symmetry requirements, the construction

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<sup>10</sup>The so called de Sitter limit.

<sup>11</sup>The slow-roll approximation.

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of these states, and the proof that there exist no quantum anomalies which block the invariance requirement. This is work in progress, though portions of it could presumably already be published.

The general strategy in the following chapters is to briefly introduce the required introductory material (terminology, notation, etc) and then proceed directly to the calculations and conclusions. The first two projects will be taken directly from previously published work while the third will draw from notes. The thesis will end with a summation of the whole work, together with an outlook for future work. The following chapters will, unfortunately, have to rely more on technical jargon than this introduction. The point of view of the thesis will switch freely from first person and third person, i.e. adopt a narrative point of view, to better delineate what I have done with what others have done.



## CHAPTER 2

# LINEAR PERTURBATION THEORY

Before extending Section 1.3 somewhat and discussing perturbation theory in general relativity, in this Chapter I start by taking a short look at the homogeneous and isotropic background spacetime. Indeed, in Section 2.1 I follow a more formal procedure (as Wald does in [3]) of rigorously defining spatial homogeneity and isotropy and showing what possible metrics are compatible with these assumptions. Next, in Section 2.2, a brief introduction to linear cosmological perturbation theory (see [4] for an extensive review) is given. One of the main features of this theory that I emphasize is that there exists a well-defined procedure to cast the basic equations of motion for energy density fluctuations in the form of a time-dependent simple harmonic oscillator, a fact which will be very important for Chapter 4 of this thesis. The classical theory of linearized perturbations is commonly cast in the Lagrangian form, and this is the language initially adopted in Section 2.2. However, the theory of quantum fluctuations as well as the quantum and classical initial value problem, is more easily treated in the Hamiltonian form and this naturally leads to the natural expression of the phenomenon of linearization stability (see [5], [6], [7], [8], [9], [10]), which is introduced in Section 2.3. Some original but ancillary results about linearization stability are presented near the end of Section 2.3 as part of the conceptual and technical build-up to Chapter 3.

Index Notation: In this thesis abstract index notation will be heavily used. In this notation, tensor equations will be written without assuming an explicit basis. However, *components* of tensors will be frequently be written in a specific coordinate basis (e.g. the comoving coordinates of FRW) in this and following chapters. Generally, Greek letters will denote the indices of basis-dependent tensor components and Latin letters their abstract, basis independent, cousins. Their ranges should be clear by context, e.g. an expression involving derivatives of spatial tensors will involve only spatial derivatives unless explicitly stated otherwise.

## 2.1 Homogeneous and isotropic spacetimes

Roughly, we may understand that spatial homogeneity implies that we do not occupy a special place in the universe, that wherever we happen to be the basic characteristics of what we observe would appear to be the same. In a similar way spatial isotropy implies that large-scale observations should not depend on what direction we look. Of course, then only one observer<sup>1</sup> at most can find the universe isotropic so it is not clear how useful this concept is. In fact, the rigorous definitions of these concepts requires a bit more thought and geometric construction.

We may imagine a spacetime endowed with the usual Lorentzian metric  $g_{ab}$  and foliate it by spatial hypersurfaces  $\Sigma_t$  labelled by successive instants of ‘time’  $t$ . The Lorentzian metric  $g_{ab}$

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<sup>1</sup>Or, technically, one *class* of observers for an expanding universe (the comoving observers).

induces a Riemannian (positive definite) metric  $h_{ab}(t)$  on each  $\Sigma_t$  by restricting the action of  $g_{ab}$  at each  $m \in \Sigma_t$  to vectors tangent to the basis vectors of  $\Sigma_t$ . Given this setup we may now rigorously say, following Wald in [3], that a spacetime is spatially homogeneous (H) and isotropic (I) if

- H: there exists an isometry, a transformation which preserves the spatial metric, such that for all  $t$  any  $m \in \Sigma_t$  can be carried to any  $n \in \Sigma_t$
- I: it is impossible to construct any geometrically preferred vectors on  $\Sigma_t$

With the aim of finding the field equations for such a symmetric space, we now ask what kind of characteristics any curvature tensor built from such a Riemannian metric can have. Thus, consider the Riemannian curvature  ${}^{(3)}R_{ab}{}^{cd}$  of  $h_{ab}(t)$ . We recall that  ${}^{(3)}R_{abcd} = {}^{(3)}R_{cdab}$  by the fundamental definition of the curvature tensor, which is defined by the commutator of two covariant derivatives on, say, the tensor  $M_{ab}$  restricted to  $\Sigma_t$ :

$$(D_a D_b - D_b D_a)M^c{}_d = -{}^{(3)}R_{\ell ad}^c M^{\ell}{}_d + {}^{(3)}R_{dab}^{\ell} M_{\ell}^c, \quad (2.1)$$

where  $D_a$  is the covariant derivative compatible with the spatial metric  $h_{ab}(t)$ . Another important property of the Riemann tensor is known as the Bianchi identity, which states that

$$D_{[a} {}^{(3)}R_{bc]de} = 0, \quad (2.2)$$

where curly (square) brackets around indices denote (anti)symmetrization of those same indices. Following Wald in [3], we note that we may view  ${}^{(3)}R_{ab}{}^{cd}$  at a point  $p \in \Sigma_t$  as a linear map,  $L$ , of a vector space  $W$  of antisymmetric, covariant, rank 2 tensors (two-forms) at  $p$  onto itself<sup>2</sup>:  $L : W \rightarrow W$ . Using the symmetry property  ${}^{(3)}R_{abcd} = {}^{(3)}R_{cdab}$ , it follows that  $L$  is symmetric with a natural, positive-definite inner product on  $W$  determined by the Riemannian metric  $h_{ab}$ . This means that  $W$  has an orthonormal eigenbasis of vectors of  $L$ , and if any the eigenvalues of these eigenvectors were distinct this would give a geometrical way in which to pick out a preferred two-form (and thus a vector). Our requirement of isotropy therefore requires that all the eigenvalues be equal, or in other words the curvature must be of the form

$${}^{(3)}R_{abcd} = K h_{c[a} h_{b]d}, \quad (2.3)$$

where  $K$  is an unspecified dimensionful scalar function. We may immediately plug equation (2.3) into the Bianchi identities (2.2) and deduce that (using metric compatibility)

$$D_e K = 0, \quad (2.4)$$

which means that  $K$  must be a (dimensionful) spatial constant. This conclusion is actually required by the assumption of homogeneity but the interesting thing here is that it was *not* a required assumption, for *the constancy of  $K$  is automatic by the Bianchi identities (2.2)*. In this way, *spatial isotropy implies spatial homogeneity* and we need only need assume isotropy.

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<sup>2</sup>It is antisymmetric map because  ${}^{(3)}R_{ab}{}^{(cd)} = 0 = {}^{(3)}R_{(ab)}{}^{cd}$ , mapping onto itself means, for 2-forms  $\omega_{ab} \in \Sigma_t$ ,  ${}^{(3)}R_{ab}{}^{cd} \omega_{cd} = \omega_{ab}$ .

Equation (2.3) implies that isotropic surfaces are spaces with a constant  $K$ , which has dimensions  $[length]^{-2}$  and we identify as the scalar curvature.  $K$  is typically scaled to be  $\pm 1, 0$ , where  $+1$  indicates a closed geometry,  $-1$  indicates a hyperbolic geometry,  $0$  a Euclidean one. Indeed, spaces of constant curvature are well-studied and have metrics that can be cast in the form

$$d\bar{s}^2 = \sum_{a,b} h_{ab} dx^a dx^b = \frac{\delta_{ab}}{(1 + Kr^2)^2} dx^a dx^b, \quad (2.5)$$

where we have used the standard summation convention of implicitly summing over repeated raised and lowered indices,  $\delta_{ab}$  is the Kronecker delta. Here I have chosen  $x^a$  to be Cartesian coordinates used in characterizing the surface and  $r^2 = x^2 + y^2 + z^2$ .

Now that we have determined the spatial geometry consistent with spatial isotropy, we may use this assumption once again to assure that any timelike observers with four velocity  $u^a$  will be orthogonal to homogeneous surfaces  $\Sigma_t$ . We may then easily write the full spacetime metric as

$$g_{ab} = -u_a u_b + h_{ab}(t) \quad (2.6)$$

since  $h_{ab}(t)u^a = 0$  for all  $u_a u^a = -1$ . By means of these timelike observers we may ‘transport’ the coordinates of a given  $\Sigma_t$  to that at another  $t + t'$ , and in that way label each  $\Sigma_t$  by the proper time  $\tau$  observed by each of those observers. This proper time and our previously chosen spatial coordinates then label each event within our coordinate chart, so one now has a general metric and set of coordinate systems for spatially isotropic and homogeneous spacetimes. Thus, finally, we may write the general metric as

$$ds^2 = g_{ab} dx^a dx^b = -d\tau^2 + a(\tau)^2 \gamma_{ij} dx^i dx^j, \quad (2.7)$$

where  $a(\tau)$  is a so-called scale factor, so that  $h_{ab}(\tau) = a(\tau)^2 \gamma_{ab}$ . We can see that there will only be dynamics with respect to this proper time parameter  $\tau$ .

Now we can try to find the field equations of GR for this particular space. The general field equations of GR, i.e. the detailed versions of (1.1), were given by Einstein some 90 years ago as

$$R^d{}_{adb} - \frac{1}{2} g_{ab} R^{\ell}{}_{m\ell}{}^m + \Lambda g_{ab} \equiv G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \quad (2.8)$$

where the usual Einstein tensor is defined through contractions of the full, four dimensional, Riemann tensor associated with  $g_{ab}$ , defined in turn completely analogously to  ${}^{(3)}R_{abcd}$  above. Here the choice  $\kappa \equiv \frac{8\pi G}{c^4}$  ensures the correct Newtonian limit to the theory. The form of the left hand side of (2.8) is dictated by the requirement that the right hand side satisfy conservation of stress-energy (i.e.  $\nabla^b T_{ab} = 0$ , where  $\nabla_a$  is the covariant derivative compatible with  $g_{ab}$ ) and  $\Lambda$  is the so-called cosmological constant which for now we set to zero.

Keeping the dimensionality of  $\Sigma_t$  general, it is easy to show that for spatially isotropic spaces

the contactions of the three-curvature are

$${}^{(3)}R_{ab} = K(n-1)h_{ab} \quad (2.9)$$

$${}^{(3)}R = Kn(n-1), \quad (2.10)$$

for  $n \equiv \dim(\Sigma_t)$ . From these relations we can actually find the normal and tangential projections of the field equations with respect to the surfaces  $\Sigma_t$ , i.e.

$$G_{ab}u^a u^b = \kappa T_{ab}u^a u^b \quad (2.11)$$

$$h^b_a G_{b\ell}u^\ell = \kappa h^b_\ell T_{\ell a}u^a, \quad (2.12)$$

using the so-called Gauss-Codazzi equations which relate the three-curvature ( ${}^{(3)}R_{abcd}$ ) and four-curvature ( ${}^{(4)}R_{abcd}$ ):

$${}^{(3)}R_{abc}{}^d = h_a{}^f h_b{}^g h_c{}^k h^d{}_j {}^{(4)}R_{fgk}{}^j - K_{ac}K_b{}^d + K_{bc}K_a{}^d \quad (2.13)$$

$$\bar{D}_a K^a{}_b - \bar{D}_b K^\ell{}_\ell = {}^{(4)}R_{cd}u^a h^c{}_b, \quad (2.14)$$

where  $K_{ab} \equiv \frac{1}{2}\mathcal{L}_u h_{ab}$  is the contracted extrinsic curvature (commonly just called the extrinsic curvature), defined by Lie-dragging the three metric along the normal to  $\Sigma_t$  (roughly, ‘taking the time derivative of  $h_{ab}$ ’). These complicated relations allow one to show, in a calculation which is not very illuminating at this stage, that

$$2G_{ab}u^a u^b = {}^{(3)}R + (K^\ell{}_\ell)^2 - K_{ab}K^{ab} \quad (2.15)$$

$$h^b_a G_{b\ell}u^\ell = h^b_a {}^{(4)}R_{b\ell}u^\ell = 2\bar{D}_{[\ell}K^\ell{}_{a]}, \quad (2.16)$$

which puts us within striking distance of the projected field equations. Indeed, we note that

$$2K_{ab} = \mathcal{L}_u h_{ab} \stackrel{*}{=} \partial_\tau h_{ab} \equiv 2Hh_{ab} \quad (2.17)$$

$$K^\ell{}_\ell = nH, \quad (2.18)$$

where  $H \equiv \frac{\partial_0 a(\tau)}{a(\tau)} \equiv \dot{a}/a$  and the overstar indicates we evaluate in the Lie derivative in the coordinate system of (2.7). Finally, we plug this into equations (2.11), (2.12), (2.15), and (2.16) to obtain

$$(n^2 - n)H^2 + n(n-1)\frac{K}{a^2} \stackrel{*}{=} 2\kappa T_{00} \equiv 2\kappa\rho, \quad (2.19)$$

where the other three equations corresponding to equations (2.12) are trivially  $0 = 0$  by three-metric compatibility. Here the assumption that the stress energy is of the form of a perfect fluid<sup>3</sup> has been used. It can be shown that such a stress energy is of the most general form compatible

<sup>3</sup>The other three equations are formally  $D^\ell(K_{\ell b} - Tr(K)h_{\ell b}) = \kappa h^b_\ell T_{\ell a}u^a$ . These are the tangential projections (along  $\Sigma_t$ ) of the field equations, and are clearly nontrivial for anisotropic models.

with spatial isotropy, namely, for a fluid with fluid velocity  $u^a$ ,

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab}, \quad (2.20)$$

where  $\rho, p$  are the energy density and pressure respectively.

Equation (2.19) only contains single derivatives in  $\tau$  and as such is not an *evolution* equation for  $a(\tau)$  but rather some type of initial value *constraint* equation. In order to obtain the evolution equation we can use stress-energy, or *conservation*, condition  $\nabla^b T_{ab} = 0$  (or equivalently, the contracted Bianchi identities  $\nabla^a G^{ab} = 0$ ) to obtain

$$\dot{\rho} + nH(\rho + p) = 0. \quad (2.21)$$

using equation (2.19) for  $\rho$ , we find after a simple calculation that

$$\frac{\ddot{a}}{a} = \frac{\kappa}{n^2 - n} \{(2 - n)\rho - np\} \quad (2.22)$$

Equations (2.22) and (2.19) represent the constrained evolution equations of general relativity for any spatially isotropic space with perfect fluid matter. Whenever  $n = 3$ , which is the only case we will consider in this thesis, the equations reduce to

$$3H^2 + 3\frac{K}{a^2} = \kappa\rho \quad (2.23)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p), \quad (2.24)$$

which are known as the Friedmann-Robertson-Walker (FRW) equations<sup>4</sup>. The solutions to the FRW equations, which are surprisingly rich despite the restriction of spatial isotropy, are extensively discussed in many texts (e.g. [3]).

To summarize, a spatially isotropic spacetime is *automatically* spatially homogeneous and has general relativistic dynamics described by equations (2.22) and (2.19), which are nonlinear ordinary differential equations in the proper time variable  $\tau$ . The spacetime geometry these equations, or rather the special cases equations (2.23) and (2.24), describe will form the background spacetime about which we will perturb the metric.

## 2.2 Linearized theory

The pioneering 1946 paper by Lifschitz, [11], effectively introduced the modern form of linear cosmological perturbation theory, which has subsequently been primarily modified by the way it treats so-called gauge ambiguities. Mathematically the problem of describing the growth of small perturbations in the context of cosmology reduces to linearizing the field equations of GR about a particular solution to equations (2.23) and (2.24). Indeed, we violate the explicit covariance of general relativity because the generalized gravitational field, or spacetime metric, is preferentially

<sup>4</sup>A curiosity for  $n = 2$ , i.e. '2 + 1' gravity, is that the energy density does not source any acceleration.

decomposed into a background metric and a perturbation. The latter is taken to be small in some ‘reasonable’ measure and the field equations for these perturbations are truncated at some order and solved. The key result which is taken to put *linearized* theory about FRW on mathematically sound ground is due to D’Eath in [7], where he shows that a large class of linear perturbations which satisfy the linearized equations of motion actually represent linearizations (in the same small parameter) of the *exact* solution. We discuss this in detail a little later in the context of linearization stability, though we note here there is some dispute over his results for the case of a closed ( $K = +1$ ) FRW background (see [12]).

Taking the background metric to be as in equation (2.27) and adopting the comoving coordinates it is expressed in, we may write the general perturbation of a FRW metric as

$$\begin{aligned}
 ds^2 = & -(1 + \epsilon A(t, \vec{x}) + \epsilon^2 \mathcal{A}(t, \vec{x}) + \dots) dt^2 + 2(\epsilon B_i(t, \vec{x}) + \epsilon^2 \mathcal{B}_i(t, \vec{x}) + \dots) dt dx^i \\
 & + a^2(t)(\gamma_{ij} + \epsilon h_{ij}(t, \vec{x}) + \epsilon^2 q_{ij}(t, \vec{x}) + \dots) dx^i dx^j,
 \end{aligned} \tag{2.25}$$

where  $\epsilon$  is the small parameter of the metric perturbations. The order of a quantity will refer to its order in  $\epsilon$  unless otherwise noted. To linear order the above expansion of the metric has some simplifying features. To illustrate this, we write the linearized perturbations in matrix form

$$\begin{aligned}
 \delta(ds^2) = & a^2(t) \underbrace{\begin{pmatrix} -2A & +2D_i B \\ +2D_i B & +2D_i D_j E - 2\psi \gamma_{ij} \end{pmatrix}}_{\text{Scalar sector}} \\
 & + a^2(t) \underbrace{\begin{pmatrix} 0 & -2V_i \\ -2V_i & +2D_{(i} F_{j)} \end{pmatrix}}_{\text{Vector sector}} + a^2(t) \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & +2t_{ij} \end{pmatrix}}_{\text{Tensor sector}}
 \end{aligned} \tag{2.26}$$

The vectors  $V^i$  and  $\mathcal{F}^i$ , along with the tensor  $t_{ij}$ , are defined to satisfy constraints of the form

$$D_i F^i = 0, \quad D_i V^i = 0, \quad D_j t^{ij} = 0, \quad t^i{}_i = 0, \tag{2.27}$$

to ensure that they do not contain parts that transform as scalars and/or vectors, i.e. that they are *purely* vectorial (transverse) and tensorial (traceless and transverse) perturbations. As we indicate above, the first part of the perturbations are the so-called scalar perturbations, the second part are the vector perturbations, and finally the tensor or gravity wave perturbations. This classification refers to the way in which these particular perturbations transform under coordinate transformations. Although the terminology is perhaps unfortunate, one should just remember that a ‘scalar’ perturbation is simply a tensorial perturbation made of the three-metric or covariant derivative acting on *scalars*, and similarly for vectorial perturbations. Thus, we see that there are *four* metric scalar functions  $A, \psi, E, B$ , *two* three-vectors  $\mathcal{F}_i, V_i$ , and *one* symmetric three-tensor  $t_{ij}$ , giving  $4(1) + 2(3) + 1(6) = 16$  functions. But there are precisely six constraints as well from  $D_i \mathcal{F}^i = 0, D_i V^i = 0, D_j t^{ij} = 0, t^i{}_i = 0$ , which leaves  $16 - 6 = 10$  independent components of the metric tensor, as required.

The various sectors of perturbations decouple and evolve independently at linear order<sup>5</sup>, and each has a separate interpretation assigned to it. The tensor perturbations, as indicated, are genuine gravitational waves while the vector perturbations describe vorticity, or rotational, types of perturbations. Scalar perturbations ultimately describe perturbations in energy densities and pressures and lead to important growths in inhomogeneities while vector and tensor perturbations, to linear order, respectively decay away kinematically in an expanding universe and do not affect inhomogeneities. For this reason we focus our emphasis exclusively to scalar perturbations but bear in mind that to second order the sectors can mix as they interact (this will come out in Chapter 4).

### 2.2.1 Gauge transformations

As compared to the general covariance of the full (infinite order) theory, which allows for arbitrary coordinate transformations, within the framework of linear perturbation theory we consider only linear infinitesimal parts of these coordinate transformations (called gauge transformations in cosmological perturbation theory). The individual perturbations of the metric or stress-energy components will change in some well-defined way under such gauge transformations. Indeed, one can write a general  $\epsilon$ -dependent coordinate transformation by

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x, \epsilon), \quad (2.28)$$

and define an associated *linearized* coordinate transformation by

$$\zeta^a = \lim_{\epsilon \rightarrow 0} \frac{\partial \tilde{x}^c(x, \epsilon)}{\partial \epsilon} \frac{\partial X^a(\tilde{x}(x, \epsilon), \epsilon)}{\partial \tilde{x}^c}, \quad (2.29)$$

where  $X^a(\tilde{x}(x, \epsilon), \epsilon) = x^a$ ,  $\forall \epsilon$ . Since the metric  $g_{ab}$  is a tensor it will in general transform by

$$g_{ab}(x, \epsilon) = \frac{\partial \tilde{x}^c}{\partial x^a} \frac{\partial \tilde{x}^d}{\partial x^b} \tilde{g}_{cd}(\tilde{x}(x, \epsilon), \epsilon), \quad (2.30)$$

one can show that under a linear coordinate transformation the linear metric fluctuations  $\delta g_{ab}$  suffer the change

$$\lim_{\epsilon \rightarrow 0} \frac{\partial g_{ab}(x, \epsilon)}{\partial \epsilon} \equiv \delta g_{ab} = \delta \tilde{g}_{ab} + \mathcal{L}_\zeta(g_{ab}(\epsilon = 0)) \equiv \tilde{g}_{ab} + \mathcal{L}_\zeta \tilde{g}_{ab}, \quad (2.31)$$

where  $\mathcal{L}_\zeta$  is the usual Lie derivative along the vector  $\zeta^a$ . To linear order it is fairly clear that transformation (2.28) cannot affect the tensor sector, while the vector and scalar sectors will in general be affected. The scalar or vector nature of the perturbations is in turn only preserved by scalar or vector gauge transformations, defined by analogy to the perturbations themselves. There will in general exist (an infinite number) of combinations of scalar perturbations which are invariant, to first order, under this restricted class of coordinate transformations [13]. Again, in what follows of this introduction we will only consider the scalar fluctuations.

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<sup>5</sup>This is nontrivial to prove. Scalar-vector and tensor-scalar decoupling is equivalent to a condition on the background curvature. Spatial isotropy is powerful enough to satisfy these curvature constraints.

In standard cosmological perturbation theory one usually makes a particular choice of  $\zeta^a$  to simplify the interpretation and fix the gauge freedom of the scalar fluctuations, called the *longitudinal gauge* [4]:

$$\zeta^0 = B - a\dot{E}, \quad (2.32)$$

$$\zeta^i = -\partial^i E, \quad (2.33)$$

which effectively sets the *potentials*  $B, E$  of the metric fluctuations  $B_{,i}, E_{|ij}$  to zero. This choice leaves no additional residual coordinate freedoms for the remaining scalar perturbations, and it is in this sense equivalent to using the so-called gauge invariant approach discussed in [13]. I will call such gauge transformations *exhaustive*. There are an infinite number of such choices available.

## 2.2.2 Equations of motion

The field equations (2.8) can be derived from an action principle. The action is defined by integration of the Lagrangian density over a chart of the four dimensional spacetime, which in the case of pure gravity (i.e. GR) is well-known as the Einstein-Hilbert action

$$S = -\frac{1}{2\kappa} \int R\sqrt{|g|}d^4x \quad (2.34)$$

Applying Hamilton's principle,  $\delta S = 0$ , in the usual way to this action leads directly to equations (2.8) where  $T_{ab} = 0$ . In this thesis we will consider perfect irrotational fluids, which we will later show can be represented by minimally coupled scalar fields (taken to be velocity potentials for the fluid) with some potential  $V(\phi)$ . One can quickly see this equivalence<sup>6</sup> by seeing that the stress-energy

$$T_{ab} = \bar{\nabla}_a\phi\bar{\nabla}_b\phi - \left(\frac{1}{2}g^{\ell m}\phi_{,\ell}\phi_{,m} + V(\phi)\right)g_{ab} \quad (2.35)$$

can be written in the form of the perfect fluid stress energy given by equation (2.20) *provided* the identification

$$u^a = \frac{\partial^a\phi}{\sqrt{-\phi^{,a}\phi_{,a}}} \quad (2.36)$$

holds. Just as the metric was perturbed, so too is the scalar field  $\phi$  perturbed as

$$\phi = \bar{\phi} + \epsilon\Phi + \epsilon^2\mathcal{F} + \dots, \quad (2.37)$$

and indeed the metric and matter fluctuations interact with each other through the perturbed field equations.

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<sup>6</sup>One should note that a subtlety with this equivalence (which is worth further investigation) lies in requiring the scalar field satisfy  $u^a u_a = -1$ . This is a strong restriction on the scalar field (velocity potential) gradients which, for example, will lead to *continuously* self-similar critical solutions found for fluid collapse. This should be contrasted with the *discrete* self-similarity of critical solutions originally found by Choptuik in Phys. Rev. Lett. **70**, 9 (1993), which assume *no* restriction on the gradients.



To obtain the linearized equations of motion in vacuum one must expand the right hand side of equation (2.34) to *second* order, take the first variation and apply Hamilton's principle, and similarly for the sum of gravitational and matter actions to get the complete equations. This procedure is straightforward but comprised entirely of cumbersome algebraic steps (see [4] for the full details). After some manipulations<sup>7</sup>, the combined second order action for the metric and matter fluctuations is (see [14] for the original calculation)

$$\begin{aligned}
 {}^{(2)}S &= \frac{1}{2\kappa} \int a(\eta)^2 \sqrt{|\bar{\gamma}|} \left[ -6\psi'^2 - 12\mathcal{H}A\psi' + 2(\bar{\Delta}\psi')(2A - \psi) - 2(\mathcal{H}' + 2\mathcal{H}^2)A^2 \right. \\
 &\quad \left. + \kappa \left( \Phi'^2 + \Phi\bar{\Delta}\Phi - a(\eta)^2 V_{,\phi\phi}\Phi^2 \right) + 2\kappa(3\bar{\phi}'\psi'\Phi - \bar{\phi}'\Phi'A - a(\eta)^2 A\Phi V_{,\phi}) \right. \\
 &\quad \left. + K \left[ -6\psi^2 + 2A^2 + 12\psi A + 2(B - E')\bar{\Delta}(B - E') \right] + 4\bar{\Delta}(B - E') \left( \frac{\kappa\bar{\phi}'\Phi}{2} - \mathcal{H}A - \psi' \right) \right] d^4x,
 \end{aligned} \tag{2.38}$$

where  $K$  is the scalar curvature parameter from equation (2.23), the primes indicate differentiation with respect to conformal time  $\eta$ , and  $\bar{\Delta}$  is the spatial laplacian  $\bar{D}^a\bar{D}_a$  for the spatial hypersurfaces. Conformal time  $\eta$  is defined via the relation  $dt = a(\eta)d\eta$  and I will interchange between comoving and conformal time often in what follows, primarily because it facilitates checking results with other work, which is mainly done in conformal time. In the above,  $\gamma_{ij}$  is the metric of the background spacetime at  $\eta = \text{constant}$ , and  $|\gamma|$  is its determinant.

The background equations of motion, in conformal time, are

$$\mathcal{H}^2 + K - \mathcal{H}' = \frac{(\bar{\phi}')^2 \kappa}{2} \tag{2.39}$$

$$2\mathcal{H}' + K + \mathcal{H}^2 = \frac{\kappa}{2}(2a^2V - (\bar{\phi}')^2) \tag{2.40}$$

$$\partial_\eta \bar{\phi}' + 2H\bar{\phi}' + a^2V_{,\phi} = 0, \tag{2.41}$$

where Equation (2.41) is the only nontrivial component of  $\bar{\nabla}_b \bar{T}^{ab} = 0$ . Varying the right hand side of (2.38) with respect to the various perturbations and setting it equal to zero will give the linearized equations of motion for scalar fluctuations. Instead of doing this directly we will proceed to cast the action into first order (Hamiltonian) form in order to get a better view of the role of the constraints and how they generate gauge-transformations. In the following development we assume a decomposition of spacetime very similar to the one given to define the background spacetime around equation (2.6).

Indeed, one may define the canonical variables

$$\pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \psi'} = \frac{2a^2\sqrt{|\bar{\gamma}|}}{\kappa} \left( -3(\psi' + \mathcal{H}A) + \bar{\Delta}(E' - B) + \frac{3\kappa\bar{\phi}'\Phi}{2} \right) \tag{2.42}$$

$$\pi_\Phi = a^2\sqrt{|\bar{\gamma}|}(\Phi' - \bar{\phi}'A) \tag{2.43}$$

$$\pi_E = \frac{2a^2\sqrt{|\bar{\gamma}|}}{\kappa} \bar{\Delta}(K(E' - B) + \psi' + \mathcal{H}A), \tag{2.44}$$

<sup>7</sup>A phrase of some endearment to the author.

where the Lagrangian density  $\mathcal{L}$  is the integrand of (2.38) times  $2\kappa$ . The conjugate momenta for the scalar lapse and shift perturbations,  $A$  and  $B$  respectively, are

$$\pi_A = 0 \quad (2.45)$$

$$\pi_B = 0, \quad (2.46)$$

and are referred to as the *primary constraints* of the theory. They signify, since e.g.  $\dot{A}$  cannot be inverted in terms of  $\pi_A$ , that  $A, B$  are in some sense not dynamical because they obey the kinematic property of not having kinetic terms in the action. Re-expressed in terms of these conjugate variables, one may show after a lengthy calculation that

$$\begin{aligned} {}^{(2)}S &= \int \left( \pi_\psi \psi' + \pi_E E' + \pi_\Phi \Phi' - {}^{(2)}H \right) d^4x \\ &= \int \left( \pi_\psi \psi' + \pi_E E' + \pi_\Phi \Phi' - \left[ J(\eta) \left( \pi_E (2\pi_\psi + \frac{3\pi_E}{\Delta}) + 2(\bar{\Delta} + 3K) \frac{\pi_\Phi^2}{\kappa} - K\pi_\psi^2 \right) \right. \right. \\ &\quad \left. \left. + \frac{a(\eta)^2 \sqrt{|\bar{\gamma}|}}{\kappa} \left[ (\bar{\Delta} + 3K)\psi^2 - \frac{\kappa}{2} ((\bar{\Delta} + 3K) - \mathcal{H}^2 - \mathcal{H}' + \frac{\bar{\phi}'''}{\bar{\phi}''}) \Phi^2 \right] + \frac{\kappa \bar{\phi}'}{2} \pi_\psi \Phi + AC_A + BC_B \right] \right) d^4x, \end{aligned} \quad (2.47)$$

where

$$\mathcal{C}_A = -\mathcal{H}\pi_\psi + \phi'_B \pi_\Phi + \frac{2a^2 \sqrt{|\bar{\gamma}|}}{\kappa} \left( -(-\ell(\ell+2) + 3K)\psi + \frac{\kappa}{2} (\mathcal{H}\phi'_B - \phi''_B)\Phi \right), \quad (2.48)$$

$$\mathcal{C}_B = \pi_E, \quad (2.49)$$

$$J(\eta) \equiv \frac{\kappa}{4a^2 \sqrt{|\bar{\gamma}|} (\bar{\Delta} + 3K)} \quad (2.50)$$

If one now varies the right hand side of equation (2.47) with respect to the canonical variables and applies Hamilton's principle, the equations of motion are given in first order form. In particular, if we vary with respect to  $A$  and  $B$ , the scalar lapse and the shift perturbations, the relations  $\mathcal{C}_A = 0, \mathcal{C}_B = 0$  result. In Dirac's language of constrained Hamiltonian theory [15], they are known as the *secondary constraints* and can be thought of as equivalent to the demand that the *primary constraints* are preserved from slice  $\Sigma_t$  to  $\Sigma_{t+t'}$ . Equation (2.48) is the linearized version of the so-called *Hamiltonian constraint* of GR, the left hand side of which is given by equation (2.15). Similarly, (2.49) is the (scalar) linearized version of the *momentum constraints* of GR. They will simply be referred to as the 'constraints' in what follows, unless otherwise noted.

If one demands that the Hamiltonian and momentum constraints are preserved from  $\Sigma_t$  to  $\Sigma_{t+t'}$ , both in the full theory and in linearized perturbation theory, no new constraints arise. In fact, again in the language of Dirac [15], the linearized Hamiltonian and momentum constraints are both said to be of *first class* in the sense that they directly generate the linearized gauge transformations<sup>8</sup> under which the canonical variables can change. In order to show how they do this and then define the gauge-fixed Hamilton equations for the cosmological perturbations, we will require additional structure of the phase space spanned by  $(A, B, \psi, E, \Phi; \pi_A, \pi_B, \pi_\psi, \pi_E, \pi_\Phi) \equiv (q_i; p_i)$  which allows

<sup>8</sup>In fact, unlike in the full theory, they form a Lie algebra.

some notion of evolution.

This additional structure will partially come from the Hamiltonian function,  $H = H(q_i; p_i)$ , which defines dynamical evolution on our five dimensional phase space via the Hamilton equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad (2.51)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (2.52)$$

We can express the Hamilton equations in a more succinct form by writing  $z = (q_i; p_i)$  and introducing the  $10 \times 10$  anti-symmetric matrix  $\Omega^{\mu\nu}$ , with  $\Omega^{\mu\nu} = 1$  when  $\nu = \mu + 5$  and  $\Omega^{\mu\nu} = 0$  when  $|\mu - \nu| \neq 5$ . Then, we obtain, for  $\mu = 1, \dots, 10$ ,

$$\frac{dz^\mu}{dt} = \sum_{\nu=1}^{10} \Omega^{\mu\nu} \frac{\partial H}{\partial y^\nu} \quad (2.53)$$

The matrix  $\Omega^{\mu\nu}$  is commonly referred to as the *inverse symplectic form* on the phase space spanned by  $z^\mu$ . The inverse symplectic form gives rise to a product structure for functions of the canonical phase space variables  $z^\mu$  called the *Poisson bracket*, which is defined by

$$\begin{aligned} \{\alpha, \beta\} &= \Omega^{\mu\nu} \nabla_\mu \alpha \nabla_\nu \beta \\ &= \sum_i \left( \frac{\partial \alpha}{\partial q_i} \frac{\partial \beta}{\partial p_i} - \frac{\partial \beta}{\partial q_i} \frac{\partial \alpha}{\partial p_i} \right) \end{aligned} \quad (2.54)$$

where  $\alpha$  and  $\beta$  are arbitrary functions of the  $(q_i; p_i)$ .

With the Hamiltonian and the Poisson bracket, we now have enough structure on the phase space to discuss evolution and gauge transformations. Using the bracket, the total time derivative of a dynamical variable  $F(z)$  is simply

$$\dot{F} = \{F, H_{total}\} + \frac{\partial F}{\partial t}, \quad (2.55)$$

which allows the standard Hamiltonian equations of motion to be compactly expressed as

$$\dot{q}_i = \{q_i, H\} = \partial_{p_i} H, \quad (2.56)$$

$$\dot{p}_i = \{p_i, H\} = -\partial_{q_i} H. \quad (2.57)$$

It is crucial to note the appearance of the partial derivative term in equation (2.55) since it makes contributions whenever there is an explicit time dependence in a canonical variable. The canonical variables will change under the general (scalar) transformation

$$\eta \rightarrow \eta + T(\eta, \vec{x}) \quad (2.58)$$

$$x^i \rightarrow x^i + \partial^i L, \quad (2.59)$$

which can be thought of as generated by an infinitesimal vector field  $\zeta^a = (T, \partial^i L)$ , as in equation (2.29). The Poisson brackets allow us to see how the constraints force the canonical variables to change in a very specific manner when transformations (2.58) and (2.59) are performed. Indeed, one may show that the infinitesimal change in some canonical variable  $z^\mu$  is

$$\delta_\zeta(z^\mu) = T\{z^\mu, \mathcal{C}_A\} + L\{z^\mu, \mathcal{C}_B\}, \quad (2.60)$$

which, when applied to the cosmological perturbations, results in e.g.:

$$\delta_\zeta \psi = T\{\psi, \mathcal{C}_A\} + L\{\psi, \mathcal{C}_B\} = T(-\mathcal{H}) + 0 = -\mathcal{H}T \quad (2.61)$$

$$\delta_\zeta \pi_\psi = \frac{2a(\eta)^2 \sqrt{|\bar{\gamma}|}}{\kappa} (\bar{\Delta} + 3K)T \quad (2.62)$$

$$\delta_\zeta \Phi = T\bar{\phi}' \quad (2.63)$$

$$\delta_\zeta \pi_\Phi = a(\eta)^2 \sqrt{|\bar{\gamma}|} (\mathcal{H}\bar{\phi}' - \bar{\phi}'')T \quad (2.64)$$

$$\delta_\zeta E = L \quad (2.65)$$

$$\delta_\zeta \pi_E = 0 \quad (2.66)$$

Using the above transformations we can deliberately select  $T$  and  $L$ , i.e. make a gauge choice, such that particular combinations of these fluctuations are zero. One popular choice was already described in equations (2.32) and (2.33), known as the longitudinal gauge. We adopt this longitudinal gauge by choosing a particular  $T$  and  $L$  such that

$$\eta \rightarrow \eta + (B(\eta, y^i) - E'(\eta, y^i)), \quad (2.67)$$

$$x^i \rightarrow x^i - D^i E(\eta, y^i), \quad (2.68)$$

which effectively sets  $B = 0 = E$ . We then solve the constraints, (2.48) and (2.49) set to zero, in the longitudinal gauge for a given set of fluctuations in terms of the remaining set. Solving for  $\pi_\Phi$  and  $\pi_\psi$ , it can be easily shown that the Hamiltonian action implies that

$$\pi_E = -\frac{\Delta}{3}\pi_\psi \Rightarrow A = \psi, \quad (2.69)$$

and we can finally write the reduced Hamiltonian for the fluctuations. For the case of  $K = 0$ , i.e. a spatially flat FRW background,

$${}^{(2)}H = \int (\pi_\nu \nu' - \tilde{\mathcal{L}}) d^3x = \frac{1}{2} \int (\pi_\nu^2 + \gamma^{ij} \nu_{,i} \nu_{,j} - \frac{z''}{z} \nu^2) \sqrt{|\bar{\gamma}|} d^3x, \quad (2.70)$$

where  $z \equiv \frac{a\bar{\phi}'}{\mathcal{H}}$ ,  $\nu \equiv a(\eta)(\Phi + \frac{\bar{\phi}'}{\mathcal{H}}\psi)$ , and we have discarded the boundary terms which naturally occur at spatial infinity. These results agree with equations (10.71)-(10.73) of [4], and the key feature to note is that the reduced Hamiltonian is of the form a time-dependent simple harmonic oscillator. It is remarkable that the second order action (2.47) can be cast in this form and the ease with which one makes the transition to treating quantum perturbations is directly related to

this simplicity.

To summarize, we have obtained the reduced Hamiltonian action for the inhomogeneous cosmological fluctuations in the longitudinal gauge-fixing. It has the form of a simple harmonic oscillator with time dependent frequency  $\frac{z''}{z}$  that depends explicitly on the background scale factor  $a(\eta)$  and matter  $\bar{\phi}(\eta)$ . I note in passing that there is a subtlety with obtaining the reduced action for the spatially homogeneous fluctuations about flat FRW [16] that is worth mentioning. Namely, if one simply takes the action (2.47) and performs the limit  $\bar{\Delta}F(\eta, \vec{x}; q_i, p_i) \rightarrow 0$  and furthermore sets  $K = 0$ , where  $F$  is any of canonical variables, then the reduced Lagrangian action is formally divergent. The proper procedure to find the reduced action is best illustrated by the following example, which involves three primary steps.

- start with a homogeneously perturbed metric and compute the only nontrivial constraint for this case (the Hamiltonian constraint);
- gauge-fix the time reparametrization freedom  $T$  in an exhaustive way, for example setting the scalar field perturbation  $\Phi = 0$ .
- solve the gauge-fixed constraint for the matter variables and only then substitute the results into the action.

In this example, the homogeneous fluctuations in a flat FRW background are described by the reduced Hamiltonian action

$${}^{(2)}S = \int \left( \pi_\psi \psi' - \pi_\psi^2 \left( \frac{\kappa V(\phi)}{6(\bar{\phi}')^2} \right) \right) d^4x. \quad (2.71)$$

It is important to note that the homogeneous modes of scalar fluctuations in flat FRW are dynamical and real, they cannot be set to zero via homogeneous gauge transformations. However typically it is best to check on a case-by-case basis as, for example, it is equally true that the  $\ell = 1$  scalar perturbations of a  $K = +1$  FRW spacetime can be set to zero by a gauge transformation<sup>9</sup>.

## 2.3 Linearization stability

Beginning in the 1970's with the work of Fischer, Marsden, Moncrief, D'Eath, and Arms (see [5], [6], [7], [8], [9], [10]) it was realized that there were situations where the linearized Einstein equations would have spurious solutions. These solutions do not correspond to linearized approximations of the desired nonlinear solution of the field equations. The existence of such spurious solutions to the linearized field equations of a given theory is referred to as its 'linearization instability'. This unfortunate terminology does not refer to any dynamical instability, it only refers to the possible existence of spurious solutions to the linearized equations.

The theory of linearization instabilities, as cast in the modern form by Moncrief et al in [5] and [6], also provided a second order condition on the linearized perturbations which is necessary (and presumably sufficient, though not proven so far) to exclude such spurious solutions. In this

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<sup>9</sup>Here,  $\ell$  is defined by  $\Delta F = -\ell(\ell + 2)F$  on a closed  $\Sigma_t$  slice of  $K = +1$  FRW.

section we will briefly review the basics of linearization instabilities via a Hamiltonian formulation for gravity, and add in some small new results along the way. The purpose of this section is to motivate and set the notation for a calculation that demonstrates that the requirement of linearization stability actually *leads* to a physical instability in a specific model called Einstein static.

Before jumping straight into formalism, however, it is perhaps useful to take a basic look at what a linearization instability is. Consider the equation  $x^2 + y^2 = 0$  and suppose we do not know its general (real) solutions. We decide to linearize it about the clear background solution  $\bar{x} = \bar{y} = 0$ , and thus attempt to solve the simpler equation  $2(\bar{x}\delta x + \bar{y}\delta y) = 0$ . Given this background solution it is clear that the perturbations  $\delta x$  and  $\delta y$  can be *any real number*, even though we know from the general solution ( $x = y = 0$ ) they can only be  $\delta x = \delta y = 0$ . Therefore the equation  $x^2 + y^2 = 0$  is said to be linearization unstable around  $\bar{x} = 0 = \bar{y}$ .

### 2.3.1 En route to linearization stability: Hamiltonian formulation

In order to discuss linearization stability it is essential to consider the linearized field equations as a set of constrained evolution equations. It is sufficient to examine only the linearized constraint equations for a linearization stability<sup>10</sup> analysis. As in Section 2.1, the gravitational field may be characterized in terms of the evolution of a spacelike surface with respect to a given foliation of spacetime into such surfaces. In Section 2.2.2 we sketched how to pass from the Lagrangian formulation to the constrained Hamiltonian formalism in perturbation theory. A similar procedure holds for the exact theory of GR, which we now outline in more detail in order to more properly setup the following, more general, linearized analysis of the constraints.

As we did previously, we choose a time function  $t$  which labels the various hypersurfaces  $\Sigma_t$ . However, we now also define a ‘time flow’ vector field  $t^a$  satisfying the normalization condition  $t^a \nabla_a t = 1$ . Using a metric  $g_{\alpha\beta}$  we can decompose  $t^a$  into normal and tangential parts with respect to the hypersurfaces  $\Sigma_t$ . In this way we define the *lapse* and *shift* functions respectively

$$N \equiv -g_{ab} t^a n^b \quad (2.72)$$

$$N^i \equiv h^a_b t^b, \quad (2.73)$$

where  $n^a$  is the unit normal to  $\Sigma_t$  (or, the normalized vector  $u^a$  for the isotropic observer of Section 2.1) and  $h_{ab}(t)$  is the induced spatial metric on  $\Sigma_t$  defined by equation (2.6). Physically, after the spacetime metric is completely known,  $N$  measures the flow of proper time  $\tau$  with respect to coordinate time  $t$  for an observer moving *normally* to  $\Sigma_t$ , while  $N^a$  measures the tangential projection of  $t^a$  onto  $\Sigma_t$ . In fact it is easy to see that the normal  $n^a$  can be expressed in terms of the lapse and shift functions via

$$n^a = \frac{1}{N}(t^a - N^a), \quad (2.74)$$

<sup>10</sup>Indeed, the standard techniques of linearization instability theory are only applicable to elliptic equations.

which allows the inverse spacetime metric to be written as

$$g^{ab} = h^{ab} - n^a n^b = h^{ab} - \frac{1}{N^2}(t^a - N^a)(t^b - N^b) \quad (2.75)$$

The form of the above inverse metric suggests that a good choice of variables will be the spatial metric  $h_{ab}$ , the lapse  $N$  and shift  $N^a$  and it is clear that the information contained in the set  $(h^{ab}, N, N^a)$  is the same as that contained in the inverse metric  $g^{ab}$ . Thus the former set seems like a good choice of field variables for any decomposed formulation of field equations, as suggested initially in [17]. Along the same lines one can show that the relation

$$\sqrt{-|g|} = N\sqrt{|h|}, \quad (2.76)$$

which when combined with the rest of the results above allows one to re-express the gravitational action in terms of the new ‘ADM’ variables to obtain a Hamiltonian functional for general relativity. One should note that since the gravitational Lagrangian density defined in equation (2.34), re-expressed in these ADM variables, does not contain any time derivatives of  $N$  or  $N^a$  their conjugate momenta vanish in precisely the same way as in perturbation theory (cf. equations (2.45) and (2.46)), forming the primary constraints of GR. This in itself implies that we should not view  $N$  and  $N^a$  as dynamical variables<sup>11</sup>.

Since we have already effectively expressed the field equations in a projected form in the development around the Gauss-Codazzi equations (2.13) and (2.14), it only remains to define a conjugate momentum to the spatial metric  $h_{ab}$  in order to achieve a Hamiltonian formulation. We define this momentum-density<sup>12</sup> on the hypersurface  $\Sigma_t$  via

$$\pi^{ab} \equiv \sqrt{|h|}(K^{ab} - h^{ab}K), \quad (2.77)$$

where  $K \equiv K^a_a$  and  $K_{ab}$  was defined below equation (2.14), which we slightly modify to  $K_{ab} \equiv \frac{1}{2}\mathcal{L}_n h_{ab}$ .

Finally, using all of the above with the results following from equations (2.13) and (2.14), we can write the Hamiltonian and momentum constraints of the exact theory of GR as

$$\mathcal{H}_\perp = \sqrt{|h|} \left( G_{abcd} \pi^{ab} \pi^{cd} - {}^{(3)}R \right) - 2\kappa\rho = 0, \quad (2.78)$$

$$\mathcal{H}^i = -2D_j \pi^{ij} - \kappa J^i = 0, \quad (2.79)$$

respectively<sup>13</sup>, where

$$G_{abcd} \equiv \frac{1}{2|h|} (h_{ad}h_{bc} + h_{ac}h_{bd} - h_{ab}h_{cd}), \quad (2.80)$$

$$\rho \equiv \sqrt{|h|} T_{\alpha\beta} n^\alpha n^\beta, \quad (2.81)$$

$$J^i \equiv -\sqrt{|h|} h^{ib} T_{bc} n^c, \quad (2.82)$$

<sup>11</sup>In this sense it is acceptable to reduce one’s configuration space to only Riemannian metrics  $h_{ab}$  on  $\Sigma_t$ .

<sup>12</sup>A density is any quantity which is weighted by a factor of a spatial or full space-time volume element.

<sup>13</sup>The constraints are actually more usefully regarded as *weakly* equal to zero [15].

and where  $\rho$  and  $J^i$  are the matter energy density and momentum flux density across  $\Sigma_t$  respectively. For the particular case of a minimally coupled scalar field, whose stress energy is given by equation (2.35), one can easily show that

$$\begin{aligned}\mathcal{L}_\phi &= -\frac{1}{2}\sqrt{-|g|} [g^{ab}\phi_{,a}\phi_{,b} + 2V(\phi)] \\ &= -\frac{1}{2}N\sqrt{|h|} \left( \left[ -\frac{1}{N^2} \right] \dot{\phi}^2 + 2 \left[ \frac{N^i}{N^2} \right] \dot{\phi}\phi_{,i} + 2V(\phi) + h^{ij}\phi_{,i}\phi_{,j} \right),\end{aligned}\quad (2.83)$$

along with

$$\rho = \frac{\pi_\phi^2}{2\sqrt{|h|}} + \sqrt{|h|} \left( V(\phi) + \frac{1}{2}h^{ij}\phi_{,i}\phi_{,j} \right) \quad (2.84)$$

$$J^i = \pi_\phi \partial^i \phi, \quad (2.85)$$

where

$$\pi_\phi \equiv \frac{\partial \mathcal{L}_\phi}{\partial \dot{\phi}} = -\frac{1}{2}N\sqrt{|h|} \left[ 2\dot{\phi} \left[ -\frac{1}{N^2} \right] + 2 \left[ \frac{N^i}{N^2} \right] \phi_{,i} \right]. \quad (2.86)$$

These results allow us to express the Hamiltonian density of the combined gravity-scalar field system using their Lagrangians as defined above. It is

$$\begin{aligned}\mathcal{H} &= \pi^{ab}\dot{h}_{ab} + \pi_\phi\dot{\phi} - \mathcal{L}_G - \mathcal{L}_\phi \\ &= N\mathcal{H}_\perp + N^i\mathcal{H}_i + N \left[ \frac{\pi_\phi^2}{2\sqrt{|h|}} + V(\phi)\sqrt{|h|} \right] + N^i\pi_\phi\phi_{,i} + \frac{N\sqrt{|h|}}{2} [h^{ij}\phi_{,i}\phi_{,j}]\end{aligned}\quad (2.87)$$

and we again emphasize that  $\mathcal{H}$  is a *density* of weight 1, and not the total Hamiltonian. An easy comparison between equation (2.87) and equations (2.84)-(2.85), (2.78)-(2.79) reveals that the Hamiltonian density is simply a linear combination of the total Hamiltonian and momentum constraints with the lapse and shift functions as coefficients. Therefore, perturbations of the total Hamiltonian density automatically correspond to perturbations of the constraints. We use this fact in studying the linearized and higher order perturbations of the constraints in the following section and chapters.

### 2.3.1.1 Hamilton's equations for gravity

To obtain the total Hamiltonian  $H$  for the gravity-scalar field system, we integrate the density  $\mathcal{H}$  given in equation (2.87) over  $\Sigma_t$ . This is

$$H = \int_{\Sigma_t} N_\alpha \mathcal{H}^\alpha d^3x, \quad (2.88)$$

where we define the shorthand  $N_\alpha = (N, N^a)$  and  $\mathcal{H}^\alpha = (\mathcal{H}_\perp, \mathcal{H}^i)$ . Because the Hamiltonian is defined as the integral over a density, it is a coordinate invariant on  $\Sigma_t$ . It is important to note that for the case of  $\Sigma_t$  topologically open,  $H$  will equal certain boundary terms [18] essentially



because the Einstein-Hilbert Lagrangian density defined by equation (2.34) also includes a surface term. We will not deal with these terms in this thesis, for reasons explained a little later.

In order to obtain the equations of motion from  $H$  we vary the action to obtain

$$\delta H = \int_{\Sigma_t} \bar{N}_\alpha \delta \mathcal{H}^\alpha + \delta N_\alpha \bar{\mathcal{H}}^\alpha d^3x \quad (2.89)$$

$$= \int_{\Sigma_t} \bar{N}_\alpha \delta \mathcal{H}^\alpha d^3x = \int_{\Sigma_t} \bar{N}_\alpha \left( \frac{\delta \mathcal{H}^\alpha}{\delta u_a} \delta u_a \right) d^3x, \quad (2.90)$$

using the background constraints (denoted by overbars, as all background quantities are in this thesis), where  $u_a$  denotes the phase space variables (field configurations)  $(h_{ab}, \pi^{ab}, \phi, \pi_\phi)$ . Similarly,  $\delta u_a$  denotes the first variation of the phase space variables.

The variations of the  $u^a, H$  will be defined as follows, following Wald in [3]. Suppose  $H[u_a]$  is a functional of  $u_a$ , or in other words a map from field configurations into numbers. Let  $u_a^{(\lambda)}$  be a smooth 1-parameter family of field configurations, starting from  $u_a^{(0)}$ , which satisfy some boundary conditions. Denote  $\delta u_a = \left. \frac{du_a^{(\lambda)}}{d\lambda} \right|_{\lambda=0}$ , and we suppose that  $\left. \frac{\delta H}{\delta \lambda} \right|_{\lambda=0}$  is well-defined, i.e. exists, for such families of  $u_a^{(\lambda)}$ . In this way we can also define a smooth 1-parameter family of connections associated with the family of 3-metrics  $h_{ab}(\lambda)$ , which is essential in varying the curvature terms in the constraints. Indeed, let  ${}^\lambda D_a$  denote the covariant derivative operator associated with  $h_{ab}(\lambda)$  and  ${}^0 D_a$  denote the covariant derivative operator associated with the background metric  $\bar{h}_{ab}$ . The difference between  ${}^\lambda D_a$  and  ${}^0 D_a \equiv \bar{D}_a$  is determined by the tensor field  $C^a_{bc}(\lambda)$  given by

$$C^a_{bc}(\lambda) = \frac{1}{2} h^{a\ell}(\lambda) [\bar{D}_b h_{c\ell}(\lambda) + \bar{D}_c h_{b\ell}(\lambda) - \bar{D}_\ell h_{bc}(\lambda)], \quad (2.91)$$

where  $C^a_{bc}(\lambda = 0) \equiv \bar{C}^a_{bc} = 0$  automatically by metric compatibility. To find the analogous difference in the Ricci curvatures associated with the two connections, substitute  $C$  for usual connection  $\Gamma$  in the definitions of the Riemann tensor<sup>14</sup> and contract to form the Ricci tensor. The result is

$$R_{ab}(\lambda) = \bar{R}_{ab} + 2C^c_{b[a} C^\ell_{\ell]c} - 2\bar{\nabla}_{[a} C^c_{c]b}, \quad (2.92)$$

where square (curly) brackets henceforth indicate antisymmetrization (symmetrization) of the indices they contain.

Finally, varying the various terms in the integrand we start with the gravitational Hamiltonian constraint terms

$$\frac{\delta \mathcal{H}_\perp}{\delta u_a} = \frac{\delta}{\delta u_a} \left[ \sqrt{|g|} G_{abcd} \pi^{ab} \pi^{cd} - {}^{(3)}R \right] \quad (2.93)$$

<sup>14</sup>Which I take to be  $[\bar{D}_a, \bar{D}_b]A^c \equiv {}^{(3)}\bar{R}^c_{\ell ab} A^\ell$ .

and find, after some straightforward manipulations, that

$$\begin{aligned} \delta \left[ \sqrt{|g|} G_{abcd} \pi^{ab} \pi^{cd} \right] &= -\frac{1}{4\sqrt{|\bar{h}|}} \delta h (2\bar{\pi}^{\ell m} \bar{\pi}_{\ell m} - \bar{\pi}^2) + \frac{1}{2|\bar{h}|} [\delta h_{ad} \bar{h}_{bc} + \bar{h}_{ad} \delta h_{bc} + \delta h_{ac} \bar{h}_{bd} + \delta h_{bd} \bar{h}_{ac} \\ &\quad - \delta h_{ab} \bar{h}_{cd} - \delta h_{cd} \bar{h}_{ab}] \bar{\pi}^{ab} \bar{\pi}^{cd} + 2\sqrt{|\bar{h}|} \bar{G}_{abcd} \bar{\pi}^{cd} \delta \pi^{ab} \end{aligned} \quad (2.94)$$

$$-\delta \left[ \sqrt{|g|} {}^{(3)}R \right] = \sqrt{|\bar{h}|} \left( {}^{(3)}\bar{G}^{ab} \delta h_{ab} - \sqrt{|\bar{h}|} (\bar{D}^m \bar{D}^a - \bar{h}^{ma} \bar{\Delta}) \delta h_{ma} \right), \quad (2.95)$$

where  $\delta h \equiv \bar{h}^{ab} \delta h_{ab}$  and  ${}^{(3)}\bar{G}^{ab} \equiv {}^{(3)}\bar{R}_{ab} - \frac{1}{2} \bar{h}_{ab} {}^{(3)}\bar{R}$ . Similarly, the variation of the gravitational parts of the momentum constraints go as

$$\frac{\delta \mathcal{H}^i}{\delta h_{ab}} \delta h_{ab} = 2(\delta C_{jc}^i \bar{\pi}^{cj} + \delta C_{jc}^j \bar{\pi}^{ij}) \quad (2.96)$$

$$\frac{\delta \mathcal{H}^i}{\delta \pi^{ab}} \delta \pi^{ab} = 2\bar{D}_j \delta \pi^{ij}, \quad (2.97)$$

where we define  $\delta C_{jc}^i$  using the definition given in equation (2.91). It is important to note that the inverse first order variation obeys  $\delta h^{ab} = -\bar{h}^{a\ell} \bar{h}^{bm} \delta h_{\ell m}$  because  $\delta(\delta_b^a) = 0$ , where  $\delta_b^a$  is the Kronecker delta.

Noting the interesting fact that (again discarding surface terms)

$$\int_{\Sigma_t} \frac{\delta \mathcal{H}^i}{\delta h_{ab}} d^3x = 2 \int_{\Sigma_t} \left( -\frac{1}{2} \delta h_{jc} \bar{h}^{im} (-\bar{D}_m \bar{\pi}^{cj}) \right) d^3x, \quad (2.98)$$

which implies that, for some spatial vector field  $X^i$ ,

$$\int_{\Sigma_t} X_i \frac{\delta \mathcal{H}^i}{\delta h_{ab}} d^3x = \int_{\Sigma_t} \delta h_{jc} \mathcal{L}_X \bar{\pi}^{jc} d^3x, \quad (2.99)$$

we may finally write the total gravitational variation of the total gravitational Hamiltonian as

$$\delta_G H = \int_{\Sigma_t} X_\alpha \frac{\delta \mathcal{H}^\alpha}{\delta u_a} \delta u_a d^3x \quad (2.100)$$

and pick off the pure gravitational Hamiltonian equations from

$$\frac{\delta H_G}{\delta h_{ab}} = N \left[ (2\bar{\pi}^{ab} \bar{\pi}_{ab} - \bar{\pi}^2) \left( \frac{-\bar{h}^{ab}}{4\sqrt{|\bar{h}|}} \right) + \frac{1}{\sqrt{|\bar{h}|}} (2\bar{\pi}_c^a \bar{\pi}^{cb} - \bar{\pi} \bar{\pi}^{ab}) + \sqrt{|\bar{h}|} ({}^{(3)}\bar{G}^{ab} + \hat{L}^{ab}) \right] + \mathcal{L}_{\bar{N}} \bar{\pi}^{ab},$$

where  $\hat{L}^{ab} \equiv -(\bar{D}^m \bar{D}^a - \bar{h}^{ma} \bar{\Delta})$ , and

$$\frac{\delta H_G}{\delta \pi^{ab}} = \frac{N}{\sqrt{|\bar{h}|}} (2\bar{\pi}_{ab} - \bar{\pi} \bar{h}_{ab}) - 2\mathcal{L}_{\bar{N}} \bar{h}_{ab} \quad (2.101)$$

Following the same procedure for the matter terms using equations (2.83)-(2.86) , we obtain

$$\begin{aligned}
\delta H_\phi &= \delta \int \left\{ N \left[ \frac{\pi_\phi^2}{2\sqrt{|\bar{h}|}} + V(\phi)\sqrt{|\bar{h}|} \right] + N^i \pi_\phi \phi_{,i} + \frac{N\sqrt{|\bar{h}|}}{2} [h^{ij} \phi_{,i} \phi_{,j}] \right\} d^3x \\
&= \int \left\{ \left[ \bar{N} \frac{\bar{\pi}_\phi}{\sqrt{|\bar{h}|}} + \bar{N}^i \bar{\phi}_{,i} \right] \delta \pi_\phi + \left[ \bar{N} \sqrt{|\bar{h}|} V_{,\phi} - \bar{h}^{ij} \bar{\pi}_\phi \mathcal{L}_{\bar{N}} \bar{h}_{ij} - \bar{N} \frac{\sqrt{|\bar{h}|}}{2} \bar{\Delta} \bar{\phi} \right] \delta \phi \right. \\
&\quad \left. + \bar{N} \left[ -\frac{\bar{\pi}_\phi^2}{4\sqrt{|\bar{h}|}} + \frac{\sqrt{|\bar{h}|}}{4} \bar{h}^{ij} \bar{\phi}_{,i} \bar{\phi}_{,j} + \frac{1}{2} V(\bar{\phi}) \sqrt{|\bar{h}|} \right] \bar{h}^{ab} \delta h_{ab} - \bar{N} \frac{\sqrt{|\bar{h}|}}{2} \bar{\phi}^{,i} \bar{\phi}^{,j} \delta h_{ij} \right\} d^3x,
\end{aligned} \tag{2.102}$$

or in other words

$$\begin{aligned}
\delta \rho &= N \left[ \left( -\frac{\bar{\pi}_\phi^2}{4\sqrt{|\bar{h}|}} + \frac{\sqrt{|\bar{h}|}}{2} \left( \frac{1}{2} \bar{h}^{ij} \bar{\phi}_{,i} \bar{\phi}_{,j} + V(\phi) \right) \right) \delta h - \frac{\sqrt{|\bar{h}|}}{2} \bar{\phi}^{,i} \bar{\phi}^{,j} \delta h_{ij} \right. \\
&\quad \left. + \sqrt{|\bar{h}|} (V(\phi)_{,\bar{\phi}} - \frac{\bar{\Delta} \bar{\phi}}{2}) \delta \phi + \frac{\bar{\pi}_\phi}{\sqrt{|\bar{h}|}} \delta \pi_\phi \right],
\end{aligned} \tag{2.103}$$

$$\delta J^i = -\bar{g}^{ij} \bar{\pi}_\phi \mathcal{L}_{\bar{N}} \bar{g}_{ij} \delta \phi - \bar{\phi} \delta \pi_\phi \bar{g}^{ij} \mathcal{L}_{\bar{N}} \bar{g}_{ij}. \tag{2.104}$$

Assembling the above results, finally we can form Hamilton's equations for the total system:

$$-\dot{\bar{\pi}}^{ab} = \frac{\delta H}{\delta h_{ab}} = \frac{\delta H_G}{\delta h_{ab}} - 2\kappa \left( \bar{h}^{ab} N \left( -\frac{\bar{\pi}_\phi^2}{4\sqrt{|\bar{h}|}} + \frac{\sqrt{|\bar{h}|}}{2} \left( \frac{1}{2} \bar{h}^{ij} \bar{\phi}_{,i} \bar{\phi}_{,j} + V(\phi) \right) \right) - N \frac{\sqrt{|\bar{h}|}}{2} \bar{\phi}^{,a} \bar{\phi}^{,b} \right) \tag{2.105}$$

$$\dot{h}_{ab} = \frac{\delta H}{\delta \pi^{ab}} = \frac{\delta H_G}{\delta \pi^{ab}} \tag{2.106}$$

$$-\dot{\bar{\pi}}_\phi = \frac{\delta H}{\delta \phi} = -2\kappa \bar{N} \sqrt{|\bar{h}|} V_{,\phi} + \kappa \bar{h}^{ij} \bar{\pi}_\phi \mathcal{L}_{\bar{N}} \bar{h}_{ij} + 2\kappa \bar{N} \frac{\sqrt{|\bar{h}|}}{2} \bar{\Delta} \bar{\phi} \tag{2.107}$$

$$\dot{\bar{\phi}} = \frac{\delta H}{\delta \pi_\phi} = -2\kappa \bar{N} \frac{\bar{\pi}_\phi}{\sqrt{|\bar{h}|}} - \kappa \bar{N}^i \bar{\phi}_{,i} \tag{2.108}$$

These dynamical equations give the background equations of motion in Hamiltonian form. I have already shown how, in an entirely analagous way, the linearized equations of motion for the fluctuations arise from the second order Hamiltonian action for the case of scalar metric and matter fluctuations in Section 2.2.2. However the advantage of the above formulation is that the Hamiltonian action (with no surface terms) is explicitly shown to be a *projection of the constraints*  $\mathcal{H}^\alpha$  *along a vector field*  $N^\alpha$ . Therefore metric and matter fluctuations will induce perturbations in the Hamiltonian action which will be of the same form, namely that of the perturbed constraints projected along some vector field. In the next section I will show how this plays a role in the analysis of linearization instabilities.

### 2.3.2 Classical Linearization Instability

As mentioned at the introduction of Section 2.3, the foundations of linearization instabilities in gravity have been worked on considerably in the past 35 years. The original, highly-mathematical, approach to the subject is due to Fischer and Marsden in [9] and [10]. Their viewpoint was that a

solution to the field equations is *linearization stable* iff each of its linear perturbations is tangent to a smooth curve of exact solutions. In the unstable case they showed there will *always be some spurious solutions* to the associated perturbed equations which are *not* tangent to any curved of exact solutions.

Their formalism was later changed to among other things better emphasize the physics<sup>15</sup>, largely by Moncrief, in [19] and [5]. In this work Moncrief basically reformulated the entire argument in a number of important ways, primarily by viewing the problem in terms of a Taylor approximation to the projected Hamiltonian and momentum constraints. A natural outcome of this new point of view is a straightforward generalization of classical linearization instabilities to the consideration of quantum linearization instabilities, which will be quite useful for later work discussed in this thesis. Another advantage to this formulation is its clear emphasis on the role of isometries in the background spacetime. I will therefore deviate strongly from a historical presentation of this subject and follow this reformulation here from the beginning. I will leave some of the more formal arguments and derivations for Appendix ? in order to more immediately focus on issues which lead to Chapter 3.

With this in mind, consider the Hamiltonian and momentum constraints, equations (2.78) and (2.79), written in the notation

$$\Phi^\alpha(u_a) \equiv (\mathcal{H}_\perp, \mathcal{H}^i) = 0, \quad (2.109)$$

where  $u_a$  are the phase space variables and the indices on  $\alpha$  run from 0 to 3. Following the developments of Section 2.3.1.1, one can imagine projecting the constraints along some general vector field  ${}^{(4)}X$ , i.e.

$$P(\Phi, {}^{(4)}X) = \int_{\Sigma_t} {}^{(4)}X^\alpha \Phi_\alpha d^3x. \quad (2.110)$$

It is important to note the absence of surface terms in equation (2.109). I will henceforth explicitly assume that  $\Sigma_t$  is topologically *closed*, i.e. has no boundary, so that no surface terms can ever arise in what follows. Therefore, the following arguments only apply when either this topological assumption or some equivalent boundary condition is used to eliminate the spatial surface terms which will in general be there. Furthermore, for simplicity I will assume a vacuum spacetime (which is what most other work on the subject also assumes) until the later sections.

Given equation (2.110), the central question now is how one can *perturbatively approximate the projection  $P$  in the neighbourhood of some exact solution  $(\bar{h}_{ab}, \bar{\pi}^{ab}; \bar{N}, \bar{N}^a)$  of the constraints*. In other words, given initial data  $(\bar{h}_{ab}, \bar{\pi}^{ab}; \bar{N}, \bar{N}^a)$  for a background spacetime, how do we approximate the (coordinate invariant) projected constraints for the perturbed spacetime? One scheme is, using

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<sup>15</sup>Their results were also strengthened. The initial papers required a maximal slice,  $\bar{K}^a_a = 0$ , for the background, and no such requirement need hold in Moncrief's work.

the intuition of Section 2.3.1.1, to *define* perturbations of the projection  $P$  via the formulae

$$\delta P = \int_{\Sigma_t} {}^{(4)}X^\alpha \delta \Phi_\alpha d^3x \quad (2.111)$$

$$\delta^2 P = \int_{\Sigma_t} {}^{(4)}X^\alpha \delta^2 \Phi_\alpha d^3x \quad (2.112)$$

⋮

where  $\delta^n \Phi$  are successive perturbations of the constraints. The key starting point of the linearization stability analysis is the result that, given the assumption of  $\Sigma_t$  closed,  $\delta P$  **vanishes iff**  ${}^{(4)}X^\alpha$  **is a Killing vector of the background**. I.e., the linearized projection of the constraints given in equation (2.111) is *automatically trivial* if it is projected along a Killing direction of the full, four-dimensional, background spacetime<sup>16</sup>. Therefore, *the first nontrivial perturbative approximation to  $P$  will occur at second order, equation (2.112)*. But, using the intuition developed in Section 2.3.1.1, this is precisely the statement that in order to faithfully approximate the Hamilton equations for the perturbations at linear order, one must take into account the second order projections of the constraints along any Killing directions of the background<sup>17</sup>. For proof of these statements I refer to Moncrief in [19] and [5], where he first gave these proofs (for the vacuum case only) in their present form.

Therefore, if the above conditions apply, the problem of solving the linearized constraint equations becomes equivalent to solving

$$\delta \Phi^\alpha = 0, \quad (2.113)$$

$$\int_{\Sigma_t} {}^{(4)}X^\alpha \delta^2 \Phi_\alpha d^3x = 0, \quad (2.114)$$

simultaneously, where  $\mathcal{L}_{{}^{(4)}X^\alpha} g_{ab} = 0$ . In fact, now one can perturb a spacetime with Killing vectors and closed spacelike slices  $\Sigma_t$  and ask what distinguishes ‘acceptable’ perturbations and spurious ones. ‘Acceptable’ perturbations, genuine linearizations, will simultaneously satisfy (2.113) and (2.114) whilst spurious solutions will only satisfy equation (2.113). There are as many equations of the type of (2.114) as there are background Killing vectors, and I will refer to these equations as *linearization stability (LS) conditions* to avoid confusion with the initial-value constraints  $\Phi$ . It is also important to note that both the LS conditions and the linearized equations must be satisfied in order for the initial data for the *second order* fluctuations be well-posed. If one ‘feeds’ spurious solutions to the second order constraint equations, their second order solutions will not be integrable.

We now discuss some properties of the LS conditions. We shall show that they are preserved from slice to slice, i.e. through ‘evolution’, and are gauge-invariant.

<sup>16</sup>By this I mean the Cauchy development of the background, to be truly consistent with the 3 + 1 language.

<sup>17</sup>If surface terms were present  $\delta P$  would not be zero, but rather equal to a surface term, and in that sense the projected constraints would simply be identities order by order in perturbation theory and yield no new information. Identical reasoning reveals that any  $\delta^2 u_a$  that occur in equation (2.113) are surface terms.

### 2.3.2.1 Properties of the LS conditions

As mentioned above, the LS conditions are preserved ([19], [5]) from  $\Sigma_t$  to  $\Sigma_{t+t'}$ . We demonstrate this for the LS conditions involving the momentum constraints to convey the basic point and flavour of the calculation, which to my knowledge is original (though the same result in the case of vacuum was known to Moncrief using different lines of reasoning). The analagous calculation for the Hamiltonian constraint is similar but more tedious and not particularly illuminating.

In section 2.3 it was shown that the momentum constraints are the generators of the spatial diffeomorphisms. We may them using the standard Poisson bracket defined previously<sup>18</sup>,

$$\{h_{k\ell}(x'), \int_{\Sigma_t} [\pi^{ij} \mathcal{L}_{N_*} h_{ij}] d^3x\} = \mathcal{L}_{N_*} h_{k\ell}(x'), \quad (2.115)$$

because of the fundamental commutation relation  $\{h_{k\ell}(x'), \pi^{k\ell}(x)\} = \sqrt{|h|} \delta^3(x' - x)$ . Furthermore, one can show that the momentum-constraint LS conditions, including the scalar matter, are

$$C^\eta = \int_{\Sigma_t} \left[ \delta\pi^{ij} \mathcal{L}_{N_*^{(\eta)}} \delta h_{ij} + \delta\pi_\phi \mathcal{L}_{N_*^{(\eta)}} \delta\phi \right] d^3x \quad (2.116)$$

for each of the  $\eta$  Killing vectors in the background (denoted by  $N_*^{(\eta)}$ ), where I have defined  $C^\eta$  to label the LS conditions. In this form it is easy to observe the time independence of the  $C^\eta$ . Indeed, after a few manipulations using the Hamilton equations (2.105) - (2.108) to get rid of the time derivatives,

$$\begin{aligned} \dot{C}^\eta &= \int_{\Sigma_t} \left[ \delta\dot{\pi}^{ij} \mathcal{L}_{N_*^{(\eta)}} \delta h_{ij} + \delta\pi^{ij} \mathcal{L}_{N_*^{(\eta)}} \delta\dot{h}_{ij} + \delta\pi^{ij} \mathcal{L}_{\dot{N}_*^{(\eta)}} \delta h_{ij} + \delta\dot{\pi}_\phi \mathcal{L}_{N_*^{(\eta)}} \delta\phi + \delta\pi_\phi \mathcal{L}_{N_*^{(\eta)}} \dot{\phi} + \delta\pi_\phi \mathcal{L}_{\dot{N}_*^{(\eta)}} \delta\phi \right] d^3x \\ &= \int_{\Sigma_t} \left[ -\frac{\delta^2 H}{\delta(\delta h_{ij})} \mathcal{L}_{N_*^{(\eta)}} \delta h_{ij} - \frac{\delta^2 H}{\delta(\delta\pi^{ij})} \mathcal{L}_{N_*^{(\eta)}} \delta\pi^{ij} - \frac{\delta^2 H}{\delta(\delta\phi)} \mathcal{L}_{N_*^{(\eta)}} \delta\phi - \frac{\delta^2 H}{\delta(\delta\pi_\phi)} \mathcal{L}_{N_*^{(\eta)}} \delta\pi_\phi \right] d^3x \\ &\quad + \int_{\Sigma_t} \delta\pi^{ij} \mathcal{L}_{\dot{N}_*^{(\eta)}} \delta h_{ij} + \delta\pi_\phi \mathcal{L}_{\dot{N}_*^{(\eta)}} \delta\phi d^3x \\ &= - \int_{\Sigma_t} \left[ D_i \left( N_*^i \delta^2 H|_{(\delta u)^2} \right) \right] d^3x = \int_{\Sigma_t} \left[ -\mathcal{L}_{N_*^{(\eta)}} \delta^2 H|_{(\delta u)^2} \right] d^3x \\ &= - \oint_{\partial\Sigma_t} \left[ n_i \left( N_*^i \delta^2 H|_{(\delta u)^2} \right) \right] d^3x = 0 \end{aligned} \quad (2.117)$$

where we have used a spatial by-parts integration to shift over the Lie derivatives and the primary constraints  $\pi_{N^*} = 0, \pi_{N_*} = 0$  to nullify the time derivatives of the lapse and shift. It is thus clear that if the constraints are obeyed on one slice they are obeyed on all, as deduced by [19], [5] for the vacuum case.

Turning to the gauge invariance of the LS conditions, consider a small shift of coordinates along the (spatial) vector field  $\vec{\zeta}$  such that  $\delta\tilde{g}_{ij} \rightarrow \delta g_{ij} + \mathcal{L}_{\vec{\zeta}} \delta g_{ij}$ ,  $\delta\tilde{\pi}^{ij} \rightarrow \delta\pi^{ij} + \mathcal{L}_{\vec{\zeta}} \delta\pi^{ij}$ ,  $\delta\tilde{\phi} \rightarrow \delta\phi + \mathcal{L}_{\vec{\zeta}} \delta\phi$ ,  $\delta\tilde{\pi}_\phi \rightarrow \delta\pi_\phi + \mathcal{L}_{\vec{\zeta}} \delta\pi_\phi$  and  $\tilde{N}^i \rightarrow \vec{N} + \mathcal{L}_{\vec{\zeta}} \vec{N}$ . Then the induced first-order difference in the constraints

<sup>18</sup>Recall that, from equation (2.54),  $\{F, G\} \equiv \frac{\delta F}{\delta g_{ij}} \frac{\delta G}{\delta\pi^{ij}} - \frac{\delta F}{\delta\pi^{ij}} \frac{\delta G}{\delta g_{ij}}$  for any two functions  $F$  and  $G$  of the gravitational  $u_a$ .

will be

$$\begin{aligned}
 \mathcal{C} - \tilde{\mathcal{C}} &= \int_{\Sigma_t} \left[ (\mathcal{L}_{\vec{\zeta}} \delta \pi^{ij}) \mathcal{L}_{\vec{N}^*} \delta g_{ij} + \delta \pi^{ij} \mathcal{L}_{\vec{N}^*} \mathcal{L}_{\vec{\zeta}} \delta g_{ij} + (\mathcal{L}_{\vec{\zeta}} \delta \pi^{ij}) \mathcal{L}_{\vec{N}^*} \mathcal{L}_{\vec{\zeta}} \delta g_{ij} + \delta \pi^{ij} \mathcal{L}_{(\mathcal{L}_{\vec{\zeta}} \vec{N}^i)} \delta g_{ij} \right] \\
 &+ \int_{\Sigma_t} \left[ (\mathcal{L}_{\vec{\zeta}} \delta \pi_\phi) \mathcal{L}_{\vec{N}^*} \delta \phi + \delta \pi_\phi \mathcal{L}_{\vec{N}^*} \mathcal{L}_{\vec{\zeta}} \delta \phi + (\mathcal{L}_{\vec{\zeta}} \delta \pi_\phi) \mathcal{L}_{\vec{N}^*} \mathcal{L}_{\vec{\zeta}} \delta \phi + \delta \pi_\phi \mathcal{L}_{(\mathcal{L}_{\vec{\zeta}} \vec{N}^i)} \delta \phi \right] d^3x \\
 &= \int_{\Sigma_t} \left[ \delta \pi^{ij} \mathcal{L}_{[\vec{N}^*, \vec{\zeta}]} \delta g_{ij} + \delta \pi^{ij} \mathcal{L}_{[\vec{\zeta}, \vec{N}^*]} \delta g_{ij} \right] d^3x + \int_{\Sigma_t} \left[ \delta \pi_\phi \mathcal{L}_{[\vec{N}^*, \vec{\zeta}]} \delta \phi + \delta \pi_\phi \mathcal{L}_{[\vec{\zeta}, \vec{N}^*]} \delta \phi \right] d^3x
 \end{aligned}$$

by definition of the Lie bracket<sup>19</sup> and spatial compactness of  $\Sigma_t$ . I note that all the surface terms in the above manipulations are zero because they are of the form  $\int_{\Sigma_t} \left[ n_i \zeta^i \delta \pi^{ab} \mathcal{L}_{\vec{N}^*} \delta g_{ab} \right] d^2\sigma$ , and  $n_i \zeta^i = 0$ . Therefore the constraints are invariant under an arbitrary (small) coordinate deformation on the slice  $\Sigma_t$ .

## 2.4 Conclusion

In this Chapter I have given a brief introduction to linearized cosmological perturbation theory, starting with a slightly more rigorous definition of the spatially isotropic FRW background space-time. Moving freely from the Lagrangian to the Hamiltonian formalism, and treating the matter as a scalar field, it was shown that the equations of motion of the scalar perturbations are that of a simple harmonic oscillator with time-dependent frequency. Then, the issue of linearization stability was introduced and discussed by introducing the so-called ADM formalism of gravity. Necessary conditions to ensure linearization stability were discussed, and some original arguments were given to show that the (nonvacuum version of these) LS conditions have preservation and gauge-invariance properties.,

Thus it is now finally possible to discuss a practical application where the LS conditions become physically important. The next chapter goes through a calculation where the LS conditions make a crucial difference in the linearized analysis of perturbations about a particular model called the Einstein static spacetime.

<sup>19</sup>Suppose  $X$  and  $Y$  are smooth vector fields in a coordinate basis  $\partial_i$ :  $X = X^i \partial_i$ ,  $Y = Y^i \partial_i$ . Then the coordinate Lie bracket is defined by  $[X, Y] = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j}$

## CHAPTER 3

# LINEARIZATION INSTABILITIES IN EINSTEIN STATIC

### 3.1 Introduction

Recently there has been renewed interest in a class of cosmological models known as the Einstein static models. This infamous class of models was designed by Einstein to predict a static spacetime (i.e.  $a(t) = a_0, a_0 \in \mathfrak{R}$  in the FRW language) since he held the view that the universe was obviously static. It was soon abandoned by Einstein after Edwin Hubble discovered the law that bears his name, which shows that the universe is actually expanding. It is also the class of models which introduced the famous cosmological constant  $\Lambda$  mentioned in equation (2.8). Another notable feature of these models was that spacetime fluctuations in them are unstable to first order in perturbation theory, as first found by Eddington in 1931 [20]. An initial perturbation of an Einstein static model, which uses a cosmological constant to ‘balance’ the gravity of a perfect fluid in much the same way a pencil balances on edge, either grows into a catastrophic collapse or expansion of the entire spacetime.

However, Barrow et al in [21] re-examined the stability of the Einstein static spacetime against arbitrary linear fluctuations in the metric and matter and found, surprisingly, that some modes are stable. More precisely, they found that given a sufficiently large speed of sound in the background, all non-gauge, spatially *inhomogeneous* scalar modes were neutrally stable (i.e. the fluctuations are not damped), and furthermore vector and tensor modes were neutrally stable on all scales irrespective of the equation of state in the background. The essence of this stability arises from the spatial compactness of the Einstein static spacetime, i.e. there exists a maximum physical wavelength in this closed space and furthermore the Jeans length is a significant fraction of this maximum scale. It turns out that for specific equations of state in the background matter all physical wavelengths fall below the Jeans wavelength and the *inhomogeneous* modes are thus stable for the usual reasons. The spatially homogeneous perturbations, however, are still always unstable.

This surprising non-Newtonian stability for a large class of fluctuations, which was pointed out earlier in different and more restricted contexts in [22], is one of the key elements of support for the Emergent Universe models proposed recently by Ellis et al. These models explicitly construct spatially closed, positively curved, cosmologies which do not bounce and in which inflation (triggered by precisely the famous homogeneous instability of Einstein static) is not preceded by an era of deceleration. This lies in stark contrast to deSitter and most other models of closed inflation. The Emergent Universe models have no initial singularity, undergo the usual inflationary period ending in the usual reheating era, and immediately solve the horizon problem owing to the staticity of the



initial state. Another key feature of these models is that a finite number of e-foldings of inflation occurs over an *infinite* time in the past. Given this, nonlinear effects can be rather important and in particular nonlinear metric and matter fluctuations will almost certainly modify the stability properties of this particular model.

In this Chapter I investigate the initial value problem for second order inhomogeneous fluctuations about the Einstein static solution. In particular, I focus on the impact of the LS conditions on the stability analysis given by Barrow et al. I shall focus entirely on the higher order constraint associated with the timelike Killing field of the Einstein static initial state. I emphasize that these remarks are entirely concerned with the initial value problem and thus do not touch on the question of the time evolution of these fluctuations.

I find, in the case of a general irrotational perfect fluid with cosmological constant, that there are no nontrivial solutions to both the linearized constraints and the LS conditions when we exclude the linear homogeneous scalar metric and matter fluctuations. In other words, the leading order linear metric and matter fluctuations must be trivial if their linear seeds do not include ‘zero mode’ homogeneous fluctuations. It is well known that these homogeneous modes are unstable, just as arbitrary perturbations of a dust model in Einstein static model are unstable. This would seem to suggest that if the universe is in a neighbourhood of the Einstein static solution then it does *not* stay there, even if perturbed with the neutrally stable modes found in reference [21]. This is one of the first instances where the linearization stability issues make a physical difference in the analysis of metric and matter stability (see also Brill and Deser in [23]).

This chapter is organized as follows. In Section 3.2 I briefly outline the details of the Einstein static background model. In Section 3.3 I define, using the constraint equations in the standard ADM decomposition of the Einstein equations defined in Section 2.3, an orthogonal decomposition (following [23]) of perturbations into transverse and longitudinal parts. In Section 3.4 I formulate and compute the nonlinear LS conditions, leaving conclusions for Section 3.5. The entire analysis is quite similar to that in Brill and Deser’s original paper [23], however there are clear differences. Firstly, Brill and Deser consider the problem with a compactified section of Minkowski spacetime as the background, whereas I examine a curved Einstein static background which in general introduces curvature terms. They also do not have to deal with matter terms, as I have emphasized above, and furthermore the irrotational perfect fluid matter with cosmological constant discussed below is more general than the minimally coupled scalar fields discussed in Barrow et al’s paper [21]. In this sense I show that there is a wider class of matter models in which linearization instability occurs than I need to consider given the claims of [21]. The bulk of the material in this Chapter is drawn from [1].

## 3.2 Einstein static spacetime and perfect fluids

Consider a FRW universe in comoving coordinates  $(t, \vec{x})$  with scale factor  $a(t)$ , with signature  $(-1, 1, 1, 1)$ , and with a perfect fluid with energy density  $\rho$  and pressure  $p$ . The equations of motion

for the scale factor  $a(t)$  are, according to the Einstein equations,

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3} [\rho(1+3w) - \Lambda], \quad (3.1)$$

$$H^2 = \frac{\kappa}{3}(\rho + \Lambda) - \frac{K}{a^2}, \quad (3.2)$$

where  $K \equiv \pm 1, 0$  is the constant curvature of the  $t = \text{const}$  slices,  $H \equiv \partial_t \ln(a)$  is the Hubble parameter,  $\Lambda$  is a cosmological constant,  $w \equiv \frac{p}{\rho}$ , and  $\kappa \equiv 8\pi G$  in units where  $c = 1$ . One can combine these equations to form  $\dot{H} = -\frac{\kappa}{2}(\rho(1+w)) + \frac{K}{a^2}$ .

As Einstein found some time ago, demanding that the universe be static ( $\dot{a} = \ddot{a} = 0$ ) obviously sets  $K$  to be positive (take it to be 1) and leads to constraints relating the initial energy density and pressure of the fluid to  $\Lambda$ . The equilibrium radius of such a static universe is set by these constraints to be

$$a_0^2 = \frac{1+3w}{\Lambda(1+w)}, \quad (3.3)$$

where  $(1+w)\rho > 0$ . In the case of dust it is seen that  $a_0^2 = 1/\Lambda$ .

However, consider the general perfect fluid equations. The equations of motion of a perfect fluid are given by the conservation equations of  $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$ , or in other words

$$(\rho + p)_{;\mu} u^\mu u_\nu + (\rho + p)u^\mu_{;\mu} u_\nu + (\rho + p)u^\mu u_{\nu;\mu} + p_{;\nu} = 0 \quad (3.4)$$

Taking the dot product of (3.4) with the timelike<sup>1</sup> fluid velocity  $u^\nu$ , we get the primary conservation law

$$\rho_{;\mu} u^\mu + (\rho + p)u^\mu_{;\mu} = 0, \quad (3.5)$$

and multiplying this law by  $u_\nu$  and subtracting from equation (3.4) we get

$$\begin{aligned} 0 &= (\rho + p)u^\mu u_{\nu;\mu} + p_{;\nu} + p_{;\mu} u^\mu u_\nu \\ &= \left(\frac{\rho + p}{\sqrt{N}}\right)(\sqrt{N}u_{[\nu];\mu]}u^\mu, \end{aligned} \quad (3.6)$$

where

$$N(\rho) = \exp\left(2 \int \frac{dp}{\rho + p}\right) \quad (3.7)$$

is a function of  $\rho$  assuming that there exists some constitutive equation  $p = p(\rho)$ . These equations are trivially satisfied if we assume that the term  $N^{\frac{1}{2}}(\rho)u_\mu$  satisfies

$$N^{\frac{1}{2}}(\rho)u_\mu = \phi_{;\mu}, \quad (3.8)$$

where  $\phi$  is a scalar potential, i.e. a velocity potential, for the flow. This is an appropriate

<sup>1</sup>Timelike here means taking normalization  $u^a u_a = -1$ .

generalisation of the irrotational conditions of non-relativistic flow [24]. Finally, multiplying this equation by itself we have that

$$N(\rho) = -\phi_{;\mu}\phi^{;\mu} \quad (3.9)$$

which is the equivalent of Bernouli's equation for non-relativistic irrotational flow. We will concern ourselves with constitutive equations where  $p$  is linearly proportional to  $\rho$ .

I wish to treat the case of an arbitrary *irrotational* perfect fluid in terms of this scalar field  $\phi$  that acts as the velocity potential of the fluid (as in [25]). Following Mukhanov and Garrigas in [25], one may do this in a clever way by choosing the scalar field to have the action

$$S = \frac{1}{2\alpha} \int (-\phi^{;\lambda}\phi_{;\lambda})^\alpha \sqrt{-|g|} d^4x, \quad \alpha \in \mathfrak{R} \quad (3.10)$$

Comparing the stress energy that results from varying the above action with respect to the metric with that of a perfect fluid, we can easily identify the corresponding energy density and pressure in terms of  $\phi$ :

$$\rho = \left(\alpha - \frac{1}{2}\right)N^\alpha, \quad (3.11)$$

$$p = \frac{N^\alpha}{2}. \quad (3.12)$$

For  $\alpha = 1$  we obtain the stiff ( $w = 1$ ) perfect fluid, i.e. a minimally coupled scalar field. However, note that the speed of sound is given by  $c_s^2 = p/\rho = (2\alpha - 1)^{-1}$  and thus causality restricts  $\alpha \geq 1$ . It turns out that stable (inhomogeneous) scalar fluctuation modes only exist when the background speed of sound satisfies the bound  $v_s^2 \geq \frac{1}{5}$  ([21], [22]) for no Jeans instability, which translates into  $\alpha \leq 3$ . I.e. there is no Jeans instability for any physical inhomogeneous modes given these conditions. Thus we take  $1 \leq \alpha \leq 3$  in what follows. We also note in passing that in order for the four velocity to be timelike the gradients of the velocity potential  $\phi$  are restricted to ones such that their temporal gradient dominates their spatial counterpart. In this model we take the background  $\phi = \left(\frac{\rho_0}{\alpha-1/2}\right)^{\frac{1}{\alpha}} t$  to satisfy equation (3.7) in the background, so the timelike condition will always be satisfied for the perturbations.

In any case the generalized Einstein static initial conditions (in terms of  $\phi$ ) become

$$N^\alpha \frac{\alpha + 1}{2} = \Lambda = \frac{\alpha + 1}{\alpha \kappa a_0^2}, \quad (3.13)$$

i.e.  $a_0^2 = (\alpha + 1)/(\alpha\Lambda)$  and  $N = (2\Lambda/(\kappa(\alpha + 1)))^{\frac{1}{\alpha}}$ . This may be interpreted physically as the equilibrium condition between the tension of a positive cosmological constant and the tendency for gravitational collapse of the fluid.

### 3.3 Initial-value constraint equations

As shown in Section 2.3, the gravitational field may be characterized in terms a constrained Hamiltonian analysis, with constraints

$$-\frac{1}{\sqrt{|g|}} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) + \sqrt{|g|} {}^{(3)}R = 2\kappa \sqrt{|g|} \rho, \quad (3.14)$$

$$(\pi^{ab})_{|b} = \kappa J^a, \quad (3.15)$$

where  ${}^{(3)}R$  is the Ricci curvature scalar for the three surface (in what follows I will drop the 3 superscript) and  $\pi \equiv g_{ab} \pi^{ab}$ . The three-metric will be referred to as  $g_{ab}$  in this Chapter, and the associated covariant derivative will be denoted  $\bar{\nabla}_a$ . All background quantities are barred.

At zeroth order, i.e. in the background Einstein static spacetime, there is only one nontrivial constraint namely

$$\bar{\rho} = \frac{3\Lambda}{2\kappa}, \quad (3.16)$$

where the background conditions  $\bar{R}_{ab} = \Lambda \bar{g}_{ab}$ ,  $\bar{\pi}^{ab} = 0$  are taken to hold. At linear order (using the background equations) the constraint equations are

$$\delta R = 2\kappa \delta \rho, \quad (3.17)$$

$$\bar{\nabla}_b \delta \pi^{ab} = \kappa \sqrt{|\bar{g}|} \delta J^a, \quad (3.18)$$

and similarly the second order equations are, using the linearized and background equations above,

$$\delta^2 R = \frac{1}{|\bar{g}|} \left( \delta \pi^{ab} \delta \pi_{ab} - \frac{1}{2} \delta \pi^2 \right) + 2\kappa \delta^2 \rho \quad (3.19)$$

$$\bar{\nabla}_b \delta^2 \pi^{ab} + 2 [\delta C_{bm}^a \delta \pi^{mb} + \delta C_{bm}^b \delta \pi^{am}] \sqrt{|\bar{g}|} = \kappa \delta^2 (J^a \sqrt{|\bar{g}|}), \quad (3.20)$$

Here  $2\delta C_{bm}^a \equiv \bar{g}^{ac} (\bar{\nabla}_m \delta g_{bc} + \bar{\nabla}_b \delta g_{mc} - \bar{\nabla}_c \delta g_{bm})$  is the perturbed connection, as defined in equation (2.91). One can rewrite equation (3.19), the equation associated with time reparametrization invariance at second order, using  $\delta^2 R = 2\delta g^{ab} \delta R_{ab} + \bar{g}^{ab} \delta^2 R_{ab} + \delta^2 g^{ab} \bar{R}_{ab}$  to obtain

$$\begin{aligned} \bar{\Delta} \delta^2 g - \bar{\nabla}^m \bar{\nabla}^b \delta^2 g_{bm} - \Lambda \bar{g}_{ab} \delta^2 g^{ab} &= \bar{\nabla}_\ell B^\ell + 2\delta C_{b[a}^c \delta C_{\ell]c}^\ell \bar{g}^{ab} + 2\delta g^{ab} \delta R_{ab} \\ &\quad - \frac{1}{|\bar{g}|} \left( \delta \pi^{ab} \delta \pi_{ab} - \frac{1}{2} \delta \pi^2 \right) - 2\kappa \delta^2 \rho, \end{aligned} \quad (3.21)$$

where by  $\bar{\nabla}_\ell B^\ell$  I denote a combination of terms that occur in the form of a total derivative. In obtaining (3.21) I have expanded the second order Ricci scalar in terms of the perturbed connection defined above and grouped second order terms and products of linear order terms. One can similarly simplify the momentum constraints (20) but in this paper our main concern will be the second order timelike, or ‘‘Hamiltonian’’, constraint.

Typically what one does is to solve the constraints for a given set of fluctuation variables and

in that way solve for the constrained variables in terms of the free variables. A convenient way to facilitate this process is through an orthogonal decomposition of the fluctuations in  $g_{ab}$  and  $\pi^{ab}$  into longitudinal and transverse parts. As pointed out in by Brill and Deser in [23] this procedure differs crucially for tensors in closed spaces from the usual one in flat space since one can now have fluctuations in global parameters such as the total volume of the space while still having no local excitations.

Following the notation of Brill et al one may decompose the metric perturbation into its various transverse, transverse traceless, and longitudinal parts via

$$\delta g_{ab} = \delta g_{ab}^{(TT)} + \frac{1}{2} (-\bar{\nabla}_a \bar{\nabla}_b + \bar{\Delta} \bar{g}_{ab}) \delta g^{(Tr)} + \tilde{\mathcal{A}}_{\delta g} \bar{g}_{ab} + 2\delta g_{(a|b)}^{(V)}, \quad (3.22)$$

where  $\bar{\Delta} \equiv \bar{\nabla}_c \bar{\nabla}^c$ . The transverse traceless and longitudinal parts are defined as usual:

$$\bar{g}^{ab} \delta g_{ab}^{(TT)} = 0 = \bar{\nabla}^a \delta g_{ab}^{(TT)}, \quad (3.23)$$

$$\bar{\nabla}_a \delta g^{(V)a} = 0, \quad (3.24)$$

and here  $\tilde{\mathcal{A}}_{\delta g} \equiv \mathcal{A}_{\delta g} + \frac{\Lambda}{2} \delta g^{(Tr)}$ . The homogeneous fluctuations  $\tilde{\mathcal{A}}_{\delta g}$  are essentially the global fluctuations (related to the volume fluctuation) unique to perturbations of closed spaces. An analogous decomposition holds for the momentum fluctuation (the homogeneous modes  $\tilde{\mathcal{A}}_{\delta \pi}$  describe the ‘time rate of change’ of the volume fluctuation).

Using the above decomposition, note that an identity due to the symmetry of the background is that the perturbations are all ‘doubly transverse’. Indeed, recover that

$$\bar{\nabla}^b \delta g_{ab} = \bar{\Delta} \delta g_a^{(V)} + \bar{g}^{bm} \left( -\bar{R}^\ell{}_{bma} \delta g_\ell^{(V)} \right) \stackrel{*}{=} (\bar{\Delta} + \Lambda) \delta g_a^{(V)} \quad (3.25)$$

$$\bar{\nabla}^a \bar{\nabla}^b \delta g_{ab} = \bar{g}^{an} [\bar{\nabla}_n, \bar{\Delta}] \delta g_a^{(V)} \stackrel{*}{=} 0 \quad (3.26)$$

Also note that one may write

$$\delta^2 R = \Lambda \bar{g}_{ab} \delta^2 g^{ab} + 2\delta g^{ab} \delta R_{ab} + 2\bar{g}^{ab} \delta C_{b|a}^c \delta C_{\ell|c}^\ell - 2\bar{\nabla}_{[a} \delta^2 \tilde{C}_{c]b}^c \bar{g}^{ab} + \bar{\nabla}_\ell B^\ell, \quad (3.27)$$

where

$$2\delta^2 \tilde{C}_{bc}^a \equiv \bar{g}^{am} (\bar{\nabla}_b \delta^2 g_{cm} + \bar{\nabla}_c \delta^2 g_{bm} - \bar{\nabla}_m \delta^2 g_{bc}) \quad (3.28)$$

One may use the decompositions above to eliminate the ‘doubly transverse’ term and obtain

$$\begin{aligned} \bar{\Delta} \delta^2 g - \Lambda \bar{g}_{ab} \delta^2 g^{ab} &= 2\delta g^{ab} \delta R_{ab} + \bar{\nabla}_\ell B^\ell + 2\delta C_{b|a}^c \delta C_{\ell|c}^\ell \bar{g}^{ab} \\ &\quad - \frac{1}{|\bar{g}|} \left( \delta \pi^{ab} \delta \pi_{ab} - \frac{1}{2} \delta \pi^2 \right) - 2\kappa \delta^2 \rho, \end{aligned} \quad (3.29)$$

where  $\delta^2 \rho$  are the second order energy density fluctuations. I explicitly insert the above decompositions into equation (3.28) in the next section.

In the case of the generalized scalar field  $\phi$  introduced in section II, the Hamiltonian formulation is more involved since the canonical momentum conjugate to  $\phi$  is generally

$$\pi_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\sqrt{|\bar{g}|}}{\mathcal{N}} (\dot{\phi} - \mathcal{N}^i \phi_{,i}) \left( \frac{\dot{\phi}^2}{\mathcal{N}^2} - \frac{2\mathcal{N}^i \dot{\phi} \phi_{,i}}{\mathcal{N}^2} - \phi^{,i} \phi_{,i} \right)^{\alpha-1},$$

where  $\mathcal{N}$  and  $\mathcal{N}^i$  are the purely kinematical lapse and shift variables of the ADM formalism. For general  $3 \geq \alpha \geq 1$  this expression cannot be inverted in closed form to find  $\dot{\phi} = \dot{\phi}(\pi_\phi)$ , however for a spatially homogeneous scalar field it is possible and the resulting Hamiltonian (energy density) is

$$H_0 = \frac{2\alpha - 1}{2\alpha} \left( \frac{(\pi_\phi)^{2\alpha}}{\sqrt{|\bar{g}|}} \right)^{\frac{1}{2\alpha-1}} + \frac{\Lambda}{\kappa} = \sqrt{|\bar{g}|} \rho, \quad (3.30)$$

where  $\bar{\pi}_\phi = \sqrt{|\bar{g}|} \dot{\phi}^{2\alpha-1}$ . I define the fluctuations  $\delta\pi_\phi$  and  $\delta^2\pi_\phi$  using the formal definition of  $\pi_\phi$  given above. For example the first order fluctuation in  $\pi_\phi$  is formally defined by

$$\delta\pi_\phi \equiv \frac{\bar{\pi}_\phi}{\sqrt{|\bar{g}|}} \left( \delta \left( \frac{\sqrt{|\bar{g}|}}{\mathcal{N}} \right) + \sqrt{|\bar{g}|} \left( \delta \dot{\phi} \frac{2\alpha-1}{\dot{\phi}} - 2(\alpha-1)\delta\mathcal{N} \right) \right), \quad (3.31)$$

and similarly for the second order fluctuation (though new cross-terms like  $\delta\mathcal{N}^i \delta\phi_{,i}$  and  $\delta\phi^{,i} \delta\phi_{,i}$  start to appear). Formally perturbing equations (3.11) and (3.12) with the above definitions of  $\delta\pi_\phi$ ,  $\delta^2\pi_\phi$  yields

$$2\kappa\delta\rho = \alpha\Lambda \frac{\delta N}{\bar{N}} = \frac{2\Lambda\alpha}{2\alpha-1} \left[ \frac{\delta\pi_\phi}{\bar{\pi}_\phi} - \frac{\delta g}{2} \right], \quad (3.32)$$

$$2\kappa\delta^2\rho = \alpha\Lambda \left[ (\alpha-1) \left( \frac{\delta N}{\bar{N}} \right)^2 + \frac{\delta^2 N}{\bar{N}} \right], \quad (3.33)$$

where the second order term is given explicitly by

$$\delta^2 N = 2(\delta\dot{\phi}^2 + \dot{\phi}\delta^2\dot{\phi}) - 4\dot{\phi}\delta\dot{\phi}\delta\tilde{N} - 2\dot{\phi}^2\delta^2\tilde{N} + 6\dot{\phi}^2(\delta\tilde{N})^2 - 2\dot{\phi}\delta\tilde{N}^i\delta\phi_{,i} - \delta\phi^{,i}\delta\phi_{,i},$$

so that, in Hamiltonian form, I finally obtain

$$\begin{aligned} 2\kappa\delta^2\rho &= \frac{2\alpha\Lambda}{2\alpha-1} \left[ \frac{\delta^2\pi_\phi}{\bar{\pi}_\phi} - \frac{\delta^2\sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}} - \frac{\delta\phi^{,i}\delta\phi_{,i}}{2} \left( \frac{\bar{\pi}_\phi}{\sqrt{|\bar{g}|}} \right)^{\frac{2}{1-2\alpha}} \right. \\ &\quad \left. + \frac{1}{(2\alpha-1)^2} \left( (2\alpha^2 + 8\alpha - 7)(\delta \ln \sqrt{|\bar{g}|})^2 + (2\alpha^2 + 4\alpha - 5)(\delta \ln \pi_\phi)^2 \right. \right. \\ &\quad \left. \left. - 4(\alpha^2 + 3\alpha - 3)(\delta \ln \sqrt{|\bar{g}|})(\delta \ln \pi_\phi) \right) \right] \end{aligned} \quad (3.34)$$

Here  $\delta g \equiv \bar{g}^{ab} \delta g_{ab}$  and  $\bar{N}^\alpha = \Lambda/(\kappa(2\alpha-1))$  by the zeroth order constraint (3.16). Inserting the above matter perturbations into equation (3.29) and using the second-order identity  $\bar{g}^{ab} \delta^2 g_{ab} +$

$\bar{g}_{ab}\delta^2g^{ab} = -2\delta g^{ab}\delta g_{ab}$ , I arrive at

$$\begin{aligned}
(\bar{\Delta} + \Lambda \frac{\alpha - 1}{2\alpha - 1})\delta^2g &= \Lambda \left( \frac{2 - 3\alpha}{2\alpha - 1} \right) \delta g^{ab}\delta g_{ab} + 2\delta g^{ab}\delta R_{ab} + 2\delta C_{b[a}^c \delta C_{\ell]c}^\ell \bar{g}^{ab} \\
&\quad - \frac{1}{|\bar{g}|} \left( \delta\pi^{ab}\delta\pi_{ab} - \frac{1}{2}\delta\pi^2 \right) - \frac{2\Lambda\alpha}{2\alpha - 1} \left[ \frac{\delta^2\pi_\phi}{\bar{\pi}_\phi} - \frac{\delta\phi^{,i}\delta\phi_{,i}}{2} \left( \frac{\bar{\pi}_\phi}{\sqrt{|\bar{g}|}} \right)^{\frac{2}{1-2\alpha}} \right. \\
&\quad \left. + \frac{1}{(2\alpha - 1)^2} \left( (-2\alpha^2 + 10\alpha - 8)(\delta \ln \sqrt{|\bar{g}|})^2 \right. \right. \\
&\quad \left. \left. + (2\alpha^2 + 4\alpha - 5)(\delta \ln \pi_\phi)^2 - 4(\alpha^2 + 3\alpha - 3)(\delta \ln \sqrt{|\bar{g}|})(\delta \ln \pi_\phi) \right) \right]
\end{aligned} \tag{3.35}$$

The various terms in  $(\delta\pi_\phi)$  can be reexpressed in terms of the metric fluctuations, using the linearized Hamiltonian constraint, via

$$\delta \ln \pi_\phi = -\frac{(2\alpha - 1)}{2\Lambda\alpha} \left( \left[ \bar{\Delta} + \Lambda \frac{\alpha - 1}{2\alpha - 1} \right] \delta g - \bar{\nabla}^c \bar{\nabla}^b \delta g_{cb} \right) \equiv \mathcal{M}\delta g, \tag{3.36}$$

so that the only explicit matter dependence in the second order hamiltonian constraint appears through the  $\delta^2\pi_\phi$  and  $\bar{g}^{ij}\delta\phi_{,i}\delta\phi_{,j}$  terms.

We are now finally in a position to state the LS conditions and derive their consequences. In the next section I show to gauge-fix the fluctuations to second order and then compute the timelike LS condition.

## 3.4 The LS conditions

### 3.4.1 Fixing the linear gauge

As is usual in relativistic perturbation theory one must fix the coordinate freedom inherent in the metric and matter fluctuations. In this particular case I want to remove the homogeneous second order matter dependence, which enters through the  $\delta^2\pi_\phi$  term, in order that equation (3.35) is of the form  $\mathcal{L}\delta^2F = S((\delta F)^2)$  (where  $\mathcal{L}$  is an elliptic operator (with only constants in its kernel) acting on the fluctuations in some quantity  $F$  in a closed space). In this form (since the timelike Killing vector component is trivial) the LS conditions (2.113) simply become  $\int S = 0$  where the integral is over the closed space [23].

Fortunately the Einstein static background crucially simplifies the relevant gauge transformation laws of the fluctuations not only at linear order, but also at second order. Indeed the term  $\delta^2\pi_\phi$  only depends on the *linear* gauge fixing essentially because  $\dot{\bar{\pi}}_\phi = 0$  knocks out any  $\dot{\bar{\pi}}_\phi^{(2)}$  term in the second order gauge transformations. Furthermore, I am only interested in the homogeneous part of this  $\delta^2\pi_\phi$  term so I pick a linear gauge-fixing to eliminate the homogeneous part and leave only the inhomogeneous part  $\delta^2\bar{\pi}_\phi$  to remain.

Consider the linear spacetime coordinate transformation  $\hat{x}^\beta = x^\beta + (T, \partial^i M + \tilde{M}^i)$ , where

$(T, M, \tilde{M}^i)$  satisfy

$$\bar{\Delta} \left( (2\alpha - 1)\bar{\Delta} + (\alpha - 1)\Lambda \right) T = -(2\alpha - 1)\delta\pi^{(Tr)} \quad (3.37)$$

$$\int_{S^3} \{2M\bar{\Delta}\delta\pi_\phi + \delta^2\pi_\phi\} \sqrt{|\bar{g}|} d^3x = 0 \quad (3.38)$$

$$\tilde{M}^i = -\delta g_{(V)}^i, \quad \bar{\nabla}_i \tilde{M}^i = 0, \quad (3.39)$$

and where the spatial dependence of the modes is understood in terms of the eigenfunctions of the spatial laplacian:  $\bar{\Delta}F = -\sum_L \frac{L(L+2)}{a_0^2} F = -\sum_L \frac{\Lambda\alpha}{\alpha+1} L(L+2)F, L \in \mathcal{Z}^+$ . In this new coordinate system  $\delta g_{(V)}^i = 0$ , the homogeneous part of  $\delta^2\pi_\phi$  is zero, and  $\delta\pi^{(Tr)} = 0$ . Thus

$$\begin{aligned} -\frac{1}{|\bar{g}|} \left( \delta\pi^{ab}\delta\pi_{ab} - \frac{1}{2}\delta\pi^2 \right) - \frac{2\Lambda\alpha}{\bar{\pi}_\phi(2\alpha-1)} \delta^2\pi_\phi &= -\frac{1}{|\bar{g}|} \left( \delta\pi_{TT}^{ab}\delta\pi_{ab}^{TT} - \frac{3}{2}(\tilde{\mathcal{A}}_{\delta\pi})^2 \right) \\ &\quad - \frac{2\Lambda\alpha}{\bar{\pi}_\phi(2\alpha-1)} \delta^2\tilde{\pi}_\phi, \end{aligned} \quad (3.40)$$

and therefore the transformations given by (3.37)-(3.39) *completely fix* the vector degrees of freedom at linear order, however we still have the freedom  $M \rightarrow M + f(t)$  in the scalar sector which one may use to eliminate the homogeneous modes  $\tilde{\mathcal{A}}_{\delta g}$  by picking a special  $f(t)$  (whose form is not particularly illuminating at this stage). Furthermore, the linearized Hamiltonian constraint (3.36) in this coordinate system implies that  $\bar{\Delta}\pi_\phi$  cannot be zero everywhere, which means one can always pick a function  $M$  such that equation (3.38) is satisfied.

### 3.4.2 Gauge-fixed LS conditions

The LS condition associated with (3.35) (an integrability condition on  $\delta^2g^{(Tr)}$ ), is effectively the integral of the right hand side of equation (3.35) set to zero:

$$\begin{aligned} 0 = \int_{S^3} &\Lambda \left( \frac{2-3\alpha}{2\alpha-1} \right) \delta g^{ab}\delta g_{ab} + 2\delta g^{ab}\delta R_{ab} + 2\delta C_{b[a}^c \delta C_{\ell]c}^\ell \bar{g}^{ab} - \frac{1}{|\bar{g}|} \left( \delta\pi^{ab}\delta\pi_{ab} - \frac{1}{2}\delta\pi^2 \right) \\ &- \frac{2\Lambda\alpha}{2\alpha-1} \left[ \frac{\delta^2\pi_\phi}{\bar{\pi}_\phi} - \frac{\delta\phi^{,i}\delta\phi_{,i}}{2} \left( \frac{\bar{\pi}_\phi}{\sqrt{|\bar{g}|}} \right)^{\frac{-2}{1-2\alpha}} + \frac{1}{(2\alpha-1)^2} \left( (-2\alpha^2 + 10\alpha - 8)(\delta \ln \sqrt{|\bar{g}|})^2 \right. \right. \\ &\left. \left. + (2\alpha^2 + 4\alpha - 5)(\delta \ln \pi_\phi)^2 - 4(\alpha^2 + 3\alpha - 3)(\delta \ln \sqrt{|\bar{g}|})(\delta \ln \pi_\phi) \right) \right] \sqrt{|\bar{g}|} d^3x \end{aligned}$$

Using by-parts integration and compactness one can show that

$$2 \int_{S^3} \sqrt{|\bar{g}|} \left( \delta g^{ab}\delta R_{ab} + \delta C_{b[a}^c \delta C_{\ell]c}^\ell \bar{g}^{ab} \right) = \int_{S^3} \delta g^{ab}\delta R_{ab}\sqrt{|\bar{g}|},$$



which implies, using the linear equations of motion and making the above gauge choice,

$$\begin{aligned} \int_{S^3} & \frac{1}{|\bar{g}|} \delta\pi_{TT}^{ab} \delta\pi_{ab}^{TT} + \delta g^{ab} \frac{\bar{\Delta} \delta g_{ab}}{2} - \frac{\Lambda}{2(2\alpha-1)} \delta g^{ab} \delta g_{ab} \\ & + \left( \frac{2\Lambda\alpha}{(2\alpha-1)^3} \left( \mathcal{M}^2(2\alpha^2 + 4\alpha - 5) - 2\mathcal{M}(\alpha^2 + 3\alpha - 3) + \frac{-2\alpha^2 + 10\alpha - 8}{4} \right) - \frac{\Lambda}{2} \right) (\delta g)^2 \\ & - 3 \left( \frac{\tilde{\mathcal{A}}_{\delta\pi}^2}{2} \right) V = 0, \end{aligned} \quad (3.41)$$

where  $V$  represents the volume of the initial static space and  $\mathcal{M}$  is defined in equation (3.35). I have eliminated the terms in  $\delta\phi_{,i}$  by using the scalar-sector momentum constraints (which are  $\partial^i (\frac{\Lambda}{2\kappa\pi_\phi} \delta\pi^{(Tr)} - \delta\phi) = 0$ ). The constraint is split into its inhomogeneous ( $L \geq 2$ ) and homogeneous ( $L = 0$ ) pieces<sup>2</sup>.

Finally, inserting the decompositions into equation (3.41) and using equation (3.36) to remove the dependence in  $(\delta\pi_\phi)^2$ , I arrive at

$$0 = \int_{S^3} A \sqrt{|\bar{g}|} d^3x - 3 \left( \frac{\tilde{\mathcal{A}}_{\delta\pi}^2}{2} \right) V, \quad (3.42)$$

where

$$\begin{aligned} A = & \sum_{L \geq 2} \left[ \alpha \Lambda^3 k^2 \left\{ \frac{\alpha^2(2\alpha^2 + 4\alpha - 5)}{2(\alpha+1)^4(2\alpha-1)} k^6 - \frac{\alpha(6\alpha^2 + 21\alpha - 20)}{4(2\alpha-1)(\alpha+1)^3} k^4 \right. \right. \\ & \left. \left. + \frac{100\alpha^3 - 194\alpha^2 + 109\alpha + 8\alpha^2 - 20}{8(\alpha+1)^3(2\alpha-1)^3} k^2 + \frac{\alpha-1}{2(2\alpha-1)(\alpha+1)} \right\} (\delta g^{(tr)})^2 \right. \\ & \left. + \left\{ \frac{1}{|\bar{g}|} \left[ \delta\pi_{ab}^{(TT)} \delta\pi_{\ell m}^{(TT)} \right] + \frac{\Lambda}{2\alpha+1} (\alpha(k^2 - 2) + \frac{\alpha+1}{2\alpha-1}) \delta g_{\ell m}^{(TT)} \delta g_{ab}^{(TT)} \right\} \bar{g}^{\ell a} \bar{g}^{mb} \right] \end{aligned} \quad (3.43)$$

and where  $k^2 \equiv L(L+2)$ . It is important to point out that equation (3.42) is *not* the integral of the second order Hamiltonian action <sup>(2)</sup> $S$  for the fluctuations since, given our gauge fixing, the symplectic terms  $\delta\pi^{(Tr)} \delta\dot{g}_{Tr}^{ij}$  do not contribute.

The main result of this Chapter is thus that the first term in equation (3.42),  $A$ , is positive definite given  $\alpha \geq 1$ ,  $L \geq 2$ . I observe that this lone LS condition is an integral over  $S^3$ , i.e. is of positive measure. This means that in the absence of the homogeneous modes, which provide a strictly negative definite contribution to the integral through the second term, there is no nontrivial solution to the LS condition *even though* there are certainly solutions to the second order equations which *only* have inhomogeneous linear seeds. In other words, if one wants to study the evolution of the second order modes one must include, as part of their source, the zero modes at linear order in order to properly satisfy the initial value constraints.

<sup>2</sup>The fluctuations corresponding to  $L = 1$  can be shown to be purely coordinate fluctuations.

### 3.5 Conclusions

I have shown that the inhomogeneous, second order scalar metric and matter fluctuations of the Einstein static spacetime are not integrable unless one includes homogeneous linear fluctuations in their source. This suggests that if the universe is in the local neighbourhood of the Einstein static solution then to second order in the fluctuations (with arbitrary linear fluctuations) the familiar homogeneous mode will, by virtue of its very presence, necessarily dominate and the spacetime fluctuations will be unstable.

It is important to note that without the LS conditions it would seem acceptable to take the existence of neutrally stable inhomogeneous modes as an indication that, to linear order, the Einstein static models are dynamically stable given sufficient pressure in the background. The LS conditions forbid this conclusion by forcing the inclusion of the unstable, homogeneous, mode in the second order initial value formulation for scalar modes. The physical significance of the LS conditions is thus clearly one of a global (and in that sense weak) constraint on the entire set of modes. Since they are formulated on a positive definite measure and the homogeneous modes, ‘zero modes’, exclusively are of negative norm, it seems natural that you cannot ignore the homogeneous modes if you insist on a nontrivial solution to the LS condition.

It is worth emphasizing that this is *not* a claim of stability or instability at second order, but rather at first order. Although the argument demands an examination of the second order equations, they show that the equations must be unstable to first order. The issue of whether or not second order perturbations can stabilize the spacetime at a sufficiently high value of the linearized solution remains completely unclear. However, the unambiguous conclusion of this chapter is that the Einstein universe is *unconditionally* unstable to first order perturbations *regardless* of the presence of a large class of neutrally stable linear modes.

Interestingly enough, the LS conditions will come up again in the completely different context of quantum backreactions in de Sitter spacetime in Chapter 5. Their close relationship to the projected constraints of general relativity, and to the generators of gauge transformations those perturbed constraints represent, will make for an interesting constraint on the possible type of symmetry the linearized quantum states of that problem can have. Hopefully, however, the reader has already been convinced that the whole issue of linearization stability is more than a formal curiosity.

## CHAPTER 4

# SUPERHUBBLE BACKREACTIONS IN SLOW-ROLL INFLATION

### 4.1 Introduction

The second project of this thesis is concerned with assessing the importance of higher order cosmological perturbations in inflationary models of the universe. These cosmological perturbations will be taken to be quantum mechanical in origin, following the intuition of inflation, and will interact with their linear counterparts and the background spacetime in a process called backreaction. This backreaction offers a glimpse of the fascinating nonlinear dynamics of metric and matter fluctuations which may have played an important role in the early universe, both in current cosmological research and even in the search for a quantum analogue of general relativity. In this chapter I will focus on the physical effects of a particular class of backreactions in general inflation models. In Section 4.2 I add slightly more detail to the brief introduction to inflation given in Section 1.2 in order to place the following work in context and define notation and terminology. Section 4.3 provides a brief literature survey, precise motivation for focusing on this particular class of backreactions, and outlines some of the constraints and assumptions of the following calculations. Next, I will draw upon the results, notation, and terminology of Chapter 2 to set up the general formalism of second order cosmological perturbations in Section 4.4. In Section 4.5 a novel set of observables for these backreactions will be defined, and in Section 4.6 I will show how one calculates these quantities in a procedure which involves controlling higher order gauge ambiguities and solving the higher order equations of motion. In Section 4.7 the effects of the backreactions are assessed, and finally I draw some conclusions.

Briefly, these conclusions suggest that there is a surprisingly large class of inflationary models which do not admit a linearized perturbative approximation of a slow-roll spacetime. As I show in the following sections, this occurs because the cumulative effects of backreactions are quite important in the limit of slower and slower roll of the background scalar field.

### 4.2 A brief look at slow-roll inflation

As briefly indicated in Section 1.2, the major success of inflationary cosmology is in simultaneously offering an explanation for the homogeneity of the universe along with a mechanism that explains its inhomogeneity. The mechanism generating inhomogeneities involves quantum fluctuations in the fields that represent the dominant form of stress-energy during the inflationary epoch, which is a postulated 'potential' dominated era of the early universe (which explains the homogeneity).

More precisely, as indicated in Chapter 1, this inflationary epoch solves some central problems of the big bang scenario by imagining the equation of state of the universe was very different at very early times. To allow for the required causal contact over the region of sky we observe, the FRW scale factor  $a(t)$  had to evolve so that its fractional rate of change is roughly a constant, i.e. has exponential dependence  $a(t) \sim e^{Ht}$ , for  $t \sim 0$ , where  $H$  is the approximately constant and positive Hubble parameter<sup>1</sup>. This rapid expansion, called *inflation*, spatially flattened and homogenized the universe because it *shrank* the causal horizon (called the Hubble horizon) sufficiently for light to traverse it in the available time. In other words, during a more typical era of evolution of the universe (where  $p \geq -\rho/3$ ), the Hubble parameter  $H$  will grow more rapidly than the scale factor  $a$ .

Using the FRW equations (2.23) and (2.24), one can immediately see precisely what kind equation of state this exponential era of expansion requires. Indeed, demanding that the Hubble parameter be *positive* in equation (2.24) and taking the curvature term in (2.23) to be negligible during this era, one can see that  $\rho > 0$  and  $\rho + 3p < 0$  must be true during inflation. For these specific reasons inflation crucially relies on being able to achieve a negative-pressure (i.e. tension) equation of state, as we intuited in Section 1.2.

This requirement naturally leads to considering *scalar fields* as the dominant component of stress-energy in the early universe. Using equation (2.35) it is easy to show that

$$\rho = \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}\partial^i\phi\partial_i\phi + V(\phi) \quad (4.1)$$

$$p = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{6}\partial^i\phi\partial_i\phi - V(\phi) \quad (4.2)$$

for a scalar field  $\phi$ , where the index  $i$  runs only spatial values. From these formulae it is clear that if the gradients of the scalar field are negligible compared to its potential  $V(\phi)$ , the approximate relation  $p \approx -\rho$  holds, or in other words a potential-dominated scalar field is a simple type of matter which leads to the required, inflationary, equation of state. It is worth noting that if  $p = -\rho$  identically the energy density  $\rho$  will not change *at all* with the expansion or contraction of the universe<sup>2</sup>, and this case is referred to as the *de Sitter* model. The inflationary era as realized by scalar fields is thus more precisely defined by the requirement that the scalar field change, 'roll', very slowly down its potential, or in other words that the kinetic terms of the scalar field be small compared to its potential terms. This is just the requirement that the energy density remain roughly constant during the inflationary era, as applied to a scalar field potential driving inflation. Treating the gradients of the scalar field as small compared to  $V(\phi)$  is called *the slow-roll approximation*, and often in the literature this is expressed in terms of the gradient of the potential itself. Indeed, the basic condition that  $V \gg (\dot{\phi})^2$  for temporal gradients can be rewritten using equation (2.21) (with  $n = 3$ , and using equations (4.1), (4.2) with no spatial gradients) as

$$\epsilon_{SR} \equiv \frac{1}{\kappa} \left( \frac{\partial_\phi V}{V} \right)^2 \ll 1, \quad (4.3)$$

<sup>1</sup>For a rigorously constant Hubble parameter and (slicing where) exponential expansion occurs, this is a deSitter model, which is very different than inflation per say.

<sup>2</sup>This is essentially because  $\dot{\rho} = 0$  by equation (2.21).

where  $\kappa \equiv 8\pi G/c^4$ . In this chapter units are such that  $G = c = 1$ . One may take derivatives of this condition to obtain further conditions but they shall not be needed in what follows. The slow-roll condition (4.2) makes good intuitive sense since it simply requires the potential to be sufficiently flat so that the scalar field may roll slowly enough for inflation to proceed. Generally the era of slow-roll is followed by a so-called era of 'reheating', where gradients become important and the field oscillates rapidly in a local or global minimum of the potential  $V(\phi)$ , but this era is not treated in any detail here.

### 4.2.1 Long-wavelength perturbations in inflation

Now that we have a handle on the slow-roll condition in scalar-driven inflation which is required to obtain a sufficient era of negative pressure, the quantum fluctuations in this scalar field can be considered. The amplification of quantum noise during the transition from the inflationary era to the 'matter dominated' era thought to follow it can be adjusted to obtain just the right level of fluctuations to describe much of the structure in our present universe. During the inflationary era quantum fluctuations have their wavelengths hugely extended, however as long as these wavelengths are shorter than the Hubble radius (which is the distance light can travel in one e-folding of expansion of the universe) these quantum fluctuations are thought to be in their vacuum state. This is a strong assumption and in general the choice of initial conditions for the fluctuations in inflation is a very contentious issue<sup>3</sup>, however in this Chapter it will be taken as valid. Once the wavelength of these fluctuations is larger than the Hubble radius, which is what I mean by long-wavelength in this thesis, causality dictates their amplitude is frozen in time (their strength also gradually becomes larger and larger during inflation and they generally are not in their vacuum state at these superhorizon scales). The scalar field values which fluctuate to higher values generally take a longer time to decay down to the reheating era mentioned above. Differences in decay time for the slowly-rolling scalar field directly lead to classical density perturbations (roughly,  $\delta\rho/\rho \sim H\delta t$  for the density contrast) because different regions of space have inflation end at different times.

It is worth noting that the quantum fluctuations can in principle affect the overall evolution of the inflationary era. Indeed, they can add an aspect of random-walk to the classical roll-down on  $V(\phi)$  such that, for example, if  $\phi$  is too close to the minimum for a long enough inflationary period then quantum fluctuations may push it up the potential. This allows further inflation and leads to a self-sustaining process which is called stochastic, or chaotic, inflation (introduced by Linde in [26], [27]).

For fluctuations in a spatially flat FRW model it is convenient to expand the spatial dependence of the fluctuations in plane wave basis of the form  $e^{i\vec{k}\cdot\vec{x}}$ , where the dot product uses the flat spatial metric in comoving coordinates as in equation (2.7). The magnitude of the vector  $\vec{k}$  divided by the scale factor  $a$ ,  $k/a$ , will be taken to be the *comoving wavenumber* of a single  $k$ -mode of a given fluctuation. The *long wavelength* approximation will be that the comoving wavenumber is always much smaller than the Hubble scale,  $H$ , in the sense that  $(k/aH)^2 \ll 1$  is negligible compared to  $(k/aH)$ . Together with the slowroll parameter  $\epsilon_{SR}$  in equation (4.3) and the strength

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<sup>3</sup>E.g., the fluctuations are in their ground state precisely when the scalar field driving inflation must be very far from equilibrium, in a false vacuum?

of the metric and matter fluctuations  $\epsilon$ , this implies one in general has three approximations in any analysis of longwavelength fluctuations in inflationary spacetimes: the slow-roll approximation, the long-wavelength approximation, and the perturbative approximation of a slowly rolling spacetime.

A complete analysis of the scalar (density) perturbations during inflation involves tracking the evolution of perturbations of the full, coupled, Einstein-scalar system, and culminates in studying consequences of equation (2.70) of Chapter 2 for the linear case. However, the basic mechanism for the origin of scalar perturbations and indeed their so-called 'scale-free' nature and short/long wavelength behaviour can be understood from the perspective of a test scalar field in a fixed, background spacetime. To get a sense of the calculations to follow in Sections 4.3 to 4.8 and to define this 'scale-free' property of the fluctuations, this will now briefly be described. A bonus will be to get a sense of what falsifiable predictions inflation *really* makes about the nature of the fluctuations and their power spectrum.

Indeed, starting with a flat spatially isotropic metric in comoving coordinates, as in equation (2.7), one may describe a Fourier mode of a free, massless, minimally coupled scalar field  $\phi$  in the above-described plane-wave basis via

$$\phi(\vec{x}, t) = \sum_k \phi_k(t) e^{i\vec{k}\cdot\vec{x}}, \quad (4.4)$$

where the wavenumbers  $k_i = n_i 2\pi/L, n_i \in \mathcal{Z}^+$  are quantized in a box of side  $L$ . Stress-energy conservation, embodied by equation (2.21) for this case, implies the equation of motion for the fourier coefficients

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \frac{k^2}{a^2}\phi_k = 0, \quad (4.5)$$

where the dots represent derivatives with respect to comoving time  $t$ ,  $H \equiv \dot{a}/a$  is the Hubble parameter. Just as equation (2.70) implies, this equation of motion is identical to that of a simple harmonic oscillator with a unit mass, a term what is commonly referred to as a 'time-dependent friction term'  $3H\dot{\phi}$ , and a spring constant  $k^2/a^2$ . It is worth emphasizing here that the phrase 'friction term' is somewhat misleading, as the equation of motion (4.5) actually stems from a time dependent, conservative, Lagrangian which in turn leads to a conserved stress-energy<sup>4</sup>. Nevertheless, one may speak of 'underdamped' and 'overdamped' regimes given a certain range of values for the proper wavelength of a mode and the Hubble parameter. When the proper wavelength of a mode,  $a/k$ , is much smaller than the Hubble radius,  $H^{-1}$ , the mode will behave as an ordinary oscillator with no damping. But, if the wavelength is much larger than the Hubble radius, the mode will behave as an overdamped oscillator. Indeed, its amplitude will 'freeze out' as  $\dot{\phi}_k$  decays rapidly to zero.

If we let the box in which we quantize  $\phi$  become arbitrarily large, then the sum given by equation (4.4) will go over to an integral that incorporates the density of states in  $k$ -space the

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<sup>4</sup>This is easily seen by rescaling the scalar field to be  $\tilde{\phi} = a(t)^{3/2}\phi$ , so that equation (4.5) becomes  $\ddot{\tilde{\phi}} + \left(\frac{k^2}{a^2} - \frac{\partial_t^2 a^{3/2}}{a^{3/2}}\right)\tilde{\phi} = 0$ , which trades a 'friction' term for a 'mass' term. I am indebted to W.G. Unruh for emphasizing this.

usual way from statistical mechanics. In this way we can go over to the quantum *field* theory of the scalar field  $\phi$  by adopting the standard continuum Fourier transform for each mode:  $\phi_k = \frac{1}{(2\pi)^{3/2}} \int \phi(x) \exp(-i\vec{k} \cdot \vec{x}) d^3x$ . Each mode acts as an *independent* harmonic oscillator and can be described using the Lagrangian density defined in general by equation (2.83):

$$\mathcal{L}_k = \frac{a^3}{2} \left( |\dot{\phi}_k|^2 - \frac{k^2}{a^2} |\phi_k|^2 \right), \quad (4.6)$$

where the factor of  $a^3$  is just the comoving coordinate volume element  $\sqrt{-|g|}$ . Here, each  $k$ -mode acts as an *independent* oscillator and we may take the ground state  $|0\rangle$  of an oscillator governed by this Lagrangian as a *Gaussian* wavefunction in  $\phi_k$ . Recalling how the argument goes for an ordinary one dimensional simple harmonic oscillator<sup>5</sup>, it is not hard to see that the test field's dispersion must go as

$$\langle 0 | |\phi_k|^2 | 0 \rangle \equiv (\delta\phi_k)^2 = \frac{a}{2a^3k}, \quad (4.7)$$

where the factors of  $a$  account for using the comoving wavenumber  $k/a$  and the normalizing with the volume element  $\sqrt{-|g|} = a^3$ . For proper wavelengths much larger than the Hubble radius the overdamped oscillator will have a frozen amplitude and  $(\delta\phi_k)$  will roughly be constant, while for wavelengths much smaller than the Hubble radius the ground state will evolve *adiabatically* and equation (4.7) will remain valid. Since the Hubble parameter (radius) is roughly constant compared to the scale factor during inflation, the proper wavelength will rapidly *exceed* the Hubble radius during inflation. This combination of assuming an adiabatic initial vacuum for the subhorizon modes, and then stretching the comoving wavelength well beyond the Hubble scale, so that their amplitudes freeze, is the heart and soul of the inflationary mechanism of generating fluctuations. It is also useful to note that, since the Hubble radius grows much more rapidly than the scale factor after inflation ends, the test scalar field can only go from being an overdamped oscillator to an underdamped one in an inflationary era followed by a matter-dominated regime.

Given the above remarks, we know that the fluctuation amplitude of subhorizon modes born in an adiabatic vacuum will freeze out at some value of the scale factor  $a_*$  such that  $k/a_* = H_*$ , where  $H_*$  is the value of the (approximately constant) Hubble parameter during inflation. The value of the frozen out amplitude of fluctuations is thus, at some later time at which they are still nevertheless superhorizon in scale, obtained by combining the definition of  $a_*$  with equation (4.7), to give

$$(\delta\phi_k)^2 = \frac{H_*^2}{k^3}, \quad (4.8)$$

which corresponds to a so-called *scale-free spectrum* of fluctuations<sup>6</sup> and which in turn is amplified by a factor of order  $(a/a_*)^2$  compared to equation (4.7). This latter fact shows how these quantum

<sup>5</sup>For an oscillator with frequency  $\omega_k$ , position operator  $x$  and momentum operator  $p$ , calculate the dispersions  $\langle \delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\omega_k} - 0$  and  $\langle \delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\omega_k}{2} - 0$  to form  $\langle \delta x^2 \rangle \langle \delta p^2 \rangle = \frac{1}{4}$ .

<sup>6</sup>Bearing in mind that  $dk/k$  is used in the Fourier conventions, not  $k^2 dk$ , which would lead to a form manifestly independent of  $k$ .

fluctuations can be amplified to classical, large-scale, scales by an inflationary era. More specifically, in order for the initial spectrum of  $\phi_k$  to induce a corresponding spectrum of classical density fluctuations, the scalar field must have a large and classical stress-energy term appearing on the right hand side of the semiclassical Einstein equations. The cross terms (in the semiclassical equations) between the fluctuations and the effective cosmological constant contribution due to  $V(\phi)$ , give rise to cosmologically significant, scale-free, density perturbations.

It is useful to note that inflation does not directly address the initial conditions of these modes because of its assertion that the subhorizon modes evolve adiabatically in their ground state, which sounds *prima facie* reasonable<sup>7</sup>. Because the negative-pressure era stretches the modes' wavelength well-beyond the causal Hubble horizon it is not necessary to deal with initial conditions in an any more formal way—one simply assumes that the modes were born in their ground state at some point prior to or during inflation. Any noninflationary scenario must directly confront the initial condition issues because superHubble modes must be present from the very beginning in order to induce cosmological perturbations. One could in fact follow Hollands and Wald in [28] and construct a noninflationary models which leads to the above scale-free spectrum of fluctuations. The existence of their construction reinforces the notion that the essential mechanism for the generation of cosmologically meaningful, scale-free, density perturbations operates with or without an inflationary era<sup>8</sup>. Therefore, the principal predictions of inflation on the nature of cosmological density fluctuations is their Gaussian nature and to a much lesser degree that their spectrum is scale-free. Another key prediction about fluctuations due to inflation is, as Andreas Albrecht refers to it in [30], their 'passivity'. This refers to the notion that perturbations evolve in an very particular *linear* manner until gravitational collapse induces nonlinearities much later in the history of the universe, or in other words the role of nonlinearities like backreaction *during* inflation is not considered important. In this sense it is unsurprising that an initial Gaussian vacuum leads to Gaussian fluctuations later on.

### 4.3 Motivation, Brief Review of Past Work

The fluctuations in the scalar inflaton field driving inflation will themselves induce metric fluctuations because of the equations of motion. For quantum fluctuations, one way to treat the problem is using the techniques outlined in Chapters 2 and 3, i.e. perturbation theory. As we just showed in Section 4.2, to linear order one of the standard results of perturbation theory is that the power spectrum of scalar (energy density) fluctuations is scale-free (see [4] for a comprehensive review), which is approximately what is observed today<sup>9</sup>.

However, the cosmological constant and dark matter/energy problems<sup>10</sup> only deepen with these same observations. There has been renewed interest in the past few years in the effect of higher order corrections to the linearized Einstein equations on both early and late-time physics in inflation. Suggestions have been made that higher order corrections to linearized theory, on superhorizon

<sup>7</sup>But arguably problematic upon closer examination.

<sup>8</sup>Their methodology was extensively criticized in Linde et al in [29], in my view unsuccessfully.

<sup>9</sup>There are serious deviations from scale-freeness in 'low  $\ell$ ' (large scales) part of the observed power spectrum.

<sup>10</sup>We cannot understand why we observe a small positive cosmological constant nor the nature of dark matter, the dominant components of our universe's energy and matter census, with current theory.



scales (i.e. larger than the Hubble radius), can take the form of a negative cosmological constant. This could produce a dynamical relaxation mechanism for the bare cosmological constant (starting from [31], to [32], [33] most recently, and many references therein). The measurability or physical reality of these superhorizon backreaction effects has been a contentious issue (see e.g. [34], [35] and reference therein), and many questions remain regarding the link between local subhorizon physics and these superhorizon backreactions.

In this Chapter I focus on explicitly evaluating the second order corrections to the homogeneous Einstein equations in perturbation theory. At *second* order in perturbation theory one might even expect that since the effect of second order contributions are *cumulative* over all wavenumbers, their relative amplitude may become comparable to that of first order. Furthermore, if one thinks about solving the perturbed Einstein constraint equations for the matter fluctuations and putting these solutions back into the perturbed evolution equations, it is not hard to see that some of the second order corrections could in fact be divided by a so-called slow-roll parameter. This only adds to the worry that the second order terms could plausibly dominate the linear ones for a ‘slow enough’ roll in the background. It is clear that in the limit as the slow-roll parameter goes to zero, so that the background universe tends to a de Sitter universe, the first order corrections go to zero and the second order fluctuations dominate in their effect on the gravitational field. At what values of the slow-roll parameter do the second order perturbations dominate over the linear ones? The most radical possibility is that the slow-roll conditions are precisely the conditions that the second order perturbations dominate. Such questions appear to be behind some of the concerns raised by L. Grischuk in [36] about the consistency of linearized perturbation theory in inflation.

The technical complications of sorting out the second order gauge issues and other nonlinear effects such tensor perturbations seeding scalar perturbations are many, but rendered tractable with the aid of packages such as GRTensor for Maple [37]. In Section 4.4 I calculate the cumulative second order contributions to the homogeneous energy density and pressure. I do not address questions of the ultraviolet (short wavelength) regularization of the fluctuations in this Chapter but focus on the superhorizon fluctuations. Indeed, I pay special attention to the case where one considers the cumulative effect of Hubble sized to nearly homogeneous contributions on the homogeneous mode. Though this is admissible using the effective methods (as I describe below), this feature would turn out to be a limitation in the arguments I present. I show explicitly how one can calculate backreactions on inhomogeneous modes in the framework of this Chapter.

In past work effective approximation methods have been used to evaluate such contributions. One popular method characterizes the backreactions in terms of an effective energy-momentum tensor  $\tau_{ab}$ . In this method there are two contributions to  $\tau_{ab}$ : the quadratic matter energy momentum tensor and the contribution of the first order gravity perturbations. Using early work by Brill, Hartle and Isaacson (see [38]) among others, the Einstein equations are expanded to second order in perturbation theory assuming the linearized equations hold (so that they drop out). Then the remaining terms are spatially averaged with respect to a given background metric and the resulting equations are interpreted as equations for a *new* homogeneous metric  $\bar{g}_{ab}$  which include the effects

of quadratic linear perturbations:

$$G_{ab}(\bar{g}_{ab}) = \kappa(\bar{T}_{ab} + \langle \tau_{ab} \rangle), \quad (4.9)$$

where  $\langle \tau_{ab} \rangle$  is the spatially averaged ‘backreaction’ stress-energy defined by

$$\tau_{ab} = T_{ab}[(\delta g_{cd})^2, (\delta\phi)^2] - \frac{1}{\kappa} G_{ab}[(\delta g_{cd})^2]. \quad (4.10)$$

Here,  $\kappa \equiv 8\pi G$  in units where  $c = 1$ ,  $\delta^n$  indicates the  $n$ -th order perturbation of the object it acts on, as in Chapters 2 and 3.

It should be noted that, by construction, the zeroth order equations are not obeyed in this formulation. In this effective scenario one is solving for a new isotropic background metric which obeys equation (4.9), and this can in itself raise difficult questions of consistency if one is interested in backreactions on inhomogeneous modes (but is ok if one looks at just the homogeneous mode). This can be seen by considering the first variation of the Hamiltonian action for a gravitational system, equations (2.88) and (2.89):

$$\delta H = \int (\delta N \bar{\mathcal{H}}_{\perp} + \delta N_i \bar{\mathcal{H}}^i + \bar{N} \delta \mathcal{H}_{\perp} + \bar{N}_i \delta \mathcal{H}^i) d^3x, \quad (4.11)$$

and assuming that the background constraints do *not* hold, i.e.  $\bar{\mathcal{H}}_{\perp} \neq 0, \bar{\mathcal{H}}^i \neq 0$ . If one has *homogeneous* variations then these linear terms will vanish *anyway*, whether or not the background equations of motion are satisfied. This is so because one can do a by-parts integration in the first and second terms above whose result will be boundary terms which can then be set to zero under reasonable assumptions [39].

Furthermore, it should also be noted that in their effective approach the full second order Einstein equations are not solved, nor are second order coordinate transformations considered. This latter fact can be of considerable concern when interpreting the significance of higher order effects, as Unruh showed in [34]. In this sense the effective approach outlined above does not appear, to me, to be able to convincingly evaluate the higher-order corrections to Einstein’s equations, simply because it never actually considers them in the context of higher-order perturbation theory. In my opinion, the full impact of second order perturbation theory can only be assessed attendant problems that come from solving the higher order equations within some second order gauge-fixing and a direct confrontation of Fourier averaging issues at higher order<sup>11</sup>.

Nevertheless, following Brandenberger and Mukhanov in [41] and [4], this effective approach can be used to evaluate the dominant long-wavelength contributions to  $\tau_{ab}$ . Defining the energy density and pressure at second order by (using  $(+, -, -, -)$  as the signature)  $\delta^2 \rho \equiv \langle \tau_0^0 \rangle$  and  $\delta^2 p \equiv -(1/3) \langle \tau_i^i \rangle$ , they find that these contributions have the effective equation of state  $\delta^2 p \approx -\delta^2 \rho$ , with  $\delta^2 \rho < 0$ . This corresponds to the equation of state of a negative cosmological constant. They also find that  $\delta^2 \rho$  grows with time, partially because as inflation proceeds more and more length scales exceed the Hubble scale and contribute to  $\delta^2 \rho$ . These two results combined

<sup>11</sup>However one should note that for super Hubble fluctuations there exist methods like Starobinski’s effective stochastic inflation methods, which although developed only for test scalar fields so far, have the promise of giving the leading order logarithmic divergences order by order in perturbation theory. See Woodard in [40]

suggest their main claim, which is that the backreactions effectively create a negative, and growing, cosmological constant which can reduce the actual cosmological constant *in the large*. Locally one might expect the situation to be different. The first order modes should in some sense, *locally*, look like simple coordinate transformations of the homogeneous solutions, and their effect on higher order metric and matter fluctuations to be again that of higher order coordinate transformations [34]. However, as stated earlier, this thesis will not focus on these difficult issues of interpretation.

Instead, the focus is on evaluating the higher order contributions to the background equation of state, and in particular calculating the quantity  $\delta^2\rho + \delta^2p$  and its dispersion. I follow a procedure of consistently (though probably not convergently) expanding the Einstein equations to second order and solving a specific subset of them assuming the zeroth and linear order equations hold. This is done about a flat FRW spacetime in which the dominant gravitating matter is a slowly-rolling, minimally coupled, scalar field  $\bar{\phi}$ , and we only study the effect of fluctuations on spatial scales exceeding the Hubble radius. In order to define the second order energy density and isotropic pressure, an invariant definition of  $\delta^2\rho$  and  $\delta^2p$  is given in terms of the eigenvalues of the stress energy tensor. Due to the mixing of tensor and scalar waves, these fluctuations will not only arise from second order scalar modes but also from quadratic combinations of scalar-scalar and tensor-tensor modes at second order. I find that in general  $\delta^2p + \delta^2\rho \neq 0$  and  $\delta^2\rho < 0$ , but that  $\frac{\delta^2\rho + \delta^2p}{\delta^2\rho}$  does become small. Perhaps surprisingly, I also find that the relative amplitude of the second order dispersion  $\langle (\delta^2\rho)^2 \rangle$  dominates over its linear counterpart  $\langle (\delta\rho)^2 \rangle$  for a wide range of slow-roll parameters in the background.

## 4.4 Long-wavelength second order perturbation theory

In Chapter 2 it was pointed out that the tensor, vector, and scalar metric modes all decouple<sup>12</sup>, which justifies examining only one class of modes at a time, which make the linear calculations far simpler to accomplish. Furthermore, depending on the initial conditions one takes for the linear metric modes at the beginning of inflation (a contentious issue), the scalar modes are typically more important than the gravitational modes at the end of inflation. Given this, the fact that scalar modes directly lead to energy density fluctuations required for structure formation, and the fact that for longwavelength perturbations gravity wave terms are typically suppressed in comparison, ‘cosmological perturbations’ have become synonymous with scalar perturbations.

At second order, however, it is well known that *the second order modes have products of linear tensor, vector, and scalar modes as sources*. For example, the effective stress-energy of linear tensor fluctuations (gravity waves) can induce second order scalar fluctuations. In order to describe this it becomes essential to include all three classes of modes (scalar, vector, and tensor) as sources for the higher order gravitational radiation. For fluctuations about a spatially flat FRW background in comoving coordinates  $(t, \vec{x})$  the perturbed metric is, according to equation (2.25),

$$\begin{aligned}
 ds^2 = & -(1 + \epsilon A(t, \vec{x}) + \epsilon^2 \mathcal{A}(t, \vec{x})) dt^2 + 2(\epsilon B_i(t, \vec{x}) + \epsilon^2 \mathcal{B}_i(t, \vec{x})) dt dx^i \\
 & + a^2(t) (\delta_{ij} + \epsilon h_{ij}(t, \vec{x}) + \epsilon^2 q_{ij}(t, \vec{x})) dx^i dx^j,
 \end{aligned} \tag{4.12}$$

<sup>12</sup>This fact is extensively discussed in [42].

where  $\epsilon$  is the strength of the linear perturbation as before,  $i, j, k, \dots$  denote purely spatial indices in this Chapter, and  $a(t)$  is the usual scale factor from Section 2. The coordinate  $t$  is here taken as the proper time of a comoving observer. Together with the matter perturbations of the scalar matter  $\bar{\phi}$ , already defined by equation (2.37), one can define the matter perturbation via

$$\phi(t, \vec{x}) = \bar{\phi}(t) + \epsilon\Phi(t, \vec{x}) + \epsilon^2\mathcal{F}(t, \vec{x}). \quad (4.13)$$

Thus the perturbations are  $(A, B_i, h_{ij}, \Phi)$  at linear order and  $(\mathcal{A}, \mathcal{B}_i, q_{ij}, \mathcal{F})$  at second order. The backreactions or higher corrections are suitably integrated quadratic combinations of terms from the former set, and they source the equations for the longwavelength part of the latter set of variables. In principle, the evolution of the fluctuations is determined by substituting the above metric and matter perturbations into the Einstein equations (2.8) and solving these at second order subject to the linearized and background equations, i.e. one solves

$$\delta^2 G_{ab} = \kappa(\delta^2 T_{ab}), \quad (4.14)$$

and also demands that the linearized and zeroth-order (background) field equations

$$\delta G_{ab} = \kappa(\delta T_{ab}) \quad (4.15)$$

$$\bar{G}_{ab} = \kappa(\bar{T}_{ab}), \quad (4.16)$$

hold. Again,  $T_{ab} = \{\phi_{;a}\phi_{;b} - g_{ab}(\frac{1}{2}\phi^{;c}\phi_{;c} + V(\phi))\}$  is the stress-energy for a minimally coupled scalar field defined by equation (2.35).

As in Section 4.2.1, the spatial dependence of the fluctuations is taken to be of the form  $e^{\pm ik_i x^i}$ , and the reader is also reminded of the longwavelength approximation

$$\left(\frac{k}{aH}\right)^2 \ll 1. \quad (4.17)$$

Here,  $H$  is the Hubble parameter that corresponds to the scale factor expansion  $a(t) \sim t^\alpha$ ,  $\alpha \gg 1$  and I take the potential of the slowly-rolling background scalar field to be

$$V(\phi) = \Lambda + \beta\phi, \quad (4.18)$$

for which the only non-trivial slow-roll condition is  $\frac{1}{\kappa}\left(\frac{V_{,\phi}}{V}\right)^2 = \frac{\kappa\beta^2}{H^4} \ll 1$ . One should note that  $\dot{H} = -\frac{\kappa\beta^2}{18H^2}$ , and we write  $\sqrt{\alpha} \sim \frac{H}{\sqrt{-\dot{H}}} = \sqrt{\frac{18H^4}{\kappa\beta^2}}$  so that the deSitter limit corresponds to  $\alpha \rightarrow \infty$ . Conservation of energy for a slowly-rolling scalar field in this potential, with initial value  $\phi_0$ , requires that  $\phi$  take the form

$$\phi = \phi_0 - \frac{\beta t}{3H}, \quad (4.19)$$

and only comoving times  $t$  such that  $0 \leq t \ll \frac{3H\phi_0}{\beta}$  are considered.

Therefore, as in Section 4.2.1, there are three small parameters in this problem:  $\epsilon$  (the strength

of the matter and metric fluctuations),  $\frac{\kappa\beta^2}{H^4}$  (slow-roll parameter associated with our choice of inflaton potential), and  $\frac{k}{aH}$ , the long-wavelength parameter. However, only two of these small parameters are independent as the order of the metric and matter fluctuations is a direct product of the physics of the slow-roll parameter, so the scale of  $\epsilon$  is in some sense dynamically set. Unless otherwise specified, all future references to the order of a quantity will refer to its order in  $\epsilon$ .

The solutions to equations (4.14) and (4.15) will be invariant under a class of diffeomorphisms which are themselves functions of  $\epsilon$ . By expanding out the diffeomorphisms order by order in  $\epsilon$ , one can apply first order, second order, etc., coordinate transformations to the solutions of the first order, second order, etc., solutions of (4.14) and (4.15). In particular, one can choose these transformations to not only simplify the form of the solutions and the equations themselves, but to also deduce what part of the metric and matter fluctuations is physical and what part is just a coordinate effect.

#### 4.4.1 Linear and second order coordinate transformations; gauge fixing

Building on the work in Section 2.2.1, second order gauge transformations are relatively straightforward to define. Recall that at linear order, using equation (2.28) to define a general  $\epsilon$ -dependent coordinate transformation, the associated *linearized* coordinate transformation is given by (2.29). At second order, one can similarly define the *second-order* coordinate transformation via

$$\chi^a = \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial \epsilon} \left[ \frac{\partial \tilde{x}^c(x, \epsilon)}{\partial \epsilon} \frac{\partial X^a(\tilde{x}(x, \epsilon), \epsilon)}{\partial \tilde{x}^c} \right]. \quad (4.20)$$

As in Section 2.2.1, since the metric  $g_{ab}$  is a tensor it will in general transform by

$$g_{ab}(x, \epsilon) = \frac{\partial \tilde{x}^c}{\partial x^a} \frac{\partial \tilde{x}^d}{\partial x^b} \tilde{g}_{cd}(\tilde{x}(x, \epsilon), \epsilon), \quad (4.21)$$

so it is not hard to show that

$$\delta^2 g_{ab} = \delta^2 \tilde{g}'_{ab} + (\mathcal{L}'_{\zeta} + \mathcal{L}'_{\chi}) \tilde{g}_{ab} + 2\mathcal{L}'_{\zeta} \delta \tilde{g}_{ab}, \quad (4.22)$$

Thus, the second order fluctuations  $\delta^2 g_{ab}$  depend on  $\zeta^a$ , i.e. they depend on the linearized coordinate transformation  $\zeta^a$  as well as the second order transformation  $\chi^a$ . The fact that second order quantities depend also the linearized gauge fixing ( $\zeta^a$ ) implies, for example, that the second order stress energy  $\delta^2 T_{ab}$  in equation (4.15) which sources the backreactions will depend on the choice of both  $\zeta^a$  and  $\chi^a$  (a gauge choice).

As shown in Chapter 2, in standard cosmological perturbation theory one usually makes a particular choice of  $\zeta^a$  to simplify the interpretation of the fluctuations. For example, the longitudinal gauge [4] makes the choice

$$\zeta^0 = B - a\dot{E}, \quad (4.23)$$

$$\zeta^i = -\partial^i E. \quad (4.24)$$

It should be noted that this longitudinal gauge only fixes the *scalar* part of the metric into diagonal form, and its simple form relies crucially on the form of the anisotropies of the perturbed stress energy. However for our problem, since we obviously cannot diagonalize the *entire* linear order metric because of the presence of TT gravity waves, we find it convenient to set a gauge choice which amounts to setting

$$B_{,i}^i = 0, \tag{4.25}$$

$$\partial^j \left( h_{ij} - \frac{\delta^{\ell m} h_{\ell m}}{3} \delta_{ij} \right) = 0, \tag{4.26}$$

which fixes, to within trivial residual gauge freedoms, the linearized metric perturbation regardless of the form of the perturbed stress energy and is sometimes known as the Poisson gauge [43]. Clearly the longitudinal gauge for the scalar sector is a special case of (4.25) and (4.26), since it restricts only the *potentials*  $B, E$  of the metric fluctuations  $B_{,i}, E_{|ij}$ . The primary physical advantage in using this generalization of the longitudinal gauge is that one can unambiguously transform to any gauge while easily keeping track of the residual freedoms, while the primary mathematical advantage is that they lead to a compact form for the perturbed hamiltonian and momentum constraints. One can always transform from this gauge to any other gauge since the transformations are *algebraic* in nature as opposed to, say, the nonlocal integrals that take one to the so-called synchronous gauge<sup>13</sup> (Bertschinger in [43] provides an excellent, more detailed, explanation).

## 4.5 Total energy density and pressure at second order

How do the classical metric and matter fluctuations at second order influence the background equation of state, and in particular how does one assess the influence of the gravitational backreactions? In classical GR it is not possible to unambiguously assign an energy to a gravitational system in the absence of some simplifying assumption (such as asymptotic flatness), and in fact as we saw in equation (2.88) the total gravitational ADM Hamiltonian is equal to zero when the universe is spatially compact. Fortunately, within finite order perturbation theory we can avoid the conceptual and technical problems involved in defining local, quasi-local, or any other practical definitions of gravitational energy-momentum because, by definition, we have a preferred decomposition of the spacetime metric. This allows us to exclusively attribute 'energy' to the 'dynamical part' ( $\delta g_{ab}, \delta^2 g_{ab}$ ) of the metric as opposed to the 'background part' ( $\bar{g}_{ab}$ ) simply because there exists a 'background derivative' ( $\bar{\nabla}_a$ ) against which to measure any such 'dynamics'. Thus we can define a 'relative gravitational stress energy'  $\tau_{ab}$  of the fluctuations with respect to the curved background, and in particular we can take combinations of  $\tau_{ab}$  and the stress energy  $T_{ab}$  to study the fluctuations in the pressure and energy density at second order.

A general formalism to define conserved quantities and conservation laws with respect to curved background spacetimes has already been developed by Katz, Bicak, and Lynden Bell in [44]. The

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<sup>13</sup>Synchronous gauge sets the shift and lapse perturbations to zero. This choice leaves a residual gauge freedom on the spatial hypersurfaces in the form of two ghost modes.

basic idea is to start from the Lagrangian

$$\begin{aligned}
 L(\epsilon) &= \frac{\sqrt{-|g|(\epsilon)}}{2\kappa} [C_{\mu\nu}^\rho(\epsilon)C_{\rho\sigma}^\sigma(\epsilon) - C_{\mu\sigma}^\rho(\epsilon)C_{\rho\nu}^\sigma(\epsilon)] \\
 &\quad - \frac{1}{2\kappa} \left( g^{\mu\nu}(\epsilon)\sqrt{-|g|(\epsilon)} - \bar{g}^{\mu\nu}\sqrt{|\bar{g}|} \right) \bar{R}_{\mu\nu} + L_M,
 \end{aligned} \tag{4.27}$$

where  $2C_{abc}(\epsilon) \equiv 2(\Gamma_{abc}(\epsilon) - \bar{\Gamma}_{abc}) = (\bar{\nabla}_b g_{ca}(\epsilon) + \bar{\nabla}_c g_{ba}(\epsilon) - \bar{\nabla}_a g_{bc}(\epsilon))$  is completely analogous to equation (2.91) and  $L_M$  is the ‘matter’ Lagrangian. Recall that  $\bar{C}_{bc}^a = 0$ , so that in the background  $\bar{L} = 0$ . The main result of [44] is that starting from  $L$  one can build vector densities  $I^\mu$  which are conserved in the sense that  $\bar{\nabla}_\mu I^\mu = 0$ . The details are interesting but tedious, and I simply quote the final result for the conserved vector density  $I^\mu$  for completeness:

$$\begin{aligned}
 I^\mu &= \left[ \sqrt{-|g|(\epsilon)}T_\nu^\mu(\epsilon) - \sqrt{|\bar{g}|}\bar{T}_\nu^\mu + \frac{1}{2} \left( g^{\rho\sigma}(\epsilon)\sqrt{-|g|(\epsilon)} - \bar{g}^{\rho\sigma}\sqrt{|\bar{g}|} \right) \bar{R}_{\rho\sigma}\delta_\nu^\mu + \sqrt{-|g|(\epsilon)}t_\nu^\mu \right] \zeta^\nu \\
 &\quad + \sqrt{-|g|(\epsilon)} \left( \sigma^{\mu[\rho\sigma]}\partial_{[\rho}\zeta_{\sigma]} + \mathcal{Z}^\mu(\zeta^\nu) \right) \\
 &\equiv \left[ \sqrt{-|g|(\epsilon)}T_\nu^\mu(\epsilon) - \sqrt{|\bar{g}|}\bar{T}_\nu^\mu + \sqrt{-|g|(\epsilon)}\tau_\nu^\mu \right] \zeta^\nu + \sqrt{-|g|(\epsilon)} \left( \sigma^{\mu[\rho\sigma]}\partial_{[\rho}\zeta_{\sigma]} + \mathcal{Z}^\mu(\zeta^\nu) \right),
 \end{aligned} \tag{4.28}$$

where  $\zeta^\mu$  is the arbitrary, smooth, vector field and  $\tau_b^a$  is the analogue of the Einstein pseudotensor (defined in terms of  $C_{bc}^a$  instead of  $\Gamma_{bc}^a$ ). Only the first term in equation (4.28), in the square brackets, will interest me in what follows. Its interpretation is that of the relative stress energy of the fluctuations with respect to a given curved background. For completeness only I mention that the last group of terms can be interpreted in terms of the relative helicity of the perturbations with respect to the background [44] and, again, I shall not consider these in this work. I denote the gravitational parts of the relative stress energy by  $\tau_{ab}$ , which of course is defined with respect to our preferred decomposition of the spacetime metric. In this sense it is not a tensor in general, however since it is exclusively constructed as a function of the difference of perturbed and background connections, its second order part  $\delta^2\tau_{\mu\nu}$  is a tensor to second order in  $\epsilon$ .

Given their definition for  $\tau_{\mu\nu}$ , whose explicit definition is essentially the same as the Landau-Lifschitz pseudostress energy [45], it is now possible to look for observables to probe the details of the effects of higher order fluctuations on the linearized theory. In the next section it is shown that the eigenvalues of the total stress energy, i.e.  $\tau_\mu^\nu + T_\mu^\nu$ , are excellent candidates for such observables. It is worth emphasizing that it will turn out that the dependence of these eigenvalues on the precise definition of  $\tau_\nu^\mu$  is **not** important, since their dominant contributions will be from terms in  $T_\nu^\mu$ .

### 4.5.1 Eigenvalues of the total stress energy

One often defines (the rotationally invariant but not boost invariant) energy density and isotropic pressure of a perfect fluid by  $-\rho \equiv g^{00}T_{00}$  and  $3p \equiv g^{ii}T_{ii}$ . While this is relatively straightforward to interpret and implement in linearized perturbation theory, at second order one gets complications such as having to subtract off shear (offdiagonal) stresses  $\sim \delta p_i \delta p_j, i \neq j$ , from the diagonal isotropic contributions  $\sim (\delta p_i)^2$ . From this and other points of view it turns out to be extremely useful to consider the eigenvalues of the mixed-valence total stress energy of the fluctuations, i.e. to consider

the eigenvalues of the tensor

$$\bar{T}_b^a + \delta T_b^a + \delta^2 T_b^a + \delta^2 \tau_b^a, \quad (4.29)$$

where the last term  $\delta^2 \tau_b^a$  is the second order part of the relative gravitational stress energy,  $\tau_{ab}$ , described above. Since it is of mixed valence, it transforms from coordinates  $x$  to  $\bar{x}$  as  $(\partial x / \partial \bar{x})(\partial \bar{x} / \partial x)$  and therefore has gauge-covariant eigenvalues  $\lambda_i$  associated to timelike and spacelike eigenvectors, which are calculated by solving the equation

$$\det(\bar{T}_b^a + \delta T_b^a + \delta^2 T_b^a + \delta^2 \tau_b^a - \lambda_i \delta_b^a) = 0 \quad (4.30)$$

It is worth emphasizing that gauge *covariance* is weaker than gauge *invariance*, since the former only refers to invariance with respect to change of coordinate basis but does **not** also guarantee invariance with respect to Lie dragging. In other words, the eigenvalues are not invariant under a general reidentification of points in the background spacetime.

At zeroth order, since we are not in deSitter but in a slow-roll spacetime, the one timelike eigenvalue and three spacelike eigenvalues (associated with their respective eigenvectors) are different and are in that sense 'sufficiently separated'. Although I am only interested in the average of the pressure eigenvalues, this 'sufficient separation' property is important since it guarantees that the perturbations obey this property as well<sup>14</sup>. I define the energy density as minus the eigenvalue of the timelike eigenvector and the cumulative isotropic pressure as the average of the distinct eigenvalues associated with their respective spacelike eigenvectors. It is worth emphasizing again that the eigenvectors at second order will in general point in different directions, but that all that matters in these calculations is the *averaged* contribution obtained after quantum averaging over  $k$ . After such averaging the terms like  $\delta\rho(k)\delta\rho(k')$  collapse to the diagonal terms  $(\delta\rho(k))^2$ .

These eigenvalues can be expressed in terms of scalars formed from the stress tensor and powers thereof. For example, to linear order one may find the averaged eigenvalues  $\delta\rho$  and  $\sum_i \delta p_i$  by perturbing the expressions

$$T_a^a = -\rho + \sum_i p_i \quad (4.31)$$

$$S_a^a S_b^b = \frac{3}{4} \left[ \rho^2 + \frac{2\rho}{3} \sum_i p_i + \frac{1}{3} \left( 4 \sum_i (p_i)^2 - \left( \sum_\ell p_\ell \right)^2 \right) \right], \quad (4.32)$$

where  $S_{ab} \equiv T_{ab} - \frac{T_{ab}^m}{4} g_{ab}$ . To linear order, these relations are equivalent to

$$\delta(T_a^a) = -\delta\rho + \sum_i \delta p_i \quad (4.33)$$

$$\frac{4}{3} \delta(S_a^a S_b^b) = 2(\bar{\rho} + \bar{p})(\delta\rho + \frac{1}{3} \sum_i \delta p_i), \quad (4.34)$$

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<sup>14</sup>Were this not the case I would need to calculate the eigenvectors as well to solve the degenerate problem.



where  $\bar{\rho} + \bar{p} = \frac{\beta^2}{9H^2}$ . Substituting the explicit expressions in terms of the metric and matter fluctuations for the left hand side, one finally obtains the desired expressions for the energy density and cumulative isotropic pressure. To higher order this procedure becomes more complicated since, as mentioned above, pressure contributions like  $(\sum_i p_i)^2$  contain both diagonal contributions like  $\sum_i p_i^2$  and offdiagonal shear contributions like  $\sum_{i \neq j} \delta p_i \delta p_j$ . However the basic strategy is similar and in any case equivalent to solving the above determinant using scalars such as the trace, ‘double trace’, and determinant. I sketch this out below.

For the second order case, the starting point is

$$\delta^2(T_a^a + \tau_a^a) = -\delta^2\rho + \sum_i \delta^2 p_i \quad (4.35)$$

$$\begin{aligned} \delta^2\left(\frac{4}{3}S_b^a S_a^b\right) &= (\delta\rho)^2 + \left(\sum_i \delta p_i\right)^2 + \frac{2}{3}\delta\rho \sum_i \delta p_i - \frac{8}{3}\sum_{i \neq j} \delta p_i \delta p_j \\ &\quad + 2(\bar{\rho} + \bar{p})(\delta^2\rho + \frac{1}{3}\sum_i \delta^2 p_i), \end{aligned} \quad (4.36)$$

where  $S_{ab} \equiv (T_{ab} + \tau_{ab}) - \frac{T_m^m + \tau_m^m}{4}g_{ab}$ . The appearance of terms like  $\sum_{i \neq j} \delta p_i \delta p_j$  and related cross-terms complicates the isolation of the desired eigenvalues  $\sum_i \delta^2 p_i$ . In order to eliminate such terms we consider the second order perturbation of the cube of the trace-free part of the total stress energy. Combined with equations (4.35) and (4.36), this will give us another equation and with it the possibility of cancelling these shear terms in terms of some function of metric and matter fluctuations. The general expression for the cube is

$$-\frac{8}{3}S_b^a S_a^c S_c^b = \rho^3 - \sum_i p_i^3 - \rho \sum_i p_i^2 + \sum_{\ell \neq m} p_\ell p_m^2 + \rho^2 \sum_i p_i + 2\rho \sum_{i \neq j} p_i p_j - 2p_1 p_2 p_3 \quad (4.37)$$

which to second order is

$$\begin{aligned} \delta^2\left(-\frac{8}{3}S_b^a S_a^c S_c^b\right) &= (\bar{\rho} + \bar{p}) \left( 3(\delta\rho)^2 - \left(\sum_i \delta p_i\right)^2 + 2\delta\rho \sum_i \delta p_i + 4 \sum_{i \neq j} \delta p_i \delta p_j \right) \\ &\quad + 3(\bar{\rho} + \bar{p})^2 \left( \delta^2\rho + \frac{1}{3}\sum_i \delta^2 p_i \right), \end{aligned} \quad (4.38)$$

and which in turn has the right form to solve for the shear terms we are desiring to eliminate from expression (4.31). Substituting in the expression of  $\delta^2(-\frac{8}{3}S_b^a S_a^c S_c^b)$ , equation (4.38), in terms of metric and matter fluctuations and solving for the four averaged eigenvalues  $\delta\rho, \delta^2\rho, \sum_i \delta p_i, \sum_i \delta^2 p_i$  in terms of the metric fluctuations, I finally get

$$\left. \begin{aligned} 4(\bar{\rho} + \bar{p})(\delta^2\rho + \frac{1}{3}\sum_i \delta^2 p_i) + \left[\frac{1}{3}(\sum_i \delta p_i)^2 + 3(\delta\rho)^2 + 2\delta\rho \sum_i \delta p_i\right] &= \delta^2\vartheta + \frac{2}{3(\bar{\rho} + \bar{p})}\delta^2\Theta \\ -\delta^2\rho + \sum_i \delta^2 p_i &= \delta^2\mathcal{T} \\ 2(\bar{\rho} + \bar{p})(\delta\rho + \frac{1}{3}\sum_i \delta p_i) &= \delta\vartheta \\ -\delta\rho + \sum_i \delta p_i &= \delta\mathcal{T} \end{aligned} \right\}, \quad (4.39)$$

where  $\theta \equiv \frac{4}{3}S_b^a S_a^b$ ,  $\Theta \equiv -\frac{8}{3}S_b^a S_a^c S_c^b$ ,  $\mathcal{T} = g^{ab}(T_{ab} + \tau_{ab})$ . The simultaneous solutions to these two sets of coupled equations are the (averaged) eigenvalues one would find directly from the matrix represented by the total stress-energy, expression (4.29). They are

$$\left. \begin{aligned} \delta p &\equiv \frac{1}{3} \sum_i \delta p_i = \frac{\delta \mathcal{T}}{4} + \frac{1}{8(\bar{\rho} + \bar{p})} \delta \theta \\ \delta p + \delta \rho &= \frac{1}{2(\bar{\rho} + \bar{p})} \delta \theta \\ \delta^2 p &\equiv \frac{1}{3} \sum_i \delta^2 p_i = \frac{1}{\bar{\rho} + \bar{p}} \left[ \frac{\delta^2 \theta}{16} + \frac{\delta^2 \Theta}{24(\bar{\rho} + \bar{p})} - \frac{3(\delta \theta)^2}{64(\bar{\rho} + \bar{p})^2} \right] + \frac{\delta^2 \mathcal{T}}{4} \\ \delta^2 \rho + \delta^2 p &= \frac{1}{\bar{\rho} + \bar{p}} \left[ \frac{\delta^2 \theta}{4} + \frac{\delta^2 \Theta}{6(\bar{\rho} + \bar{p})} - \frac{3(\delta \theta)^2}{16(\bar{\rho} + \bar{p})^2} \right] \end{aligned} \right\} \quad (4.40)$$

It is worth pointing out that the definitions of these eigenvalues do *not* make any additional assumptions about being defined in a comoving frame of reference, which in general is not consistent with our gauge fixing (it is only consistent with dust, i.e. no pressures). The fluid velocities are certainly *not* zero and in fact represented by the timelike eigenvector associated with  $\rho$ .

#### 4.5.1.1 Linear contributions to the energy density and pressure

At linear order the two values  $\delta \rho, \sum_i \delta p_i$  of the total stress energy  $\delta T_b^a$ , comprised of only the stress energy of matter, can easily be found. Assuming the longitudinal gauge-fixing, we find

$$\left. \begin{aligned} \delta p &= -\frac{\beta}{3H} (\partial_t - 3H) \Phi - \frac{\beta^2}{18H^2} A \\ \delta \rho + \delta p &= 2\beta \Phi \end{aligned} \right\}, \quad (4.41)$$

To linear order, only scalar modes can induce energy density and pressure fluctuations. In the longitudinal gauge fixing specified by equations (4.23),(4.24), the (constrained) equation of motion for the spatial diagonal metric perturbation  $\psi$  in the long-wavelength limit, assuming slow-roll, is simply

$$(\partial_t^2 + H\partial_t)\psi(t) = 0, \quad (4.42)$$

whose nondecaying solution can be taken to be a nonzero constant ( $\equiv \psi$ ). This is to be distinguished from the pure deSitter case where this constant is precisely zero, since there are no physical linear scalar modes in pure dS<sup>15</sup>. The corresponding matter perturbation  $\Phi$  is easily found via the constraint equations (namely  $\Phi = (3H^2/\beta)\psi$ ). Using this and the constraint equations from (4.15) to express the result in terms of  $\psi$ , we find the dominant contributions to  $\delta p$  and  $\delta \rho$  are

$$\delta p \approx -\frac{3\alpha^2}{\kappa t^2} \psi \stackrel{\alpha \rightarrow \infty}{=} -\frac{3H^2}{\kappa} \psi \quad (4.43)$$

$$\delta \rho + \delta p \approx \frac{\psi H^2}{9\kappa} (54\epsilon_{LW} - (6Ht - 1)\epsilon_{SR}), \quad (4.44)$$

<sup>15</sup>One can set up a scalar field with initial velocity on a flat potential, which will rapidly decelerate due to Hubble friction and drive the spacetime to dS. During this deceleration, which will violate slow-roll, there will of course be scalar gravitational perturbations but the physical (noncoordinate) part of these fluctuations will smoothly tend to zero as  $\dot{\phi} \rightarrow 0$ . In pure de Sitter the statement that the only nongauge excitations of the metric are TT is true on all physical length scales, as shown in the next chapter.

where  $\left(\frac{k}{aH}\right)^2 \equiv \epsilon_{LW}$ ,  $\frac{\kappa\beta^2}{H^4} \equiv \epsilon_{SR}$ . Although the right hand of the latter equation is in some sense small, it is not zero. Therefore the linearized contribution to the equation of state, though highly suppressed, will still depend on the details of the small parameters. This point may perhaps be more obvious if one considers how the time evolution equation for the scalar field fluctuations ( during slow-roll ) is modified by the inclusion of the metric fluctuations, namely

$$\ddot{\Phi} + 3H \left( 1 - \frac{1}{9 \left( \frac{\kappa(\phi_0 - \phi)}{\partial_\phi \ln(V(\phi))} - 9 \frac{\epsilon_{LW}}{\epsilon_{SR}} \right)} \right) \dot{\Phi} + 3H^2 \epsilon_{LW} \left( 1 - \frac{2}{27} \frac{\epsilon_{SR}}{\epsilon_{LW}} \right) \Phi = 0, \quad 0 \leq t \ll \frac{3H\phi_0}{\beta}. \quad (4.45)$$

This equation is the perturbed analogue of the conservation equation (2.21). It shows how the gravitational fluctuations, at the linearized level, effectively induce a negative effective mass for the fluctuating scalar field as well as modify the effective Hubble parameter - all in a way which depends on what values we take for  $\epsilon_{SR}, \epsilon_{LW}$ . Since the longitudinal gauge (or equivalently, a longitudinal choice of gauge invariant variables) admits no residual linearized coordinate transformations, these fluctuations cannot be associated with coordinate modes and are hence physical.

It is also important to note that the right hand side of (4.43) contributes at a given, fixed, wavenumber  $k$  to the linear fluctuations  $\delta\rho, \delta p$ . At higher order we generically expect that contributions to the nearly homogeneous *second* order modes of  $\delta^2\rho$  and  $\delta^2 p$  will be cumulative over a broad range of  $k$  of the *linear* modes. It is this enhanced, cumulative, contribution to the second order modes that can make the nonlinear contributions to the equation of state nontrivial. We turn to this now.

#### 4.5.1.2 Second order energy density and pressure perturbations at fixed $k$

Considering for a moment the situation at second order with fixed  $k$ , the partially gauge fixed eigenvalues of the total stress-energy are, for the generic case with no gauge-fixing,

$$\begin{aligned} \delta^2 p = & -\frac{\beta}{3H} (\partial_t + 3H) \mathcal{F} + \frac{\beta^2}{18H^2} \mathcal{A} + \frac{1}{2} (\partial_t \Phi)^2 - \frac{12H^2}{32\kappa} \psi^2 - \frac{3H^2}{32\kappa} C^2 \\ & + \frac{3k^2}{2a^2} \Phi^2 + \frac{\beta\Phi}{3Ha^2} k_i B^i + \frac{\beta^2}{18H^2} \left( -\frac{B_i B^i}{a^2} + A^2 \right) + \frac{\beta}{3H} A \partial_t \Phi \end{aligned} \quad (4.46)$$

$$\delta^2 \rho = -\delta^2 p + 2\beta\mathcal{F} - \frac{93H^2}{4\kappa} \psi^2 + \frac{93H^2}{\kappa} C^2, \quad (4.47)$$

where  $C^2$  is the squared amplitude of the linear tensor fluctuations. As in the linear case, the matter fluctuations  $\mathcal{F}$  are related to the metric fluctuations via the second order constraints from (4.14). The behaviour of the second order scalar perturbations will be influenced by not only the constant scalar modes at linear order, but also the linear tensor-tensor terms. One would thus expect that the scalar modes at second order will become time dependent. In fact, if we set a gauge according to equations (4.25) and (4.26) in Section 4.4.1( so that at linear and second order

the scalar sectors of the metric are diagonal) it can be shown this implies

$$\begin{aligned}
 t &\rightarrow t + (B - a\dot{E}) \\
 &+ \left[ \mathcal{B} - a^{(2)}\dot{E} + 3E(3B - 10a\dot{E})k^2 + 4(a^2\ddot{E} + H\dot{E})(2a\dot{E} - B) + 2a\dot{E}(A - \frac{5}{2}a\dot{B}) \right. \\
 &\left. + B(a\dot{B} - 2A) - 2aE\dot{\psi} \right] \tag{4.48}
 \end{aligned}$$

$$x^i \rightarrow x^i + \partial^i \left\{ -E + \left[ -^{(2)}E - \frac{33}{2}k^2E^2 - 2\psi E + \frac{9}{2}a^2\dot{E}^2 - \frac{1}{2}B^2 - 4a\dot{E}B \right] \right\} \tag{4.49}$$

then we obtain, using the spatial second order scalar field equations in this new coordinate system,

$$\mathcal{A} = -\mathcal{Q} + \underbrace{\frac{2}{3}[(h_+)^2 + h_+h_- + (h_-)^2]}_{\text{TT-TT sector}} + \underbrace{\psi^2 - 2\kappa\Phi^2}_{\text{Scalar-Scalar sector}}, \tag{4.50}$$

$$\mathcal{B} = 0, \tag{4.51}$$

$${}^{(2)}E = 0, \tag{4.52}$$

where  $q_{ij} \equiv \mathcal{Q}\delta_{ij} + (\partial_i\partial_j - \delta_{ij}\frac{\Delta}{3})^{(2)}E$  and  $h_+, h_- \in \mathfrak{R}$  denote the two TT independent degrees of freedom of  $h_{ij}$ . Note that the second-order lapse  $\mathcal{A}$  contains contributions from the TT-TT gravitational wave contributions at linear order. In effect, this gauge choice is the longitudinal gauge at second order in the scalar sector. It can easily be shown that it admits no nontrivial residual scalar coordinate freedoms to second order. Within this gauge, under the slow-roll and longwavelength approximations and using the constraints to express everything in terms of the metric fluctuations<sup>16</sup>, the equations of motion for  $\mathcal{Q}$  are

$$(\partial_t^2 + H\partial_t)\mathcal{Q}(t) = H^2 \left[ 24Ht - 162\frac{k^2}{a^2H^2} \left( \frac{H^4}{\kappa\beta^2} \right) \right] \psi^2 + \left[ \frac{70k^2}{3a^2} - \frac{2\kappa\beta^2t}{9H} \right] C^2, \tag{4.53}$$

where  $C^2 \equiv ((h_+)^2 + h_+h_- + (h_-)^2) \in \mathfrak{R}$ . The growing solution for  $\mathcal{Q}(t)$  is, for a fixed  $k$  mode of  $\psi$ ,  $C$ , and assuming that  $a \sim a_0t^\alpha$  with  $\alpha \gg 1$ , is

$$\mathcal{Q}(t) \approx \left\{ 24\alpha^2 \ln(t) - \frac{81k^2\alpha^2}{\kappa\beta^2} \left( \frac{t^{-2\alpha}}{a_0^2} \right) \right\} \psi^2 + \frac{35k^2}{3\alpha^2} \left( \frac{t^{-2\alpha}}{a_0^2} \right) C^2 + D, \tag{4.54}$$

where  $D$  is a constant of integration deduced by the initial conditions of  $\mathcal{Q}$ , implicitly set to zero by analogy with the linearized sector. This shows that the effect of a given linearized mode at some  $k$  on the nearly homogeneous mode of  $\mathcal{Q}(t)$  is to make it time-dependent, as expected.

When  $\delta^2 p$  and  $\delta^2 \rho$  are computed in this exhaustive coordinate system, using the second and

<sup>16</sup>See Appendix A for more details on the constraints and how they are used.

linear order constraints, I obtain the following result for the dominant longwavelength contributions:

$$\delta^2 p \approx -\frac{3H(t)}{\kappa} (H(t) + \partial_t) \mathcal{Q}(t) - \frac{54H(t)^6 \psi^2}{\kappa^2 \beta^2} - \frac{3H^2 C^2}{16\kappa} \quad (4.55)$$

$$\delta^2 p + \delta^2 \rho \approx +\frac{216\psi^2 H(t)^4 k^2}{a^2 \kappa^2 \beta^2} + \frac{45H^2 C^2}{4\kappa} \quad (4.56)$$

At first glance, it is immediately apparent that if we set  $C^2 = 0$ ,  $\mathcal{Q}(t) = 0$  (as Brandenberger et al do in [41] and references therein) then we obtain  $\delta^2 \rho < 0$  but that  $\delta^2 p + \delta^2 \rho$  is not in general zero because of terms that involve ratios of small parameters:

$$\delta^2 \rho \approx -54 \frac{\alpha^6 \psi^2}{\kappa^2 t^6 \beta^2} \quad (4.57)$$

$$\delta^2 p + \delta^2 \rho \approx +216 \frac{k^2 \alpha^4 \psi^2}{a_0^2 t^{2\alpha} t^4 \kappa^2 \beta^2}, \quad (4.58)$$

However, I do find that, given  $C^2 = 0$ ,  $\mathcal{Q}(t) = 0$ , quadratic scalar-scalar backreaction contributions generally do mimic that of a negative cosmological constant since  $(\delta^2 p + \delta^2 \rho)/\delta^2 \rho \sim (k/aH)^2$ , which is set to zero in their analysis. Nevertheless, for sufficiently slow-roll it seems worrisome that for Hubble-scale size fluctuations the second order contributions  $\delta^2 p$  and  $\delta^2 \rho$  can be comparable to  $\delta \rho$  and  $\delta p$  at a given  $k$ , since as they stand equations (4.57) and (4.58) show that the second order contributions carry 'extra factors' of slow-roll enhancement compared to equations (4.43) and (4.44).

However, I would like to emphasize that it is clearly inconsistent to set  $\mathcal{Q}(t)$  to zero in our approach since it is also of second order and in fact contains terms proportional to  $\psi^2$ . Such a tactic also violates the second order field equations, which take into account that the metric fluctuation  $\mathcal{Q}(t)$  at a given scale will receive contributions from *all* of the Fourier linear modes under consideration (in our case, as we explain below, from the horizon to an approximately homogeneous cutoff). Furthermore since the purely second order contributions to the energy density and pressure at these superhorizon scales are not constants but can evolve in comoving time, the contributions may change significantly during the slow-roll era to the end of inflation.

In the following section I will examine the cumulative effect of the quadratic combinations of linear modes, averaged over superhubble scales, onto a given scale of the purely second order modes. For simplicity I will initially focus on the effect on the second order homogeneous mode alone. I strongly emphasize that the main gauge-fixing of the paper, as described by relations (4.25) and (4.26), is not well-defined in the strictly homogeneous limit  $k = 0$ . An appropriate gauge transformation must be made to sensibly take the homogeneous limit and make these manipulations, and we discuss this in the next section.

#### 4.5.1.3 IR (super-Hubble) contributions from the backreactions

We now consider the cumulative contributions to the energy density and pressure at second order due to the superhorizon modes. Considering only super Hubble fluctuations we know from Section 4.2 that the dominant linear modes are independent of time since the background equation of

state, during slow-roll, is approximately time-independent. In terms of the Fourier-decomposed  $\psi_k$ , the quantum fluctuations (which we take to be Gaussian) during this era depend on  $k$  in such a way that the fluctuations per decade are a constant, which is just a fancy way of saying that something like equation (4.8) will hold. Although strictly speaking cosmological fluctuations are quantized in terms of a reduced variable such as e.g. the Mukhanov-Sasaki (MS) variable  $\nu = a(\Phi - \frac{\beta}{3H^2}\dot{\psi})$  (as described in [4]), one can always use the linearized constraints to simply relate (in the longwavelength limit)  $\nu$  and  $\psi$  up to time-dependent factors. Indeed, the spatial two-point correlation function of  $\psi$  is (after an angular integration)

$$\langle 0|\hat{\psi}(t, \vec{x})\hat{\psi}(t, \vec{x} + \vec{r})|0 \rangle \equiv \langle \psi^2 \rangle = \int_{k_{min}}^{aH} \frac{dk}{k} \frac{\sin(kr)}{kr} \left[ \frac{k^3}{4\pi^2} |\psi_k(t)|^2 \dot{\phi}^2 \right], \quad (4.59)$$

where  $\hat{\psi}(t, \vec{x})$  is the quantum operator associated with  $\psi$ , expanded in the classical basis of plane waves. The Fourier transform of the two-point function is known as the power spectrum, and completely characterizes Gaussian fluctuations in the sense that all higher correlation functions can be expressed in terms of it. Here,  $|0 \rangle$  is the vacuum chosen so that the modes of the reduced MS variable  $\nu_k$  obey  $\nu_k(t_0) \sim k^{-1/2}$ ,  $\dot{\nu}_k \sim ik^{1/2}$  at some initial time  $t_0$ , which in turns implies a set of more complicated conditions on  $\psi_k$  which are not illuminating at this stage ( see [4] for more details ). The metric fluctuations  $\langle \psi^2 \rangle$  at the horizon scale are related to the density contrast fluctuations  $\langle (\delta\rho/\bar{\rho})^2 \rangle$  by equations (4.37) and (4.38), and one can easily show that  $\langle (\delta\rho)^2 \rangle = 36(H^4/\kappa^2) \langle \psi^2 \rangle$ . Using all of this, it is relatively straightforward to show that

$$k^3 |\psi_k|^2 = \frac{1}{4} \frac{H^4}{(2\pi\dot{\phi})^2} = \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2, \quad (4.60)$$

( see [46] for more details ). I take this relation to hold up to some almost homogeneous scale, say  $k = k_{min} \ll aH$ , or in other words the infrared divergence of the linearized fluctuations is cut off at some scale  $k = k_{min}$ , so that for  $k \sim 0$ ,  $|\psi_k|^2 \sim 0$ . The factors of 4 on the right hand side come from using equations (4.42) and (4.43) to relate the fluctuations in the density contrast to the fluctuations in  $\psi$ . Again, this is equivalent to the usual statement made about the power spectrum in terms of the reduced Mukhanov-Sasaki variable  $\nu$  (namely that  $k^3 |\nu_k|^2 \sim \frac{H^2}{\epsilon_{SR} m_{pl}^2}$ , for  $m_{pl}$  Planck mass). It is worthwhile to notice that the corresponding result for the tensor amplitudes will not be enhanced by a slow-roll factor, so we will ignore them in what follows.

Using equation (4.60) we can average over the quadratic combinations of linear fluctuations to solve the constrained equation of motion for the diagonal second order homogeneous metric fluctuation  $\mathcal{Q}_0(t)$ , equation (4.48). Indeed, keeping only the dominant terms,

$$(\partial_t^2 + H\partial_t)\mathcal{Q}_0(t) = 4\pi \int_{k_{min}}^{aH} {}^{(2)}S(k)k^2 dk, \quad (4.61)$$

where  ${}^{(2)}S(k) = H^2 \left[ 24Ht - 162 \frac{k^2}{a^2 H^2} \left( \frac{H^4}{\kappa\beta^2} \right) \right] |\psi_k|^2$  and  $k_{min} \ll aH$ . Carrying out the integral over  $k$  and solving equation (4.55), it is relatively straightforward to show that the dominant

solution takes the form

$$\mathcal{Q}_0(t) \approx \frac{\kappa H^2 \alpha}{4\pi \epsilon_{SR}^2} \left( \frac{27^2}{10} - 6N \right) - \frac{36N\kappa H^2}{4\pi \epsilon_{SR}^2} \ln\left(\frac{aH}{k_{min}}\right), \quad (4.62)$$

where  $N = \int H dt = \alpha \ln(t)$  is the number of e-foldings, and  $\alpha = Ht \gg 1$  as described in Section 4.3. Using this result and performing similar integrations for the remaining terms in equations (4.54) and (4.55), we finally obtain expressions for the contributions to the homogeneous mode of  $\delta^2 \rho$  and  $\delta^2 p$

$$\begin{aligned} \delta^2 \rho_{IR} &\approx -\frac{3H}{\kappa} (H + \partial_0) \mathcal{Q}_0(t) - \int_{k_{min}}^{aH} \frac{54H^6}{\kappa^2 \beta^2} 4\pi |\psi_k|^2 k^2 dk \\ &\approx \frac{H^2}{\kappa} \left( \frac{\kappa H^2}{4\pi \epsilon_{SR}^2} \right) \left[ -3\alpha \left( \frac{27^2}{10} - 6N \right) + 36 \left( N - \frac{27}{2} \right) \ln\left(\frac{aH}{k_{min}}\right) \right] \end{aligned} \quad (4.63)$$

$$\begin{aligned} \delta^2 p_{IR} + \delta^2 \rho_{IR} &\approx \int_{k_{min}}^{aH} \frac{216H^4}{a(t)^2 \kappa^2 \beta^2 \epsilon_{SR}} (4\pi k^4 |\psi_k|^2) dk \\ &\approx 2\pi \frac{216H^4}{a^2 \kappa^2 \beta^2 \epsilon_{SR}} (aH)^2 \left( \frac{9\kappa H^2}{16\pi^2} \right), \end{aligned} \quad (4.64)$$

We can immediately compare the magnitude of the leading homogeneous backreaction term in, say,  $\delta^2 \rho_{IR}/\bar{\rho}$ , to the root mean square of the density contrast  $\sqrt{\langle (\delta\rho/\bar{\rho})^2 \rangle}$  during inflation by using equation (4.63). One can demand a *consistency condition* for linearized theory, namely that the second order contributions be subdominant compared to that of the linearized sector, i.e.  $\delta^2 \rho_{IR} < \sqrt{\langle (\delta\rho)^2 \rangle}$ . If we use the expressions (4.60), (4.63) and (4.64) above, this demand is crudely equivalent to the condition that, for  $N \gtrsim 70$ ,

$$\epsilon_{SR} > (4\kappa H^2)^{1/4} N^{3/4}, \quad (4.65)$$

i.e. the usual slow-roll condition will in general be violated if  $(\kappa H^2)^{1/5} N^{2/5} > 1$ . Thus inequality (4.65) suggests that the breakdown of the linearized approximation, in the strict sense of back-reactions having a relatively larger amplitude, occurs when one assumes what may well be the slow-roll condition for the background spacetime. Furthermore it is apparent that, although the right hand side of (4.64) is not zero, the form of these dominant contributions is approximately that of a cosmological constant since  $(\delta^2 p + \delta^2 \rho)/\delta^2 \rho \sim -1/\ln(k_{min}/aH) \sim 0$ .

As I alluded to above, in the current gauge fixing I have chosen the homogeneous limit of the fluctuations is not well defined since, by equation (4.60) and the linear order equation  $\psi = -A$ , the lapse  $A$  will diverge as  $k \rightarrow 0$ . However, the results are valid even if one makes a gauge transformation which renders the superhorizon fluctuations well defined in the homogeneous limit. Indeed, after making such a gauge transformation, then the homogeneous limit is taken and the total lapse goes to the value 1 while the offdiagonal terms go to 0, the central feature of ‘slow-roll’ enhancement remains and the above arguments still apply. Let us consider how this works explicitly by looking at the linear case. Choosing the infinitesimal gauge transformations to be  $\zeta^a = (T(t, k), kL(t, k), kL(t, k), kL(t, k))$ , we first note that  $T_k(t)$  must be, by equation (4.60) and

the fact that  $\delta g'_{00} = \delta g_{00} - 2\dot{T}$ , so that

$$T_k(t) = -\frac{3 \ln a}{4\pi} \sqrt{\frac{\kappa}{k^3 \epsilon_{SR}}} \quad (4.66)$$

ensures that  $|A_k| \rightarrow 0$  for  $k \rightarrow 0$ . Given this divergent gauge transformation of the comoving time, we can ask for what  $L_k(t)$  we can ensure the rest of the perturbations that appear in the metric will be well-defined for  $k \rightarrow 0$ . Since

$$\mathcal{L}_\zeta \bar{g}_{0i} = -T_{,i} + a^2 \dot{L}_{,i}, \quad (4.67)$$

in position space, then choosing  $L_k(t) = -\int \frac{\ln a}{k^{3/2}} \frac{3}{4a^2\pi} \sqrt{\frac{\kappa}{\epsilon_{SR}}} dt$  will render the shift  $B$  induced by equation (4.67) to zero. The off-diagonal terms  $E$  induced in turn by this choice of  $L$  will be well posed in the homogeneous limit since  $k^2 E$  is what appears in the metric, and this will decay as  $\sqrt{k}$ . However, the diagonal spatial metric terms will in general receive a contribution of  $2Ha^2T$ , which will get large as  $k \rightarrow 0$ . Since I implicitly use an IR cutoff of  $k_{min}$ , beyond which equation (4.60) will not hold and may in fact be replaced by a relation which does not diverge with  $k$ , these large contributions can be considered in some sense regulated<sup>17</sup>. I note in passing that  $k_{min}$  may be related to the spatial Hubble scale at the start of inflation.

To second order the argument is very similar, only more tedious. Given the above choice for the linear gauge-fixing as  $k \rightarrow 0$ , we must pick a second order  $\chi^a = ({}^{(2)}T(t, k), 0, 0, 0)$  such that the second order shift and offdiagonal spatial terms go to zero in the limit  $k \rightarrow 0$ . Since

$$\delta^2 \tilde{g}'_{0i} = \delta^2 g_{0i} + \delta g_{00} k_i T + \delta g_{ii} k^i \dot{L} + \delta g_{ij}^{(i \neq j)} k^j \dot{L} + \delta g_{i0} (\dot{T} - 2k^2 L) + T \partial_0 \delta g_{0i} - k ({}^{(2)}T + {}^{(2)}S)$$

where  $({}^{(2)}S \equiv a^2 k \dot{L} (\dot{T} - 12k^2 L) - kT(4\dot{T} - 4a\dot{a}\dot{L} - 6k^2 L - a^2 \ddot{L}))$ , one can show that the choice

$${}^{(2)}T = \frac{1}{k} \left[ ({}^{(2)}S + AkT + 3\psi k \dot{L}) \right] \quad (4.68)$$

will yield  $\delta^2 \tilde{g}'_{0i} = 0$ . The offdiagonal, second order, spatial terms induced by the above transformation are

$$\delta^2 \tilde{g}'_{ij} \stackrel{i \neq j}{=} 2k^2 T^2 - 4ak^2 (2\dot{a}L + a\dot{L})T + 18a^2 k^4 L^2, \quad (4.69)$$

which at worst diverge as  $k^{-1}$ . Since only the expression  $k^2 ({}^{(2)}E)$  appears in the metric, these offdiagonal terms in the metric smoothly go to zero in the homogeneous limit. Once again the spatial diagonal contributions to  $\mathcal{Q}$  will grow large as  $k \rightarrow 0$ .

Now we can finally address the total effect of all these transformations on the quantities of interest,  $\delta^2 \rho, \delta^2 p$ . To second order, for example,  $\delta^2 \rho$  will be Lie-dragged along  $\zeta^a$  and  $\chi^a$  according

<sup>17</sup>Of course, the functional form of the divergences may change with regulation, but the physics (the relative amplitudes of  $\delta^2 \rho, \delta^2 p$  compared to  $\delta \rho, \delta p$ ) should not.



to the transformation

$$\begin{aligned}\delta^2\tilde{\rho}' &= \delta^2\rho + (\mathcal{L}_\zeta^2 + \mathcal{L}_\chi)\bar{\rho} + 2\mathcal{L}_\zeta\delta\rho \\ &= \delta^2\rho + 2(T\partial_0\delta\rho - 3k^2L\delta\rho) + {}^{(2)}T\dot{\rho} + \left(T^2\ddot{\rho} + T\dot{T}\dot{\rho} - \dot{\rho}k^2LT\right)\end{aligned}\quad (4.70)$$

and similarly for the averaged pressures  $\delta^2p$ . Choosing the above expressions for  $T, L, {}^{(2)}T$ , we can see that the total effect on  $\delta^2\rho$  and  $\delta^2p$  will be to introduce terms that diverge like  $k$  but are suppressed by factors of  $\beta$ . Such terms will be subdominant compared to the terms already present in the original gauge, or in other words, the gauge transformation which renders the metric diagonal in the homogeneous limit does not undo the dominant contributions to the energy density and pressure at second order.

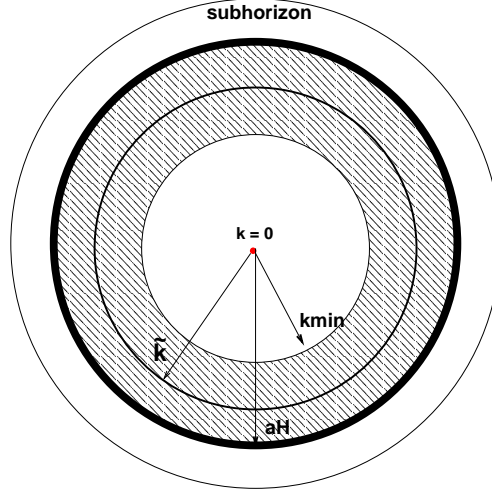
### 4.5.2 Comment on backreaction on inhomogeneous second order modes

An important loophole in the above analysis resides in the fact that I compare the horizon scale (frozen) amplitudes of the linearized fluctuations to that of the homogeneous sector of the second order fluctuations. It is far from clear that this is an acceptable comparison to make, not least because we are directly cutting off the divergence of the linearized fluctuations by imposing an IR cutoff at  $k = k_{min} \ll aH$  and then comparing the amplitude of the second order fluctuations with that of the linearized fluctuations well beyond the cutoff. A valid criticism of this result is thus that it would be natural for the second order perturbations to dominate at the homogeneous scale simply because we have cut off the linearized fluctuations long before comparing their amplitude to that at second order, or in other words the spatial dependence induced by evaluating the (quantum averaged) second order amplitude at some  $k_{min} < k = \tilde{k} \ll aH$  may alter the conclusions of inequality (4.65). See Figure 1. Furthermore, since in cosmology one does not typically compare quantum averages<sup>18</sup> but rather *dispersions*, we must actually compare the dispersions  $\rho, p$  at first and second order.

Therefore, we need to compare the quantity  $\sqrt{\langle \left(\frac{\delta^2\rho_{IR}}{\bar{\rho}}\right)^2 \rangle}$  at some scale  $\tilde{k}$  such that  $k_{min} \lesssim \tilde{k} \ll aH$  to the horizon-scale amplitude of the linearized fluctuations  $\sqrt{\langle \left(\frac{\delta\rho}{\bar{\rho}}\right)^2 \Big|_{k=aH} \rangle}$ , and see if the the amplitude of the second order modes may *still* dominate over that of the linear modes assuming slow-roll. Since the second order quantity will contain functions that are the square of the square of the  $k$ -mode it is necessary to compute these quantities in a thorough manner. The computation of the fluctuations of the operator  $\delta^2\rho_{IR}$  which are inherent in the expressions for the dominant terms of  $\langle \left(\frac{\delta^2\rho_{IR}}{\bar{\rho}}\right)^2 \rangle$  is actually fairly involved, and in the next series of relatively technical sections I first derive the expressions for the inhomogeneous case ( $k \neq 0$ ) and then take the homogeneous limit ( $k = 0$ ).

En route to computing the required quantities, we consider the two and four point functions of

<sup>18</sup>The quantum average of the linearized energy density fluctuations  $\langle \delta\rho \rangle$  is just zero, which is not very interesting. However  $\langle \delta\rho^2 \rangle \neq 0$ .



**Figure 4.1:** All linearized modes in the shaded region, spanning from  $k = aH$  to  $k = k_{min} \ll aH$  in spatial scale, are taken to seed second order modes at a particular value of  $k$ . The thick black boundary indicates the Hubble scale, where the amplitudes of the linearized fluctuations freeze out during slow-roll. We compare the amplitudes of the linearized fluctuations at the thick black boundary to that of the second order fluctuations at  $k = \tilde{k}$  and  $k = 0$  (and ignore the influence of suitably renormalized subhorizon ( $k > aH$ ) modes at second order) during slow-roll.

the operator  $\psi$ . The case of the 2-point function, as treated broadly in Section 4.5.1.3 is relatively simple to consider. Expanding  $\psi_k$  in terms of the creation and annihilation operators  $a, a^\dagger$  one can write

$$\psi_k = \omega_{\tilde{k}} a_k + \omega_{\tilde{k}'}^* a_{k'}^\dagger, \quad (4.71)$$

where  $\omega_k$  is equal to ( by analogy to the familiar Minkowski result  $\omega_k \sim \left( \frac{e^{ikx}}{\sqrt{2k}} \right)$ )

$$\omega_k = \sqrt{\frac{1}{a^3}} \sqrt{\frac{a}{2k}} e^{ik/aH} \left( 1 + \frac{iaH}{k} \right)$$

The first term is a normalization term that goes as  $1/\sqrt{V}$  since  $a^3$  corresponds to the volume measure of a comoving observer. This solution to equation (4.42) (with the spatial gradient term restored) is valid to within a couple of Hubble times on either side of the horizon exit, and during this time the variation of  $H$  is negligible. Therefore it makes sense that, *up to some phase factor that varies slowly on the Hubble timescale*, the expression of  $\omega_k$  has this simple form compared to the flat space result.

Defining the 2-point function as

$$\langle \psi^2 \rangle \equiv \langle \int_{\Omega_{k'}} \psi_{(k'-k)} \psi_{k'} d^3 \vec{k}' \rangle$$

we find that

$$\begin{aligned}
 \langle \psi^2 \rangle &= \left\langle \int_{\Omega_{k'}} (a_{k'-k} \omega_{(\vec{k}'-\vec{k})} + a_{k'-k}^\dagger \omega_{(\vec{k}'-\vec{k})}^*) (a_{k'} \omega_{\vec{k}'} + \omega_{\vec{k}'}^* a_{k'}^\dagger) d^3 \vec{k}' \right\rangle \quad (4.72) \\
 &= \left\langle \int_{\Omega_{k'}} a_{k'-k} a_{k'}^\dagger \omega_{(\vec{k}'-\vec{k})} \omega_{\vec{k}'}^* d^3 \vec{k}' \right\rangle \\
 &= \delta(-k) \left\langle \int_{\Omega_{k'}} \omega_{(\vec{k}'-\vec{k})} \omega_{\vec{k}'}^* d^3 \vec{k}' \right\rangle,
 \end{aligned}$$

where  $\Omega_{k'}$  indicates the superHubble range of integration for  $|\vec{k}'| \equiv k'$ , namely  $k' \in [k_{min} + k, aH]$ ,  $k_{min} \ll aH$ . In other words, the quantum average of the 2-point function is only non-trivial for homogeneous contributions, as suggested by the Poincare invariance of the linearized fluctuations. In the following we will assume that the spectrum of modes is discrete by imposing periodic boundary conditions on the spatially flat slicing of the background. In this manner the delta functions that appear in the commutation relations of ladder operators simply become Kronecker deltas and the volume normalizations are implicitly carried by the above definition of  $\omega_k$ .

For the four-point function the situation is different. More specifically, we wish to compute the ‘square of the two-point function’, i.e. we do not technically want the four-point function but rather the *fluctuations* in the operator  $\psi^2$ . Indeed, denoting these fluctuations by  $\langle \psi^4 \rangle$  we define (in thte continuum limit)

$$\langle \psi^4 \rangle \equiv \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \right\rangle,$$

one may verify that

$$\begin{aligned}
 \langle \psi^4 \rangle &= \left( \left\langle \int_{\Omega_{k'}} \psi_{(k'-k)} \psi_{k'} d^3 \vec{k}' \right\rangle \right)^2 \quad (4.73) \\
 &+ \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{(k'-k)} \psi_{(k''-k)} d^3 \vec{k}' d^3 \vec{k}'' \right\rangle \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{k'} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \right\rangle \\
 &+ \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{(k'-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \right\rangle \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{(k''-k)} \psi_{k'} d^3 \vec{k}' d^3 \vec{k}'' \right\rangle,
 \end{aligned}$$

where similarly  $\Omega_{k''}$  indicates the superHubble range of integration for  $k''$ , namely  $k'' \in [k_{min} + k, aH]$ . Assuming that  $k \neq 0$  we find that, using equation (4.72),

$$\begin{aligned}
 \langle \psi^4 \rangle &= \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} a_{k'} a_{k''}^\dagger \omega_{\vec{k}'} \omega_{\vec{k}''}^* d^3 \vec{k}' d^3 \vec{k}'' \right\rangle \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} a_{k'-k} a_{k''-k}^\dagger \omega_{(\vec{k}'-\vec{k})} \omega_{(\vec{k}''-\vec{k})}^* d^3 \vec{k}' d^3 \vec{k}'' \right\rangle \\
 &+ \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} a_{k'-k} a_{k''}^\dagger \omega_{(\vec{k}'-\vec{k})} \omega_{\vec{k}''}^* d^3 \vec{k}' d^3 \vec{k}'' \right\rangle \left\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} a_{k''-k} a_{k'}^\dagger \omega_{(\vec{k}''-\vec{k})} \omega_{\vec{k}'}^* d^3 \vec{k}' d^3 \vec{k}'' \right\rangle,
 \end{aligned}$$

which straightforwardly simplifies to

$$\langle \psi^4 \rangle = \left( \int_{\Omega_{k''}} |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \left[ \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2 4\pi \ln\left(\frac{aH}{k_{min}}\right) + \int_{\Omega_{k'}} |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right] \quad (4.74)$$

using equation (4.60) for  $k > k_{min}$ . It turns out that we shall also require the expressions  $\langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 \rangle$  and  $\langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k''^2 k'^2 \rangle$  in what follows, so, using the above, we find (for  $k \neq 0$ )

$$\begin{aligned} \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 \rangle &= \left( \int_{\Omega_{k'}} k'^2 |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} k''^2 |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} k'^2 |\omega_{\vec{k}'}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} |\omega_{\vec{k}'}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} k''^2 |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \quad (4.75) \end{aligned}$$

and

$$\begin{aligned} \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k''^2 k'^2 \rangle &= \left( \int_{\Omega_{k'}} k'^2 |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} k''^2 |\omega_{\vec{k}''}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} k'^4 |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} |\omega_{\vec{k}''}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} k''^4 |\omega_{\vec{k}''}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} k'^2 |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} k''^2 |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} k'^4 |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \\ &+ \left( \int_{\Omega_{k'}} |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' \right) \left( \int_{\Omega_{k''}} k''^4 |\omega_{(\vec{k}''-\vec{k})}|^2 d^3 \vec{k}'' \right) \quad (4.76) \end{aligned}$$

Once again taking equation (4.60) to give the amplitude of the linearized quantum fluctuations at horizon crossing and assuming this frozen amplitude for  $k', k'' \ll aH$  (up until  $k', k'' = k_{min}$ ,

where we cut it off) we complete the calculation, using

$$\begin{aligned}
 \int_{\Omega_{k'}} |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' &= \int_{\Omega_{k'}} \left[ \int_0^\pi \int_0^{2\pi} \frac{1}{(k^2 + k'^2 - 2kk' \cos(\theta))^{3/2}} d\phi \sin(\theta) d\theta \right] k'^2 dk' \\
 &= \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2 4\pi \left( \frac{1}{2} (\ln(k' - k) + \ln(k' + k)) \right) \Big|_{k_{min}+k}^{aH} \\
 &\approx \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2 4\pi \left( \ln \frac{aH}{k_{min}} - \frac{1}{2} \ln(1 + 2\frac{k}{k_{min}}) \right), \tag{4.77}
 \end{aligned}$$

and similarly

$$\begin{aligned}
 \int_{\Omega_{k'}} k'^2 |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' &= \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2 2\pi \left( k'^2 + k^2 (\ln(k' - k) + \ln(k' + k)) \right) \Big|_{k_{min}+k}^{aH} \tag{4.78} \\
 &\approx \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2 2\pi \left( (aH)^2 + 2k^2 \left( \ln \frac{aH}{k_{min}} - \frac{1}{2} \ln(1 + 2\frac{k}{k_{min}}) \right) \right),
 \end{aligned}$$

along with (remembering again that  $k \ll aH$ )

$$\begin{aligned}
 \int_{\Omega_{k'}} k'^4 |\omega_{(\vec{k}'-\vec{k})}|^2 d^3 \vec{k}' &= \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2 \pi \left( k'^4 + 2k'^2 k^2 + 2k^4 (\ln(k' - k) + \ln(k' + k)) \right) \Big|_{k_{min}+k}^{aH} \\
 &\approx \frac{1}{4} \frac{9\kappa}{\epsilon_{SR}} \left( \frac{H}{2\pi} \right)^2 \pi \left( (aH)^4 + 4k^4 \left( \ln \frac{aH}{k_{min}} - \frac{1}{2} \ln(1 + 2\frac{k}{k_{min}}) \right) \right) \tag{4.79}
 \end{aligned}$$

to obtain finally (using the fact that the  $k', k''$  ranges of integration are identical)

$$\langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} \rangle \approx \eta \left( 2(\ln(\sigma))^2 - \frac{3}{2} \ln(\sigma) \ln(\gamma) + \frac{1}{4} (\ln(\gamma))^2 \right) \tag{4.80}$$

$$\begin{aligned}
 \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 \rangle &\approx \eta (aH)^2 \left( \left[ 1 + 2 \left( \frac{k}{aH} \right)^2 \ln \left( \frac{\sigma}{\sqrt{\gamma}} \right) \right] \ln \left( \frac{\sigma^2}{2\sqrt{\gamma}} \right) \right. \\
 &\quad \left. + \frac{1}{2} \ln \left( \frac{\sigma}{\sqrt{\gamma}} \right) \right) \tag{4.81}
 \end{aligned}$$

$$\begin{aligned}
 \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 k''^2 \rangle &\approx \eta (aH)^4 \left[ \left( \frac{1}{2} + \frac{1}{4} (\ln(\sigma) + 3 \ln \frac{\sigma}{\sqrt{\gamma}}) \right) + \left( \frac{k}{aH} \right)^2 \left( \frac{3}{2} \ln \frac{\sigma}{\sqrt{\gamma}} \right) \right. \\
 &\quad \left. + \left( \frac{k}{aH} \right)^4 \left( 3 \left( \ln \frac{\sigma}{\sqrt{\gamma}} \right)^2 + \ln \sigma \ln \frac{\sigma}{\sqrt{\gamma}} \right) \right] \tag{4.82}
 \end{aligned}$$

In the above, we define the dimensionless factors  $\sigma, \gamma, \eta$  as

$$\sigma \equiv \frac{aH}{k_{min}} \tag{4.83}$$

$$\gamma \equiv 1 + \frac{2k}{k_{min}} \tag{4.84}$$

$$\eta \equiv \left( \frac{9\kappa\pi}{\epsilon_{SR}} \right)^2 \left( \frac{H}{2\pi} \right)^4, \tag{4.85}$$

and once again we assume  $k_{min} \ll aH$ . We retain powers of  $k/aH$  for now simply for generality. The long-wavelength approximation will kill off these terms later on in the calculation.

Finally, one can take the homogeneous limit ( $k \rightarrow 0$ ) of expressions (4.80)-(4.82), bearing in mind that the squares of the two-point functions now contribute as shown by equation (4.72). When we take the homogeneous limit of the above equations we find

$$\langle \psi^4 \rangle_{k=0} \approx 3\eta (\ln \sigma)^2 \quad (4.86)$$

$$\lim_{k \rightarrow 0} \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 \rangle \approx 3\eta (aH)^2 \ln \sigma \quad (4.87)$$

$$\lim_{k \rightarrow 0} \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 k''^2 \rangle \approx \frac{3}{2} \eta (aH)^4 \left( \frac{1}{2} + \ln \sigma \right) \quad (4.88)$$

which provides a coarse but useful check on the algebra to this stage. Now that the explicit fluctuations of  $\langle \psi^2 \rangle$  have been computed we can move on to look at the other terms that occur in  $\langle (\delta^2 \rho_{IR})^2 \rangle$ .

### 4.5.3 Second order equations of motion and correlation functions at second order

In order to compute the full quantity  $\delta^2 \rho(k) \delta^2 \rho^\dagger(k)$  we shall formally encounter two-point functions not only involving  $\psi_k$ , but also those involving the second order fluctuations  $\mathcal{Q}_k$ , e.g.

$$\langle \int_{\Omega_{k'}} \mathcal{L} \mathcal{Q}_{k'-k} \mathcal{L} \mathcal{Q}_{k'} d^3 \vec{k}' \rangle, \quad (4.89)$$

where  $\mathcal{L} \equiv (\partial_0 + H)$ . Of course, by solving the second order equations of motion these sorts of expressions can be reduced to four-point functions involving only  $\psi_k$ . We now show how this reduction is accomplished.

Using equation (4.61) and the fact that  $\dot{H} = -H^2 \frac{\epsilon_{SR}}{18}$ , one can easily show via by-parts integration<sup>19</sup> that

$$\mathcal{L} \mathcal{Q}_k = \int_{\Omega_{k'}}^t \int_{\Omega_{k'}}^{(2)} S(t', k'; k) d^3 \vec{k}' dt' + \frac{1}{18} \int_{\Omega_{k'}}^t \int_{\Omega_{k'}} \epsilon_{SR} H(t')^2 \mathcal{Q}(t', k'; k) dt' d^3 \vec{k}', \quad (4.90)$$

i.e. we compute the first integral of the reduced second order equation of motion. A tedious integration reveals that the leading terms of the latter integral over  $\mathcal{Q}$  are suppressed by a factor of  $\epsilon_{SR}/\alpha$  compared to those of the first term. Therefore we ignore the latter terms and write

$$\mathcal{L} \mathcal{Q}_k \approx \int_{\Omega_{k'}}^t \int_{\Omega_{k'}}^{(2)} S(t', k'; k) d^3 \vec{k}' dt' \quad (4.91)$$

---

<sup>19</sup>In operating on expressions involving  $H, a$ , etc., we always first assume the time dependence of  $a(t) = a_0 t \alpha, H(t) = \alpha/t$  before the operation, and only after the operation take the limit of  $\alpha \gg 1, \alpha/t \rightarrow H \in \mathfrak{R}$ . So, for example,  $\int H dt = \alpha \ln(t) \neq Ht$  before taking the limit  $\alpha \gg 1$ .

which immediately leads us to the expression, again using equation (4.61),

$$\langle \mathcal{LQ}_k \rangle = \langle \int^t \int_{\Omega_{k'}} H(t')^2 \left( 24H(t')t' - 162 \frac{k'^2}{a(t')^2 H(t')^2} \frac{H(t')^4}{\kappa\beta^2} \right) \psi_{(k'-k)} \psi_{k'} d^3 \vec{k}' dt' \rangle \quad (4.92)$$

Equation (4.92) allows us to evaluate the relevant quantities which will appear in the expression for  $\delta^2 \rho(k) \delta^2 \rho(k)$ , such as (4.89) and  $\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{LQ}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle$ . Notice that  $\mathcal{LQ}$  has units inverse seconds, as makes sense since the metric fluctuations are defined as dimensionless. Using our earlier results, we find that

$$\begin{aligned} \langle \int_{\Omega_{k'}} \mathcal{LQ}_{k'-k} \mathcal{LQ}_{k'} d^3 \vec{k}' \rangle &= \langle \int^t H^2 \left( 24Ht' - 162 \frac{k'^2}{a^2 H^2} \frac{H^4}{\kappa\beta^2} \right) \int_{\Omega_{k'}} \psi_{(k'-k)} \psi_{k'} d^3 \vec{k}' dt' \rangle \quad (4.93) \\ &\quad \times \langle \int^t H^2 \left( 24Ht'' - 162 \frac{k''^2}{a^2 H^2} \frac{H^4}{\kappa\beta^2} \right) \int_{\Omega_{k''}} \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}'' dt'' \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{LQ}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle &= \langle \left[ \int^t H^2 \left( 24Ht' - 162 \frac{k'^2}{a^2 H^2} \frac{H^4}{\kappa\beta^2} \right) \right. \\ &\quad \left. \int_{\Omega_{k'}} \psi_{(k'-k)} \psi_{k'} d^3 \vec{k}' dt' \right] \times \int_{\Omega_{k''}} \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}'' \rangle, \quad (4.94) \end{aligned}$$

where the appropriate time dependences of  $a(t) \sim a_0 t^\alpha$ ,  $H(t) \sim \alpha/t$  are assumed above (as properly shown in equation (4.92)). Expanding the above expressions (assuming once again that the ranges of integration for  $k'$ ,  $k''$  are the same, as above), we obtain

$$\begin{aligned} \langle \int_{\Omega_{k'}} \mathcal{LQ}_{k'-k} \mathcal{LQ}_{k'} d^3 \vec{k}' \rangle &= \int^t \int^t H(t')^3 H(t'')^3 (24)^2 t' t'' \langle \psi^4 \rangle dt' dt'' \quad (4.95) \\ &\quad - \int^t \int^t H(t')^4 H(t'')^3 (24t') \left( \frac{162}{a(t')^2 \kappa\beta^2} \right) \langle \psi^4 k'^2 \rangle dt' dt'' \\ &\quad - \int^t \int^t H(t'')^4 H(t')^3 (24t'') \left( \frac{162}{a(t'')^2 \kappa\beta^2} \right) \langle \psi^4 k''^2 \rangle dt' dt'' \\ &\quad + \int^t \int^t (162)^2 H(t')^4 H(t'')^4 \left( \frac{1}{a(t')^2 a(t'')^2 \kappa^2 \beta^4} \right) \langle \psi^4 k'^2 k''^2 \rangle dt' dt'' \end{aligned}$$

and similarly

$$\begin{aligned} \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{LQ}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle &= 24 \int^t H^2(Ht') \langle \psi^4 \rangle dt' \quad (4.96) \\ &\quad - 162 \int^t H^2 \frac{1}{a^2 H^2} \frac{H^4}{\kappa\beta^2} \langle \psi^4 k'^2 \rangle dt', \end{aligned}$$

where we define  $\langle \psi^4 \rangle \equiv \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} \rangle$ ,  $\langle \psi^4 k'^2 \rangle \equiv \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 \rangle$  and  $\langle \psi^4 k''^2 k'^2 \rangle \equiv \langle \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} k'^2 k''^2 \rangle$  all integrated over  $k'$ ,  $k''$ , as in the above sections.

Inserting expressions (4.80) through (4.82) (and using (4.77) and (4.79) to perform the temporal product integrations in (4.95) [which amount to symmetrization in  $t'$ ,  $t''$ ]) into (4.95) and (4.96) we obtain (for  $k \neq 0$ )

$$\begin{aligned}
 & \langle \int_{\Omega_{k'}} \mathcal{L} \mathcal{Q}_{k'-k} \mathcal{L} \mathcal{Q}_{k'} d^3 \vec{k}' \rangle = \int^t \int^t H(t')^3 H(t'')^3 (24)^2 t' t'' \bar{\eta} \left( 2(\ln(\bar{\sigma}))^2 - \frac{3}{2} \ln(\bar{\sigma}) \ln(\gamma) + \frac{1}{4} (\ln(\gamma))^2 \right) dt' dt'' \\
 & - \int^t \int^t H(t')^4 H(t'')^3 (24t') \left( \frac{162}{a(t')^2 \kappa \beta^2} \right) \bar{\eta}(\bar{\alpha} \bar{H})^2 \left( \left[ 1 + 2 \left( \frac{k}{\bar{\alpha} \bar{H}} \right)^2 \ln \left( \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right] \ln \left( \frac{\bar{\sigma}^2}{2\sqrt{\gamma}} \right) + \frac{1}{2} \ln \left( \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right) dt' dt'' \\
 & - \int^t \int^t H(t'')^4 H(t')^3 (24t'') \left( \frac{162}{a(t'')^2 \kappa \beta^2} \right) \bar{\eta}(\bar{\alpha} \bar{H})^2 \left( \left[ 1 + 2 \left( \frac{k}{\bar{\alpha} \bar{H}} \right)^2 \ln \left( \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right] \ln \left( \frac{\bar{\sigma}^2}{2\sqrt{\gamma}} \right) + \frac{1}{2} \ln \left( \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right) dt' dt'' \\
 & + \int^t \int^t (162)^2 H(t')^4 H(t'')^4 \left( \frac{1}{a(t')^2 a(t'')^2 \kappa^2 \beta^4} \right) \bar{\eta}(\bar{\alpha} \bar{H})^4 \left[ \left( \frac{1}{2} + \frac{1}{4} (\ln(\bar{\sigma}) + 3 \ln \frac{\bar{\sigma}}{\sqrt{\gamma}}) \right) + \left( \frac{k}{\bar{\alpha} \bar{H}} \right)^2 \left( \frac{3}{2} \ln \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right. \\
 & \quad \left. + \left( \frac{k}{\bar{\alpha} \bar{H}} \right)^4 \left( 3 \left( \ln \frac{\bar{\sigma}}{\sqrt{\gamma}} \right)^2 + \ln \bar{\sigma} \ln \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right] dt' dt'' \tag{4.97}
 \end{aligned}$$

and

$$\begin{aligned}
 & \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{L} \mathcal{Q}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle = 24 \int^t H^2(H t') \eta \left( 2(\ln(\sigma))^2 - \frac{3}{2} \ln(\sigma) \ln(\gamma) + \frac{1}{4} (\ln(\gamma))^2 \right) dt' \\
 & - 162 \int^t H^2 \frac{1}{a^2 H^2} \frac{H^4}{\kappa \beta^2} \eta(\alpha H)^2 \left( \left[ 1 + 2 \left( \frac{k}{\alpha H} \right)^2 \ln \left( \frac{\sigma}{\sqrt{\gamma}} \right) \right] \ln \left( \frac{\sigma^2}{2\sqrt{\gamma}} \right) + \frac{1}{2} \ln \left( \frac{\sigma}{\sqrt{\gamma}} \right) \right) dt', \tag{4.98}
 \end{aligned}$$

where we remind the reader that  $\sigma, \gamma, \eta$  are defined in definitions (4.83)-(4.85) and the  $\sim$  symbol denotes symmetrization in  $t', t''$ . One can once again verify that the dimensions of all of the terms in equations (4.97) and (4.98) are inverse seconds squared and inverse seconds respectively, as required.

Once again we can immediately take the homogeneous limit directly from the above equations (again remembering to add in the contributions from the squares of the averages of the two point functions, as before), or by using the relations (4.86)-(4.88) directly in (4.97) and (4.98). For example, the result for  $\lim_{k \rightarrow 0} \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{L} \mathcal{Q}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle$  is, putting in the explicit time dependence of the background,

$$\begin{aligned}
 \lim_{k \rightarrow 0} \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{L} \mathcal{Q}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle & = \left[ 3(24) \frac{(9\kappa\pi)^2}{(2\pi)^4} \frac{\alpha^{15}}{\kappa^2 \beta^4} \int^t \frac{dt'}{t'^{14}} \left( \ln \frac{a_0 t'^{\alpha} \alpha}{t' k_{min}} \right)^2 \right. \\
 & \left. - 3(162) \frac{(9\kappa\pi)^2}{(2\pi)^4} \frac{\alpha^{18}}{\kappa \beta^2 \kappa^2 \beta^4} \int^t \frac{dt'}{t'^{18}} \ln \frac{a_0 t'^{\alpha} \alpha}{t' k_{min}} \right] \tag{4.99}
 \end{aligned}$$

and similarly for  $\lim_{k \rightarrow 0} \langle \int_{\Omega_{k'}} \mathcal{L} \mathcal{Q}_{k'-k} \mathcal{L} \mathcal{Q}_{k'} d^3 \vec{k}' \rangle$ . The key issue is now one of simplifying (4.97)-(4.98) to extract their dominant parts (those 'most enhanced' by factors of the slow-roll parameter). It should be noted that this is complicated by the fact that the dependence on the slow-roll parameter is not just carried through factors of  $V' \sim \beta$ , but also through  $\alpha$ , which in the Section 4.3 we defined to be proportional to  $1/\sqrt{\epsilon_{SR}}$ . Therefore we must, to be safe, carry out the time integrals *before* we make the approximation that  $\alpha \gg 1, \epsilon_{SR} \ll 1, \alpha/t \rightarrow H \in \mathfrak{R}$  especially since, just as differentiation in comoving time of the scale factor generally introduces factors of the slow-roll parameter in the numerator of a given expression, integration introduces factors in the denominator. This is the reason it is not immediately obvious the second term in equation (4.90) is subdominant to the first term.



We begin in the simpler case of the homogeneous limit. In that case, one can for example show that the dominant terms for the following integrals are (for  $\alpha \gg 1$ ):

$$\int \frac{dt'}{t'^{14}} \left( \ln \frac{a_0 t'^{\alpha} \alpha}{t' k_{min}} \right)^2 \approx -\frac{2\alpha^2}{2197t^{13}} - \frac{2\alpha}{2197t^{13}} (13 \ln \sigma - 2) \quad (4.100)$$

$$\int \frac{dt'}{t'^{18}} \left( \ln \frac{a_0 t'^{\alpha} \alpha}{t' k_{min}} \right) \approx -\frac{\alpha}{289t^{17}} + \frac{1}{289t^{17}} (-17 \ln \sigma + 1) \quad (4.101)$$

and similarly for the other integrals required. Using these sorts of integrals in equations (B11) for example, one can show that

$$\lim_{k \rightarrow 0} \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{LQ}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle = -3(162) \frac{(9\kappa\pi)^2}{(2\pi)^4} \frac{H^5}{289\epsilon_{SR}^3} [-\alpha^2 + \alpha(-17 \ln(\sigma))], \quad (4.102)$$

and similarly for  $\lim_{k \rightarrow 0} \langle \int_{\Omega_{k'}} \mathcal{LQ}_{k'-k} \mathcal{LQ}_{k'} d^3 \vec{k}' \rangle$ .

For the full inhomogeneous problem we can make excellent use of the long-wavelength approximation to rewrite equations (4.97) and (4.98) as

$$\begin{aligned} \langle \int_{\Omega_{k'}} \mathcal{LQ}_{k'-k} \mathcal{LQ}_{k'} d^3 \vec{k}' \rangle &= \int^t \int^t H(t')^3 H(t'')^3 (24)^2 t' t'' \bar{\eta} \left( 2(\ln(\bar{\sigma}))^2 - \frac{3}{2} \ln(\bar{\sigma}) \ln(\gamma) + \frac{1}{4} (\ln(\gamma))^2 \right) dt' dt'' \\ &- \int^t \int^t H(t')^4 H(t'')^3 (24t') \left( \frac{162}{a(t')^2 \kappa \beta^2} \right) \bar{\eta}(a\bar{H})^2 \left( \ln \left( \frac{\bar{\sigma}^2}{2\sqrt{\gamma}} \right) + \frac{1}{2} \ln \left( \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right) dt' dt'' \\ &- \int^t \int^t H(t'')^4 H(t')^3 (24t'') \left( \frac{162}{a(t'')^2 \kappa \beta^2} \right) \bar{\eta}(a\bar{H})^2 \left( \ln \left( \frac{\bar{\sigma}^2}{2\sqrt{\gamma}} \right) + \frac{1}{2} \ln \left( \frac{\bar{\sigma}}{\sqrt{\gamma}} \right) \right) dt' dt'' \\ &+ \int^t \int^t (162)^2 H(t')^4 H(t'')^4 \left( \frac{1}{a(t')^2 a(t'')^2 \kappa^2 \beta^4} \right) \bar{\eta}(a\bar{H})^4 \left[ \left( \frac{1}{2} + \frac{1}{4} (\ln(\bar{\sigma}) + 3 \ln \frac{\bar{\sigma}}{\sqrt{\gamma}}) \right) \right] dt' dt'' \end{aligned} \quad (4.103)$$

and

$$\begin{aligned} \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{LQ}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle &\approx 24 \int^t H^2(Ht') \eta \left( 2(\ln(\sigma))^2 - \frac{3}{2} \ln(\sigma) \ln(\gamma) + \frac{1}{4} (\ln(\gamma))^2 \right) dt' \\ &- 162 \int^t H^2 \frac{H^4}{\kappa \beta^2} \eta \left( \ln \left( \frac{\sigma^2}{2\sqrt{\gamma}} \right) + \frac{1}{2} \ln \left( \frac{\sigma}{\sqrt{\gamma}} \right) \right) dt' \end{aligned} \quad (4.104)$$

It turns out the dominant terms in these integrals are, for  $\alpha \gg 1$  and  $\epsilon_{SR} \ll 1$ ,

$$\langle \int_{\Omega_{k'}} \mathcal{LQ}_{k'-k} \mathcal{LQ}_{k'} d^3 \vec{k}' \rangle \approx -\frac{531441\kappa^2 H^6}{360448\pi^2 \epsilon_{SR}^4} [192\alpha + 7040 \ln(\sigma) - 2112 \ln(\gamma)] \quad (4.105)$$

$$\langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \mathcal{LQ}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3 \vec{k}' d^3 \vec{k}'' \rangle \approx \frac{\kappa^2 H^5}{4624\pi^2 \epsilon_{SR}^3} [32805\alpha^2 - 6561\alpha(-85 \ln(\sigma) + 51 \ln(\gamma))], \quad (4.106)$$

where we note that  $k$  dependence comes in solely from the  $\ln(\gamma)$  terms and that the dimensions of equations (4.105) and (4.106) are respectively inverse seconds squared and inverse seconds, as required.

We are now finally in a position to collect all of these results, namely, equations (4.105)-(4.106) and (4.73) (and (4.80)-(4.82), (4.90)), to fully evaluate the quantity  $\sqrt{\langle \left( \frac{\delta^2 \rho_{IR}}{\bar{\rho}} \right)^2 \rangle}$  at some scale  $\tilde{k}$

such that  $k_{min} < \tilde{k} \ll aH$  (and also at  $\tilde{k} = 0$ ). The goal, once again, is to compare the magnitude of this term to the horizon-scale amplitude of the linearized fluctuations  $\sqrt{\langle \left(\frac{\delta\rho}{\bar{p}}\right)^2 \Big|_{k=aH} \rangle}$ .

Using all of these above results, one may write the averaged square of the dominant contributions to the second order IR pressure contribution (or energy density, etc.) as

$$\begin{aligned} \langle \delta^2 p_{IR}(k) \delta^2 p_{IR}^\dagger(k) \rangle &\approx \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \left(\frac{54H^2}{\kappa\epsilon_{SR}}\right)^2 \psi_{(k'-k)} \psi_{k'} \psi_{(k''-k)} \psi_{k''} d^3\vec{k}' d^3\vec{k}'' \rangle \\ &+ \langle \int_{\Omega_{k'}} \left(\frac{3H}{\kappa}\right)^2 \mathcal{L}\mathcal{Q}_{k'-k} \mathcal{L}\mathcal{Q}_{k'} d^3\vec{k}' \rangle \\ &+ \langle \int_{\Omega_{k'}} \int_{\Omega_{k''}} \left(\frac{6H}{\kappa} \frac{54H^2}{\kappa\epsilon_{SR}}\right) \mathcal{L}\mathcal{Q}(t', k'; k) \psi_{(k''-k)} \psi_{k''} d^3\vec{k}' d^3\vec{k}'' \rangle \end{aligned} \quad (4.107)$$

Putting in the results from all of the above appendices we find that the dominant terms are of the form

$$\langle \delta^2 p_{IR}(k) \delta^2 p_{IR}^\dagger(k) \rangle \approx \left(\frac{H^2}{\kappa}\right)^2 \frac{\kappa^2 H^4}{\epsilon_{SR}^4 \pi^2} (A_1 \alpha^2 + \alpha (B_1 \ln(\gamma) + C_1 \ln(\sigma))), \quad (4.108)$$

where

$$A_1 \equiv \frac{2657205}{1156\pi^2} \quad (4.109)$$

$$B_1 \equiv \frac{-1594323}{68\pi^2} \quad (4.110)$$

$$C_1 \equiv \frac{2657205}{68\pi^2} \quad (4.111)$$

It is important to note that the details of this calculation confirm that the naive guess ventured in the Section 4.3 holds: there are solutions of the diffeomorphism constraints of general relativity, at second order in perturbation theory about a slowly rolling background, which introduce a factor of  $1/\epsilon_{SR}$  into any expression of  $\delta^2\rho_{IR}$  and  $\delta^2 p_{IR}$ , and these factors survive through quantum averaging. A second order gauge transformation cannot eliminate the presence of such slow-roll enhanced terms.

Thus, finally, comparing the amplitudes of the second order fluctuations at  $k_{min} < k = \tilde{k} \ll aH$  to the amplitudes linearized fluctuations at the horizon scale,  $k = aH$ , we see that

$$\sqrt{\langle \left(\frac{\delta p}{\bar{p}}\right)^2 \Big|_{k=aH} \rangle} > \sqrt{\langle \left(\frac{\delta^2 p_{IR}}{\bar{p}}\right)^2 \Big|_{k=\tilde{k}} \rangle}$$

is equivalent to demanding that

$$\frac{\tilde{A}_2 \sqrt{\kappa} H}{\sqrt{\epsilon_{SR} \pi}} > \frac{\kappa H^2}{\epsilon_{SR}^2 \pi^2} \sqrt{A_1 \frac{N}{\epsilon_{SR}} + \sqrt{\frac{N}{\epsilon_{SR}}} (B_1 \ln(\gamma) + C_1 \ln(\sigma))}, \quad (4.112)$$

where  $\alpha^2 = N/\epsilon_{SR}$ ,  $\tilde{A}_2 = \frac{9}{16}$ . We can see that the spatial dependence is subdominant in the sense that it is multiplied by a lower power of  $\alpha$ : in other words, the homogeneous fluctuations (mean) are not, in the slow-roll limit, altered by the finite corrections which occur when one evaluated at the finite spatial scale of  $k = \tilde{k} \ll aH$ . We also note that (4.112) is positive definite, as it should be, since  $k \ll aH$ . Thus, we find that

$$\epsilon_{SR} > \frac{2}{3}(\kappa H^2)^{\frac{1}{4}}(A_1 N)^{\frac{1}{4}} \quad (4.113)$$

is the *consistency condition* for linearized theory at second order. Given that  $N \sim 70$ ,  $A_1 \sim 200$ , (4.113) implies e.g. that for  $\kappa H^2 \sim 1 \leftrightarrow H^2 \sim m_{plank}^2$  the slow-roll condition **must** be violated. We see that the amplitude of the second order fluctuations in our quantity  $\delta^2 \rho_{IR}$  dominates the corresponding linearized amplitude if the background spacetime is rolling slowly enough and  $\kappa H^2$  is large enough, as it is in many models. Note also the appearance of the number of e-foldings  $N$ , which indicates, as previously shown, that this effect is a *cumulative* effect which depends on the growth of the phase space of superhorizon modes. It is worth emphasizing that bound (4.113) provides a *quantitatively stronger* restriction than a similar bound given by Abramo, Mukhanov, and Brandenberger in equations (106)-(108) of [47]. This is so because

- bound (4.113) is already strongly saturated (violates the slow-roll condition) at the backreaction scales given by [47];
- bound (4.113) is stronger than what would equivalently be found in the effective approach used in [47] because here one evaluates the relative amplitudes of the first and second order fluctuations at an *inhomogeneous* scale which is *well before* any IR cutoff. Since the effective method cannot address backreactions on inhomogeneous modes, as discussed in Section 4.3, it is perhaps not surprising that the bounds obtained are different.

The violation of this inequality (4.113) in any models of inflation casts doubt on the viability of the linearized approximation to those slowly rolling spacetimes, simply because nonlinear effects dominate any linear effects. However, as I imply in the caption to Figure 1 above, this calculation would ultimately contain formally divergent subhorizon contributions to  $\delta^2 \rho$ ,  $\delta^2 p$  and it is this real physical effect of the coupling between subhorizon (UV) modes with superHubble (IR) modes that will reveal the observable importance of backreaction effects for local observers.

## 4.6 Summary and Conclusion

To summarize, I study the perturbations of the inflationary slow-roll spacetime which are at second order in the metric and matter fluctuations. I follow a procedure of consistently (though probably not convergently) expanding the Einstein equations to second order and solving them assuming the zeroth and linear order equations hold. In order to isolate the physical degrees of freedom in the second order fluctuations, a longitudinal gauge-fixing procedure is used at second order. Namely, I specify *two* independent infinitesimal, inhomogeneous, coordinate transformations (gauges), one at linear order and one at second order, which admit no residual coordinate freedoms. Within this

coordinate system we evaluate the fluctuations of two independent background scalars formed from the stress energy and from them define the fluctuations, to second order, of the isotropic energy density  $\delta^2\rho$  and pressure  $\delta^2p$ . These fluctuations will not only arise from second order scalar modes but also from quadratic combinations of scalar-scalar and tensor-tensor modes at second order. Whereas the nondecaying linear scalar and tensor modes are constants at these scales (the vector modes die away), the nondecaying second order modes are time dependent and this leads to time-dependent  $\delta^2\rho$  and  $\delta^2p$  which depend sensitively on the initial conditions of the second-order modes. Furthermore, given that we have three effective small parameters in this problem (a slow-roll parameter, the strength of the metric and matter fluctuations, and the longwavelength approximation ( $H^2/\lambda^2 \ll 1$ ), we find that our  $\delta^2\rho$  and  $\delta^2p$  depend sensitively on the hierarchy of small parameters one assumes in the sense that ambiguous terms like  $(k^2/H^2a^2)(H^4/\kappa\beta^2)$  appear in the mode expansion of  $\delta^2\rho$ ,  $\delta^2p$ . For the (incomplete) case of just scalar-scalar backreactions and no genuinely second order metric or matter fluctuations, we find that  $\delta^2\rho < 0$  but that  $\delta^2p + \delta^2\rho \neq 0$  in general. We find that the second order contributions to the energy density and pressure can, with the assumption of slow-roll, dominate over the second order linear contributions to the energy density and isotropic pressure given a broad range of initial conditions.

I conclude that when one truly goes to second order and solves the Einstein equations for the higher order classical fluctuations, they do approximately lead to a cosmological constant type of contribution in this gauge. Furthermore, it seems that these higher order corrections dominate the linear terms if slow-roll holds in the background, suggesting the breakdown of the linearized approximation. Some previous calculations of higher order superhorizon effects (of which we are aware) have used a procedure which effectively takes the expectation value of the gauge-fixed metric *before* forming some sort of ‘invariant’ measure of the expansion with which to probe ‘local’ backreaction, i.e. gauge fixing before taking the expectation value. Such a procedure suffers from higher order gauge ambiguities, and at least for a model with massless, minimally coupled scalar with quartic self-interaction (no gravity) one can first form a desired ‘invariant’ and then take expectation values and gauge fix this result (see for example [35], and also [32]). Further investigations along this line will almost certainly prove useful, and one pay-off seems to be new ideas for observables in backreactions, such as recently described in [48]. In any event, perhaps some use can be made of the point of view advanced in this thesis, which is that the *eigenvalues* of the mixed valence total stress energy of to second order in IR fluctuations can give useful insights into backreactions without the additional complication of disentangling the physics from the gauge properties of the *tensor*  $\langle \delta^2 T_{\mu\nu} \rangle$ .

I would also like to note an ancillary result which is a consequence of the above work. Namely, in second order in perturbation theory about pure de-Sitter (no-roll), the scalar sector is *nontrivial*. This fact can be of possible use in the literature, where it is sometimes a confusing issue. This fact can also be seen by e.g. examining the second order gauge fixing (4.48)-(4.49) in the main paper or the reduced equation of motion for  $\mathcal{Q}$ , equation (4.61), which shows that the TT sector will mix with and source a nontrivial scalar sector at second order when  $\epsilon_{SR} \rightarrow 0$ . It is perhaps worth emphasizing that the extra metric functions one can fix in flat, vacuum, spacetime and special spacetimes like de Sitter are generic, but obey field equations. Though they correspond to residual

degrees of freedom specified only on an initial value surface, they are completely determined by equations of motion they *also* satisfy. Therefore any residual gauge fixing makes crucial use of the equations of motion to set additional terms to zero via a gauge transformation. This is implicitly shown in the formal gauge-fixing considerations of Chapter 5.

Finally, as indicated in Section 4.3, the procedure used here does not say anything about what an observer would measure as the averaged cosmological constant in his own observable patch of the universe. In other words, the suitably renormalized subhorizon contributions to  $\delta^2 p$  and  $\delta^2 \rho$  will allow a probe of the possibly crucial physics of the coupling of subhorizon and superhorizon modes in inflation and possibly shed some light on what we even mean by local modifications to a cosmological constant. Certainly the proof of the appearance of nonanalytic terms such as  $\frac{N}{\epsilon_{SR}}$  in the second order expansions of perturbation theory in slow-roll spacetimes is only a small step in this direction. The reader is left to judge the significance of the suggestive calculations in this Chapter in this light.

## CHAPTER 5

# BACKREACTIONS IN DE SITTER

### 5.1 Introduction

The final project of this thesis deals with backreactions in de Sitter space. In Section 5.2 I set the context of the calculation up by briefly extending the comments below equation (4.2) to discuss some of the properties of deSitter spacetimes, following closely the excellent discussion in [49]. This is done first in a geometric way, by reviewing the symmetries and briefly indicating why I pick a particular slicing, and then by examining the equations of motion for a massive, free particle in de Sitter. In Section 5.3 I motivate the backreaction problem in pure de Sitter and contrast it with the same problem in inflationary slow-roll spacetimes as discussed in Chapter 4. The equations of motion are derived for the backreactions, which are expressed in terms of various approximations which are also defined. Also, I go through the procedure of classical gauge-fixing and show that there are no quantum anomalies which arise in the quantum case (heavily relying on recent results of Hollands and Wald in ([50], [51])). In Section 5.4 the quantum analogues of the LS conditions considered in Chapters 2 and 3 are formulated for this case, and are proven to have no quantum anomalies. In Section 5.5 I discuss the constraints that the LS conditions impose on the physical states (namely, de Sitter invariance) of the quantized gravitational perturbations and outline ways in which to satisfy them. Section 5.6 summarizes the Chapter. The main results are the imposition of strict de Sitter invariance on *all* the physical states (not just the vacuum) and a sketch of how to construct de Sitter invariant states.

### 5.2 de Sitter space

We know from elementary particle physics that physical processes are invariant to great accuracy under the Poincare group (the Lorentz group of Special Relativity plus spatial translations). However, one of the key ideas that apparently led Einstein to general relativity also asserts that the very existence of local inertial frames which are related by Lorentz transformations is due to the large scale distribution of matter in the universe. This idea is known as *Mach's Principle*, and is in many ways encapsulated by Einstein's field equations (2.8). One cosmological consequence of Mach's principle is that it prevents one from imagining the universe rotating as a whole, since it has nothing to rotate with respect to<sup>1</sup>. Large scale observations suggest strongly that the matter distribution is isotropic on cosmological scales, and application of Mach's holistic principle suggests that local Poincare invariance is a logical result. However, we also know from recent observations that the universe is not globally flat and is in fact expanding. If we also assume uniformity in time

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<sup>1</sup>Whereas in Newtonian physics one could imagine some global rotation with respect to a rigid axis.

(in addition to spatial isotropy), then it turns out that Einstein's equations (2.8) have two classes of solutions discovered by and named after the Dutch astronomer Wilhelm de Sitter. In both classes the de Sitter solutions describe an expanding universe where the worldlines of celestial objects are geodesics whose radial velocity is exactly proportional to their radial distance away from any point in the space, and as intimated below equation (4.2) the energy density of the matter and space does not change *at all* with this expansion. In the language of Chapter 4 (specifically, equations (4.1) and (4.2)) this implies that if we think of the matter sourcing this expansion as a scalar field, then it has no gradients and can only have a constant potential, or in other words the only possible stress energy is proportional to the metric times a (cosmological) constant  $\Lambda$ . In this Chapter I will only study the case with a positive  $\Lambda$ , which is just called the de Sitter spacetime. The case with negative  $\Lambda$  is called the anti de Sitter spacetime.

The de Sitter spacetimes are invariant under a ten-parameter group called the de Sitter group, often labelled  $SO(4,1)$  in the de Sitter case and  $SO(3,2)$  in the anti de Sitter case. In the limit of zero curvature  $SO(4,1)$  reduces (or more properly, 'contracts') to the Poincare group, just as the Galilei group is the contraction of the Poincare group in the limit of an infinite speed of light. The replacement of the Poincare group in de Sitter spacetimes means e.g. that the regular translation group is not valid in de Sitter, implying a loss of the usual laws of momentum and energy conservation. These laws will be replaced by similar laws corresponding to the notion of translation in de Sitter, just as the finiteness of the speed of light introduces Lorentz contraction/dilation factors in the usual laws of Newton. Furthermore, the centre of the de Sitter group (its so-called Casimir invariants) will provide additional important information on what it precisely means to be invariant under the de Sitter group. Generically we may intuit that the new laws will mix time and space in some manner, and this expectation is borne out below. We start by using the fact that the de Sitter solutions are maximally symmetric, which technically means they are conformally flat (i.e. conformal to Minkowski spacetime). Maximal symmetry can be shown in much the same way the spatially isotropic spacetimes were investigated in Section 2.1, the principal difference being that the *full, four dimensional* Riemann curvature tensor  $R_{abcd}$  will now be proportional to the quantity  $g_{c[a}g_{b]d}$  since temporal isotropy is also required. The conformal part of this quantity, which is by definition the traceless part, is zero by simple simultaneous contraction of the first and third and second and fourth indices. In fact, the condition of conformal flatness and maximal symmetry is equivalent to demanding  $R_{ab} - \frac{1}{4}Rg_{ab} = 0$ , which means the Riemann tensor is determined by the Ricci *scalar* ( $R = 4\Lambda$ ) alone. In that sense one may regard the stress energy content of the maximally symmetric spacetimes as that of a perfect fluid with equation of state  $p = -R/4\kappa = -\rho$ .

The metrics of maximally symmetric, conformally flat, spacetimes can be cast by definition into the form

$$ds^2 = \Phi^2 \eta_{\mu\nu} dx^\mu dx^\nu, \quad (5.1)$$

where  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  is the Minkowski metric in global inertial coordinates and  $\Phi$  is the so-called conformal factor. One can further define two Lorentz invariants:  $d\tau^2 = -dx^\mu dx_\mu$  and

$\sigma^2 \equiv x^\mu x_\mu$ , in terms of which the conformal factor may be written conveniently as

$$\Phi(\sigma^2) = \left(1 + \frac{\sigma^2}{4R^2}\right)^{-1}, \quad (5.2)$$

$R$  being the radius of curvature of the de Sitter spacetime (dS) (so that  $R \rightarrow \infty$  implies flatness, i.e.  $\Phi \rightarrow 1$ ). Now, owing to maximal symmetry, dS as defined by the metric (5.1) can be completely embedded in a *flat* five dimensional Minkowskian space with coordinates  $\zeta_i$  such that

$$\zeta_a \zeta^a = -\zeta_0 \zeta^0 + \zeta_1 \zeta^1 + \zeta_2 \zeta^2 + \zeta_3 \zeta^3 + \zeta_4 \zeta^4 = R^2 \equiv \frac{3}{\Lambda}, \quad (5.3)$$

where latin indices range from 0 to 4. We can find out how  $x^\mu$  and  $\zeta^a$  are related by demanding equation (5.3) hold. Indeed, supposing we set (using equation (5.2))

$$\zeta_\mu = \Phi x_\mu \quad (5.4)$$

$$\zeta_4 = B, \quad B \in \mathfrak{R}, \quad (5.5)$$

then we arrive at the condition that (using  $g^{44} = +1$ )

$$B^2 = R^2 \Phi^2 \left(1 - \frac{\sigma^2}{4R^2}\right)^2, \quad (5.6)$$

meaning one can write equations (5.4) and (5.5) as

$$\zeta_\mu = \Phi x_\mu \quad (5.7)$$

$$\zeta_4 = \pm R \Phi \left(1 - \frac{\sigma^2}{4R^2}\right) \quad (5.8)$$

One may invert these relations to find

$$x_\mu = \frac{2\zeta_\mu}{1 + \frac{\zeta_4}{R}} \quad (5.9)$$

$$\frac{\sigma^2}{4R^2} = \frac{1 - \frac{\zeta_4}{R}}{1 + \frac{\zeta_4}{R}}, \quad (5.10)$$

which simply represent the four dimensional coordinates  $x^\mu$  as stereographic projections from the five dimensional embedding surface onto the de Sitter space. Selecting the  $\zeta_4$  coordinate to represent a spacelike dimension and insisting that the ratio  $\frac{\zeta_4}{R}$  is real specializes to the case of postive  $\Lambda$ , which is what I exclusively discuss in this thesis.

The generators of the de Sitter group are thus the generators of ‘rotations’ (the Lorentz group) of the five dimensional embedding surface. One may write these rotations as orthogonal, five dimensional, matrices  $R_b^a$  such that

$$\zeta'^a = R_b^a \zeta^b, \quad (5.11)$$



i.e.  $R_b^a R_a^c = \delta_b^c$  where the  $\delta$  function is a five dimensional Kronecker identity matrix. This implies  $\zeta'^a \zeta'_a = \zeta^a \zeta_a$ , and furthermore the six dimensional subgroup under which

$$\zeta'^\mu \zeta'_\mu = \zeta^\mu \zeta_\mu \quad (5.12)$$

also induces the Lorentz transformations under which  $\sigma^2 = \sigma'^2$ , where Greek indices run over four dimensions. However, the key point here is that the remaining four transformations, which mix the spacelike  $\zeta_4$  and the one timelike and three spacelike  $\zeta_\mu$  induce *nonlinear* ‘boost’ transformations of the stereographic coordinates  $x^\mu$ . These turn into four simple translations in the limit of  $R \rightarrow \infty$ , and in this way the de Sitter group ‘contracts’ to the Poincare group in the limit of zero curvature.

This subtle point is best seen by calculating the limit of zero curvature for the so-called ‘centre’ of the de Sitter group, i.e. the quantities which are invariant under de Sitter transformations. To this end it is useful to try to find an explicitly de Sitter invariant quantity which reduces to the standard interval between two points  $x$  and  $x'$  of the Poincare group. Namely, we wish to generalize

$$z(x, x') \equiv (x_\mu - x'_\mu)(x^\mu - x'^\mu) \quad (5.13)$$

for the case of the de Sitter group. It is not hard to guess by the above comments that

$$\begin{aligned} Z(x, x') &\equiv (\zeta_a - \zeta'_a)(\zeta^a - \zeta'^a) \\ &= \frac{3}{\Lambda} + \frac{3}{\Lambda} - 2\zeta'^a \zeta_a \end{aligned} \quad (5.14)$$

is a good candidate, where in the second line I have used equation (5.3) to deduce that  $Z$  is only a function of the cross-term  $\zeta'^a \zeta_a$ . Using equation (5.7), it is then easy to see that

$$Z(x, x') = \Phi(\sigma^2)\Phi(\sigma'^2)z(x, x') = \frac{z(x, x')}{(1 + \frac{\sigma^2}{4R^2})(1 + \frac{\sigma'^2}{4R^2})} \quad (5.15)$$

is the de Sitter invariant we desire. An important notion throughout this chapter will be one of whether or not a given function is de Sitter invariant, and using the above I will call *any quantity which is a function of  $Z$  alone de Sitter invariant*.

### 5.2.1 Classical dynamics in de Sitter spacetime

To get a better handle on the key differences between de Sitter spacetimes and flat spacetimes, consider the dynamics of a massive test particle in de Sitter. The classical equations of motion are derived from extremizing the proper time along the worldline of the particle, i.e.

$$\delta \int ds = 0 = \delta \int \Phi d\tau \quad (5.16)$$

which will be extremized if the Euler-Lagrange equations

$$\frac{d}{d\tau} \left( \Phi \frac{dx_\mu}{d\tau} \right) = \partial_\mu \Phi \quad (5.17)$$

hold. Letting  $p_\mu \equiv m\Phi \frac{dx_\mu}{d\tau}$ , where  $m$  is the mass of the particle, we may rewrite equations of motion (5.17) as

$$p_{\mu,\tau} = -\frac{m\Phi^2 x_\mu}{2R^2} \quad (5.18)$$

Now, as I indicated in the Introduction to this Chapter it is reasonable to intuit that space and time will be unified in a certain sense that generalizes the concept in flat spacetime that a free particle has a constant linear and angular momentum. In de Sitter this statement translates to a free particle having a constant ‘angular momentum 10-vector‘ [49]. By analogy with flat spacetime, one may define this angular momentum to be

$$\ell_{ab} \equiv m(\zeta_a \zeta_{b,s} - \zeta_b \zeta_{a,s}), \quad (5.19)$$

and further let

$$\pi_\mu = \frac{\ell_{4\mu}}{R} \quad (5.20)$$

Using equations (5.7) and (5.8) one can show (recalling again that  $p_\mu \equiv m\Phi \frac{dx_\mu}{d\tau}$ )

$$\begin{aligned} \ell_{\mu\nu} &= m \left[ \Phi x_\mu \left( \Phi \frac{d}{\Phi d\tau} \right) x_\nu - \Phi x_\nu \left( \Phi \frac{d}{\Phi d\tau} \right) x_\mu \right] \\ &= x_\mu p_\nu - x_\nu p_\mu \end{aligned} \quad (5.21)$$

and, in complete detail,

$$\begin{aligned} \pi_\mu &= \frac{m}{R} \left[ \left( R\Phi \left( 1 - \frac{\sigma^2}{4R^2} \right) \right) \left( \Phi \frac{d}{\Phi d\tau} \right) x_\mu - \Phi x_\mu \left( \Phi \frac{d}{\Phi d\tau} \right) \left( R\Phi \left( 1 - \frac{\sigma^2}{4R^2} \right) \right) \right] \\ &= \left( \Phi \left( 1 - \frac{\sigma^2}{4R^2} \right) \right) p_\mu - \frac{m\Phi}{R} x_\mu R \left( \Phi \frac{d}{\Phi d\tau} \right) \left( R\Phi \left( 1 - \frac{\sigma^2}{4R^2} \right) \right) \\ &= \left( \Phi \left( 1 - \frac{\sigma^2}{4R^2} \right) \right) p_\mu - m\Phi x_\mu \left[ -(x^\nu)_{,\tau} \frac{x_\nu}{2R^2} \right] \\ &= \left( \Phi \left( 1 - \frac{\sigma^2}{4R^2} \right) \right) p_\mu + \frac{p^\nu x_\nu x_\mu}{2R^2} \end{aligned} \quad (5.22)$$

Now, since  $\ell^{\mu\nu} x_\nu = x^\mu p^\nu x_\nu - x^\nu p^\mu x_\nu$ , then one can reexpress the last term in (5.22) in terms of the angular momentum tensor in four dimensions  $\ell_{\mu\nu}$ . Indeed, one finds

$$\begin{aligned} \pi_\mu &= \left( \Phi \left( 1 - \frac{\sigma^2}{4R^2} \right) \right) p_\mu + \frac{1}{2R^2} [\ell_\mu{}^\nu x_\nu + \sigma^2 p_\mu] \\ &= \frac{p_\mu}{\Phi} + \frac{\ell_{\mu\nu} x^\nu}{2R^2} \end{aligned} \quad (5.23)$$

and furthermore it is obvious that  $\lim_{R \rightarrow \infty} \pi_\mu = p_\mu$ . For finite  $R$  however we see that energy and momentum conservation are not defined using  $p_\mu$  alone as they were for flat spacetime, but

rather the combination of terms in  $\pi_\mu$  above. So, substituting  $p_\mu = p_\mu(\pi_\mu, \ell_{\mu\nu})$  into the equation of motion (5.18) we see that *a free particle in de Sitter can be described by a constant angular momentum 10-vector*, i.e.

$$\ell_{ab} = \alpha, \quad \alpha \in \mathfrak{R} \quad (5.24)$$

A consequence of this is that a linear momentum can be transformed in to an angular momentum by a simple displacement and vice-versa, which should be contrasted with the translation invariance of  $p_\mu$  in flat spacetime. Similarly energy (and momentum) as defined just under equation (5.17) is not conserved with respect to de Sitter transformations, only energy and momentum defined by equation (5.23) are. Perhaps unsurprisingly, the energy defined by (5.23) is not positive-definite.

## 5.2.2 Casimir invariants for de Sitter

As indicated by equation (5.11), the generators of the de Sitter group are the 10 dimensional, antisymmetric, rotation generators  $J_{ab}$ . By analogy to the Lorentz rotations and boosts in regular four-dimensional Minkowski space, their algebra must be of the form

$$[J_{ab}, J_{cd}] = +i[\eta_{ad}J_{bc} - \eta_{ac}J_{bd} + \eta_{bc}J_{ad} - \eta_{bd}J_{ac}], \quad (5.25)$$

where again latin indices span the full five dimensions and  $\eta_{ab} = \text{diag}(-1, 1, 1, 1, 1)$  is the flat 5d Minkowski metric. Breaking up these relations in a ‘4 + 1’ format, they become

$$[J_{\alpha\beta}, J_{\gamma\delta}] = +i[\eta_{\alpha\delta}J_{\beta\gamma} - \eta_{\alpha\gamma}J_{\beta\delta} + \eta_{\beta\gamma}J_{\alpha\delta} - \eta_{\beta\delta}J_{\alpha\gamma}] \quad (5.26)$$

along with the two relations

$$[J_{4\lambda}, J_{\mu\nu}] = i(\eta_{\lambda\mu}J_{4\nu} - \eta_{\lambda\nu}J_{4\mu}) \quad (5.27)$$

and

$$[J_{4\lambda}, J_{4\nu}] = -i\eta_{44}J_{\lambda\nu} \quad (5.28)$$

Setting  $\Pi_\alpha \equiv \frac{J_{4\alpha}}{R}$  and demanding that it reduce to the energy momentum operators  $P_\mu$  in flat spacetime, i.e. that the operator relations  $\lim_{R \rightarrow \infty} \Pi_\mu = P_\mu$  hold, we can express the last two commutation relations above as

$$[\Pi_\alpha, J_{\beta\gamma}] = i\frac{R}{R}(\eta_{\alpha\beta}\Pi_\gamma - \eta_{\alpha\gamma}\Pi_\beta) = i(\eta_{\alpha\beta}\Pi_\gamma - \eta_{\alpha\gamma}\Pi_\beta) \quad (5.29)$$

and

$$[\Pi_\alpha, \Pi_\beta] = -\frac{i}{R^2}\eta_{44}J_{\alpha\beta} = -\frac{i}{R^2}J_{\alpha\beta} \quad (5.30)$$

It is apparent that in the limit of zero curvature ( $R \rightarrow \infty$ ) the Poincare relations are recovered, and in this sense the Poincare group is a contraction of the de Sitter group.

However, this is even clearer if we look at the centre of the de Sitter group, i.e. compute its Casimir invariants. The form of these special elements can be directly determined by simply applying what we know about the flat spacetime case in 4d. Indeed, as with the Poincare group, one expects one of the Casimir invariants will be of the form

$$C_1 = \alpha(R) J_{ab} J^{ab} = M_{dS}^2, \quad (5.31)$$

i.e. it will give us the de Sitter invariant rest mass in analogy with the Poincare rest mass invariant. Expanding out equation (5.31) using the definition of I find

$$\begin{aligned} C_1 &= \alpha(R) [J_{5\mu} J^{5\mu} + J_{\mu\nu} J^{\mu\nu}] \\ &= \alpha(R) [R^2 \Pi^\mu \Pi_\mu + J_{\mu\nu} J^{\mu\nu}] \end{aligned} \quad (5.32)$$

$$\begin{aligned} &= \alpha(R) R^2 \left[ \frac{P_\mu}{\Phi} + \frac{x^\nu L_{\mu\nu}}{2R^2} \right] \left[ \frac{P^\mu}{\Phi} + \frac{x_\nu L^{\mu\nu}}{2R^2} \right] + \alpha(R) J_{\mu\nu} J^{\mu\nu} \\ &= \alpha(R) R^2 \left[ \frac{P^\mu P_\mu}{\Phi^2} + \frac{x^\nu L_{\mu\nu} L^{\mu\beta} x_\beta}{2R^2} \right] + \alpha(R) J_{\mu\nu} J^{\mu\nu}, \end{aligned} \quad (5.33)$$

where  $\alpha(R)$  is some arbitrary function of the curvature,  $P_\mu \sim \partial_\mu$ , and<sup>2</sup>  $L_{\mu\nu} \sim x_{[\mu} P_{\nu]}$ . However, in order to recover the flat space result of  $C_1 \sim P^\mu P_\mu$  one must require that this function go as  $\alpha(R) = \gamma/R^2$ ,  $\gamma \in \mathfrak{R}$ . Putting this in I find that, after some trivial factoring,

$$C_1 = \frac{\gamma}{\Phi^2} P_\mu P^\mu + \frac{\gamma}{R^2} \left[ \delta^{\mu\alpha} \delta^{\nu\beta} + \frac{\delta^{\mu\alpha} x^\beta x^\nu}{2R^2} \right] L_{\mu\nu} L_{\alpha\beta} \quad (5.34)$$

Since  $P^\mu P_\mu = -m^2 \Phi^2$  by the above arguments, this implies that  $\gamma = -1$  if we want the right flat spacetime limit. Putting this in we finally obtain the first Casimir invariant for de Sitter group as

$$\begin{aligned} C_1 &= M_{dS}^2 = m_{\text{Poincaré}}^2 - \frac{1}{R^2} \left( \delta^{\mu\alpha} \delta^{\nu\beta} + \frac{\delta^{\mu\alpha} x^\beta x^\nu}{2R^2} \right) \ell_{\mu\nu} \ell_{\alpha\beta} \\ &= m_{\text{Poincaré}}^2 - \frac{\Lambda}{3} \left( \delta^{\mu\alpha} \delta^{\nu\beta} + \Lambda \frac{\delta^{\mu\alpha} x^\beta x^\nu}{6} \right) \ell_{\mu\nu} \ell_{\alpha\beta}, \end{aligned} \quad (5.35)$$

where in the second line the definition  $3/R^2 \equiv \Lambda$  from equation (5.2) has been used. It is clear that for finite curvature one sees the Poincare rest mass is modified by angular momentum terms involving the particle in four dimensions.

The form of the second invariant, which I only briefly mention here, can also be guessed by analogy with the Poincare invariant, i.e. we guess the form of the ‘generalized Pauli-Lebanski’ Casimir invariant via

$$C_a \equiv \frac{\alpha}{R} \epsilon_{abcde} J^{bc} J^{de}, \quad (5.36)$$

<sup>2</sup>I am glossing over the fact that these are now operators, this will be covered in greater detail later on in this Chapter.

where  $\alpha \in \mathfrak{R}$  and  $\epsilon$  is the totally antisymmetric levi-civita symbol in 5 dimensions. The reason a  $\frac{1}{R}$  dependence appears in the prefactor is to anticipate one of the  $J$ 's assuming a 4 index, and thus having a term like  $J^{4\mu}/R$  which is just the generalized momentum term we saw earlier. In any case one could not put this in and then figure out later that it is necessary to get the right zero curvature limit. Nevertheless, computing this with the prefactor included one obtains

$$C_4 = \frac{\alpha}{R} \epsilon_{4\mu\nu\lambda\sigma} J^{\mu\nu} J^{\lambda\sigma} \quad (5.37)$$

The only other term that is possible given the antisymmetry of the products is

$$\begin{aligned} C_\mu &= \frac{\alpha}{R} \epsilon_{4\mu\alpha\beta\gamma} J^{4\alpha} J^{\beta\gamma} \\ &= \alpha (\epsilon_{5\mu\alpha\beta\gamma} \Pi^\alpha J^{\beta\gamma}), \end{aligned} \quad (5.38)$$

which means the second Casimir invariant must have the form

$$C_a C^a = C_\mu C^\mu + C^4 C_4 \quad (5.39)$$

Its physical interpretation is that of the spin magnitude for massive particles in de Sitter, just as the Pauli-Lebanski pseudovector does for the Poincare group. Since  $C_1 \sim J^2$  it is clearly also a function of  $Z$  alone and is therefore a de Sitter invariant quantity.

Now that we have computed the de Sitter invariant analogue of the Poincare rest mass (Casimir invariant) and seen that it is purely a function of the de Sitter invariant  $Z$  of equation (5.14) it is clearer how to think of de Sitter invariance in general. It is also now clearer, on account of equation (5.11), how to view elements of the de Sitter group: namely, an element of the de Sitter group is a mapping of one orthonormal basis in five-dimensional Minkowski spacetime to another orthonormal basis. In the next section I discuss how to conveniently represent the most general element and find the group volume, and I also briefly justify why I choose a particular set of coordinates to cover de Sitter spacetime.

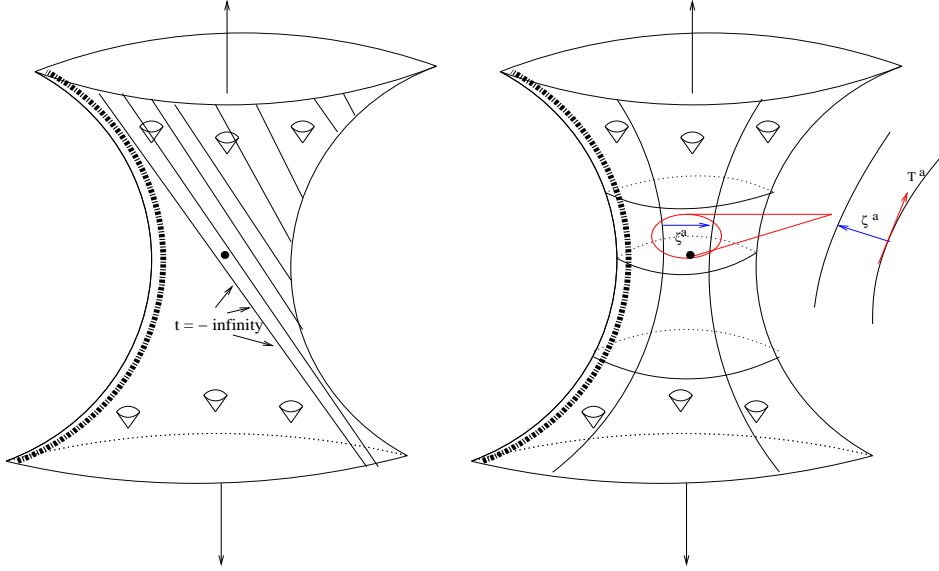
### 5.2.3 Slicing

One can think of the hyperboloid defined by equation (5.4) in the five-dimensional embedding space in terms of a metric, namely

$$ds^2 = -d\zeta_0^2 + d\zeta_1^2 + d\zeta_2^2 + d\zeta_3^2 + d\zeta_4^2 \quad (5.40)$$

One may introduce coordinates on this surface in any number of ways. One popular choice is to implicitly define the coordinates via

$$\left. \begin{aligned} t &= R \ln \frac{(\zeta_0 + \zeta_4)}{R} \\ x &= \frac{R\zeta_1}{\zeta_0 + \zeta_4} \\ y &= \frac{R\zeta_2}{\zeta_0 + \zeta_4} \\ z &= \frac{R\zeta_3}{\zeta_0 + \zeta_4} \end{aligned} \right\} \quad (5.41)$$



**Figure 5.1:** The flat slicing only covers half of de Sitter (left figure). The thickly coloured timelike geodesic does not cross the  $t = \text{constant}$  surfaces of the given flat patch at all. Only two flat patches (plus technically a third one on their mutual boundary at  $t = \pm\infty$ ) can cover the entirety of de Sitter spacetime. The closed slicing (right figure) uses three spheres, pictured by circles at constant  $0 \leq \chi \leq \pi$ , to foliate *all* of de Sitter. The geodesic deviation vector, labelled  $\zeta^a$ , obeys  $\dot{\zeta}^a = \Lambda\zeta^a$ .

on the hyperboloid, in terms of which the metric (5.40) becomes

$$ds^2 = -dt^2 + e^{\frac{t}{R}} \delta_{ij} dx^i dx^j \quad (5.42)$$

This is the *flat slicing* of de Sitter: the surfaces of  $t = \text{constant}$  are flat three-spaces. As shown in Figure 5.2, this flat slicing only covers half of the de Sitter spacetime. In fact, the thickly coloured worldline of a timelike observer in this Figure does not intersect even one flat slice as she travels from the asymptotic past of de Sitter to another asymptotic future region. In this Chapter I find it problematic to discuss large scale effects which may force one to consider events outside one of the flat patches envisioned above. Issues which stem from this choice and in turn give rise to a view on the importance (or even attainability) of de Sitter invariance itself may in fact be problematic, as emphasized by Woodard in [52], however in this Chapter I will proceed anyway having noted this. In this spirit, I choose the coordinates

$$\left. \begin{aligned} \zeta_0 &= R \sinh\left(\frac{t}{R}\right) \\ \zeta_4 &= R \cosh\left(\frac{t}{R}\right) \cos(\chi) \\ \zeta_1 &= R \cosh\left(\frac{t}{R}\right) \sin(\chi) \cos(\theta) \\ \zeta_2 &= R \cosh\left(\frac{t}{R}\right) \sin(\chi) \sin(\theta) \cos(\phi) \\ \zeta_3 &= R \cosh\left(\frac{t}{R}\right) \sin(\chi) \sin(\theta) \sin(\phi) \end{aligned} \right\}, \quad (5.43)$$

in which the metric (5.40) now has the form

$$ds^2 = -dt^2 + R^2 \cosh^2\left(\frac{t}{R}\right) (d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)) \quad (5.44)$$

This metric is singular for  $\chi = 0, \pi$  and  $\theta = 0, \pi$  however these singularities are just the usual polar coordinate singularities. Indeed, this coordinate system covers all of the de Sitter spacetime given  $-\infty < t < \infty$ ,  $0 \leq \chi \leq \pi$ ,  $0 \leq \theta \leq \pi$ , and  $0 \leq \phi \leq 2\pi$ , and a surface of constant time  $t$  is shown in Figure 5.2 above as a circle in the 'neck' region. These spatial sections are three spheres  $S^3$  of constant positive curvature, i.e. the slicing shows the natural topology of de Sitter spacetime in the sense that it is naturally of the form  $\mathfrak{R} \times S^3$ . This choice will be called the closed slicing of de Sitter. Implementing closed slicing also has the advantage of emphasizing the essential point that the timelike geodesics are constantly radially accelerating away from each other at a rate proportional to their separation<sup>3</sup> in a *continuous* fashion from  $\chi = 0$  to  $\chi = \pi$ , without any of the discontinuities which would occur at  $t = \pm\infty$  in the flat slicing.

I note that in the closed slicing it is easy to see why there are 10 parameters in the de Sitter group because the 10 isometries of de Sitter come from the 6 spatial symmetries of the three spheres plus four boosts for four dimensions. Furthermore, the spatial projections (as defined by the above coordinates, the  $t = \text{constant}$  parts) of the four boosts are actually conformal isometries (Killing vectors) of de Sitter spacetime proper, with conformal factor equal to the (constant) Hubble parameter. Indeed, a general element of the  $\text{SO}(4,1)$  dS group can be constructed as a product of rotations about the three angles  $\phi, \theta, \chi$  and a boost  $\beta$ .

More precisely, let  $e_{[0]}^\mu$  be the time-like and future directed member of the standard orthonormal basis  $e_{[A]}^\mu$  for 5d Minkowski spacetime (where  $A$  goes from 0 to 4), and take the  $e_{[i]}^\mu$  be spacelike. Then a general element  $M$  of  $\text{SO}(4,1)$  is the product of an element which fixes  $e_{[0]}^\mu$  with an element that brings  $e_{[0]}^\mu$  into a generic position in the future light cone. Indeed, one may write  $M$  as the product

$$\begin{aligned} M &= R_\phi \times R_\theta \times R_\chi \times B_\beta \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(\phi) & -\sin(\phi) \\ 0 & 0 & 0 & \sin(\phi) & \cos(\phi) \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(\chi) & -\sin(\chi) & 0 & 0 \\ 0 & \sin(\chi) & \cos(\chi) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cosh(\beta) & \sinh(\beta) & 0 & 0 & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & & & & \\ & \mathbf{O} & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}, \end{aligned} \quad (5.45)$$

where  $\mathbf{O}$  is a  $4 \times 4$  real, orthogonal matrix and the empty spaces in the entries are understood to

<sup>3</sup>This is easy to prove: Since  $R_{abcd} \sim g_{c[a}g_{b]d}$ , then the relative acceleration of one geodesic from another along the tangent vector  $T^a$  is simply  $T^e \nabla_e (T^c \nabla_c \zeta^a) \equiv \ddot{\zeta}^a = -\Lambda \zeta^a T^d T_d + \Lambda T_c T^a \zeta^c = \Lambda \zeta^a$ , as claimed in Figure 5.2, using  $\zeta^a T_a = 0$ ,  $T^a T_a = -1$ .

be zeroes (see Gursev in [49] for more details and justification, as well as Higuchi in [53]). Thus, for example the image of  $\tilde{e}_{[1]}^\mu$  under this transformation is

$$\begin{aligned} \tilde{e}_{[1]}^\mu &= e_{[0]}^\mu \sinh(\beta) + e_{[1]}^\mu \cos(\chi) \cosh(\beta) + e_{[2]}^\mu \cos(\theta) \sin(\chi) \cosh(\beta) \\ &\quad + e_{[3]}^\mu \cos(\phi) \sin(\theta) \sin(\chi) \cosh(\beta) + e_{[4]}^\mu \sin(\phi) \sin(\theta) \sin(\chi) \cosh(\beta), \end{aligned} \quad (5.46)$$

which clearly shows the ‘mixing’ of space and time in a general de Sitter transformation. Furthermore, using expression (5.45) for  $M$  it is possible to determine the ‘line element’ of the dS group (see Higuchi in [53]) and from that the group volume element. Using Maple it is straightforward to prove that the group volume element, which will be used in the following sections, is

$$dV = \sinh(\beta)^3 \sin(\chi)^2 \sin(\theta) d\beta d\chi d\theta d\phi \quad (5.47)$$

Formally, the dS group volume  $V = \int_{-\infty}^{\infty} d\beta \int_0^\pi d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi$  is infinite.

With all of these properties of dS in hand, I move on to discuss the backreaction problem in the next section.

### 5.3 Backreactions

In Chapter 4 I compared the effect of a particular class of backreactions, namely super Hubble backreactions, to that of linearized fluctuations by comparing the dispersions  $\langle (\delta^2 \rho_{IR})^2 \rangle$  and  $\langle (\delta \rho)^2 \rangle$ . As mentioned in Section 4.3, I did not address questions of the subhorizon fluctuations. Furthermore, the main result of Chapter 4 was that for small-enough, but finite, values of the slow-roll parameter the second order super Hubble fluctuations could dominate the linear ones in their effect on the gravitational field. Given this, it is interesting to consider both the role of subhorizon fluctuations and superhorizon fluctuations in when the slow-roll parameter is exactly zero. In this situation, as mentioned above, the scalar field driving inflation becomes trivial and its potential becomes proportional to the positive cosmological constant of de Sitter space. Furthermore, a natural approximation of the backreaction problem occurs in the pure de Sitter (i.e.  $\epsilon_{SR} = 0$ ) case. Indeed, consider the kinetic terms of the Klein-Gordon stress energy equation (2.35), which one may write schematically as

$$T \sim \underbrace{\dot{\phi}_B^2}_{\text{negligible by slow-roll}} + \underbrace{(\dot{\phi}_B)(\nabla\delta\phi)}_{\text{Linear quantum correction}} + \underbrace{(\nabla\delta\phi)(\nabla\delta\phi) + \dots}_{\text{Nonlinear quantum corrections}} .$$

As  $\dot{\phi}_B \rightarrow 0$ , which is the same as  $\epsilon_{SR} \rightarrow 0$ , the ‘nonlinear corrections’ will start to become important until  $\dot{\phi}_B$  is precisely zero, when they will be the *only* corrections. In this chapter I study the situation where  $\dot{\phi}_B = 0$  identically, which is de Sitter spacetime. The goal is to solve equation (5.48) for both subhorizon fluctuations, where treatment of their UV divergences becomes essential, and also for superhorizon fluctuations. To further the latter goal, the scalar field fluctuations will be assumed to be massless, which also has the advantage of not setting another scale in the problem. The ultimate goal is to better probe the physics of the *limit* of  $\epsilon_{SR} \rightarrow 0$ , which appears to be



mathematically singular or at least problematic in the sense of Chapter 4 (Grishchuk seems to have been the first to articulate this in [36]), by studying a simple example of backreactions at  $\epsilon_{SR} = 0$ .

Taking the ‘right hand side’ of the field equations to be composed of products of quantum fluctuations I can set up the field equations to ask what the *leading order* effect will be on the gravitational field. More concretely, I write down the Einstein equations

$$G_{ab}(g_{ab}) = \kappa T_{ab}(g_{ab}, \phi), \quad (5.48)$$

where once again  $\kappa \equiv 8\pi$  in units where  $G = c = 1$ . The *leading order* gravitational and scalar field perturbations may be defined via

$$g_{ab} = \bar{g}_{ab} + \epsilon^2 \delta^2 g_{ab} \quad (5.49)$$

$$\phi = \bar{\phi} + \epsilon \delta \phi, \quad (5.50)$$

so the equations of motion that describe the backreaction of the scalar field fluctuations on the background metric are the perturbed Einstein equations

$$\underbrace{\mathcal{L}_{ab}[\delta^2 g_{ab}]}_{\text{Linearized gravity}} = \underbrace{\kappa \mathcal{Q}_{ab}[(\delta\phi)(\delta\phi)]}_{\text{Nonlinear source}}, \quad (5.51)$$

where  $\mathcal{L}_{ab}$  is a *linear* second order hyperbolic operator and  $\mathcal{Q}_{ab}$  is an operator of mixed character acting on the quadratic collection of matter fluctuations. Equation (5.48) as it stands is missing a term on the left hand side which corresponds to purely gravitational wave interactions, i.e. tensor fluctuations. For now I will neglect these terms<sup>4</sup>, which makes this analysis incomplete but tractable. Also, equation (5.48) as it stands is formally divergent in the sense that its right hand side is a product of quantum distributions. In this brief section I will ignore this and treat  $\delta\phi$  as classical, however in Section 5.3.2 these divergences will be treated.

Perturbing the Einstein equations of motion about a de Sitter background using equations (5.46) and (5.47) to define the perturbations, I obtain the equations of motion easily by a straightforward application of equations (2.91) and (2.92) and  $\bar{R}_{ab} = \Lambda g_{ab}$ . Since only the last two terms of equations (2.92) can contribute to first order, the algebra is straightforward for  $\delta R_{ab}$  and results in

$$\begin{aligned} \delta R_{ac} &= -\bar{\nabla}_a \delta C^b_{bc} + \bar{\nabla}_b \delta C^b_{ac} \\ &= -\bar{\nabla}_a \left( \frac{1}{2} \bar{\nabla}_c \delta g \right) + \bar{\nabla}_b \left( \frac{1}{2} \bar{g}^{bm} (\bar{\nabla}_a \delta g_{mc} + \bar{\nabla}_c \delta g_{ma} - \bar{\nabla}_m \delta g_{ac}) \right) \\ &= -\frac{1}{2} \bar{\square} \delta g_{ac} - \frac{1}{2} \bar{\nabla}_a \bar{\nabla}_c \delta g + \frac{1}{2} \bar{\nabla}^m [\bar{\nabla}_a \delta g_{mc} + \bar{\nabla}_c \delta g_{ma}], \end{aligned} \quad (5.52)$$

where  $\bar{\square} \equiv \bar{\nabla}^a \bar{\nabla}_a$ . Using the simple form of the Riemann curvature one can commute through

<sup>4</sup>That there are only TT-TT like terms missing from the left hand side is emphasized at the end of Chapter 4 and later on in this Chapter, where it is proven in detail that there are no scalar or vector modes to linear order in de Sitter.

derivatives with relations like

$$\begin{aligned}\bar{\nabla}^m \bar{\nabla}_a \delta g_{mc} &= \bar{\nabla}_n \bar{\nabla}_a \bar{g}^{mn} \delta g_{mc} \\ &= -\frac{4\Lambda}{3} \bar{g}_{ca} \delta g + \frac{4\Lambda}{3} \delta g_{ac} + \bar{\nabla}_a \bar{\nabla}^m \delta g_{mc}\end{aligned}\quad (5.53)$$

Combining these results and their simple consequences it is easy to show that

$$\begin{aligned}\delta \left( R_{ac} - \frac{R}{2} g_{ac} + \Lambda g_{ac} \right) &= \delta R_{ac} - \bar{g}_{ac} \frac{\delta R}{2} - \frac{1}{2} \delta g_{ac} (\bar{R}) + \Lambda \delta g_{ac} \\ &= -\frac{1}{2} \bar{\square} \delta g_{ac} + \frac{\Lambda}{3} (4 - 6 + 3) \delta g_{ac} + \frac{\Lambda}{3} \bar{g}_{ac} \delta g \left( -1 + \frac{3}{2} \right) \\ &\quad + \left\{ -\frac{\bar{g}_{ac}}{2} \left( -\bar{\square} \delta g + \bar{\nabla}^\ell \bar{\nabla}^m \delta g_{m\ell} \right) \right\} \\ &\quad + \left\{ -\frac{1}{2} \bar{\nabla}_a \bar{\nabla}_c \delta g + \frac{1}{2} \left( \bar{\nabla}_a \bar{\nabla}^m \delta g_{mc} + \bar{\nabla}_c \bar{\nabla}^m \delta g_{ma} \right) \right\},\end{aligned}\quad (5.54)$$

which then straightforwardly leads to the left hand side of equation (5.48)

$$\begin{aligned}\mathcal{L}_{ab}[\delta^2 g_{ab}] &= \left( \bar{\square} + \frac{2\Lambda}{3} \right) \delta^2 g_{ac} + \left( \frac{\Lambda \bar{g}_{ac}}{3} - \bar{\nabla}_a \bar{\nabla}_c + \bar{\square} \bar{g}_{ac} \right) \delta^2 g - \bar{g}_{ac} \bar{\nabla}^\ell \bar{\nabla}^m \delta^2 g_{\ell m} \\ &\quad + 2 \bar{\nabla}_{(a} \bar{\nabla}^m \delta^2 g_{c)m},\end{aligned}\quad (5.55)$$

where  $\delta^2 g \equiv \bar{g}^{ab} \delta^2 g_{ab}$  is the trace<sup>5</sup> of  $\delta^2 g$ . The right hand side is simple enough to derive by inspection of equation (2.35), bearing in mind that that when  $\epsilon_{SR} = 0$ ,  $V(\phi) = \text{const} \sim \Lambda$ :

$$\kappa \mathcal{Q}_{ab}[(\delta\phi)(\delta\phi)] = 2\kappa (\bar{\nabla}_a \delta\phi \bar{\nabla}_c \delta\phi - \frac{\bar{g}_{ac}}{2} \bar{\nabla}_m \delta\phi \bar{\nabla}^m \delta\phi),\quad (5.56)$$

As before, the solutions  $\delta^2 g_{ab}, \delta\phi$  of equations (5.51) will be invariant under a group of gauge transformations. We can again strategically choose the gauge condition to simplify either the interpretation of the fluctuations or the amount of work required in getting to the solutions. In the next section I outline what my gauge choice is to simplify equations (5.51) themselves. It is worth noting immediately, however, that because of the de Sitter background, the scalar field perturbations will be gauge-invariant to first order because the background scalar field is a constant.

### 5.3.1 Gauge-fixing the classical equations of motion

Given the formality of the following arguments it will be useful to have a clear idea of the exact goal of this section. The idea is to simplify the form of equations (5.51) so that their solutions have a clear physical meaning. I show in this section that it is possible to transform to a system of coordinates wherein the divergence of the metric perturbation  $\bar{\nabla}^a \delta^2 g_{ab}$  obeys a simple condition which renders the trace of the field equations free of any matter terms. However, this choice still leaves some residual coordinate freedom which one can use. I show that I can use this residual

<sup>5</sup>The notation is slightly confusing here. Even though equation (5.51) explicitly refers to the linearization of the Einstein tensor, for example, so does equation (5.52). I label the linear fluctuation in the metric  $\delta^2 g_{ab}$  in order to show that this *linear fluctuation is at second order in  $\epsilon$* . This is self-consistent when the matter term goes as  $\epsilon^2$ .

freedom to furthermore set the trace of the metric fluctuation  $\bar{g}^{ab}\delta^2 g_{ab} = 0$  as well as the 'shift'  $\delta^2 g_{0i}$  to zero. The imposition of these additional conditions, while consistent in the case of this particular calculation about a maximally symmetric background, is not possible in general. The benefit of going through this procedure is a clear view of the physical degrees of freedom of the fluctuations, which is important for quantization.

Indeed, starting from equations (5.51) it is convenient to fix the linearized fluctuations (linearized gauge sector) such that the four conditions

$$\bar{\nabla}^m \delta^2 \tilde{g}_{\ell m} - \frac{\bar{\nabla}_\ell}{2} (\delta^2 \tilde{g}) = \frac{\kappa}{2} \bar{\nabla}_\ell (\delta\phi)^2 \quad (5.57)$$

hold. The tilde denotes that these particular conditions have been fixed. The main advantage of this choice is that equations (5.51) assume the simple form

$$(\bar{\nabla}^c \bar{\nabla}_c - \frac{2\Lambda}{3}) \delta^2 \tilde{g}_{ab} + \frac{2\Lambda}{3} \bar{g}_{ab} \delta^2 \tilde{g} = -2\kappa \delta\phi \bar{\nabla}_a \bar{\nabla}_b \delta\phi, \quad (5.58)$$

in this gauge. As discussed extensively in Chapters 2-4, the perturbed metric may be transformed by picking a specific form for  $\xi_T^a$  in the law  $\delta^2 \tilde{g}_{ab} = \delta^2 g_{ab} + \mathcal{L}_{\xi(T)} \bar{g}_{ab}$ . Setting the conditions (5.57) is thus equivalent to solving for  $\xi_{(T)}^\beta$  from the equations

$$(\bar{\nabla}^b \bar{\nabla}_b + \Lambda) \xi_a^{(T)} = -\bar{\nabla}^b \delta g_{ab} - \frac{1}{2} \bar{\nabla}_a (\bar{\nabla}_b \xi_{(T)}^b) + \frac{\kappa}{2} \bar{\nabla}_a (\delta\phi)^2 + \frac{\bar{\nabla}_a}{2} \delta^2 g. \quad (5.59)$$

Once a solution with particular initial conditions is obtained for  $\xi_a^{(T)}$ , one can furthermore demand tracelessness by observing that the *trace* of equations (5.51) in the gauge (5.54) is

$$(\bar{\square} + 2\Lambda) \delta^2 \tilde{g} = 0 \quad (5.60)$$

via the condition of masslessness for the  $\delta\phi$  matter fluctuation<sup>6</sup>. This motivates constructing an object  $\delta^2 \tilde{g}'_{ab} \equiv \delta^2 \tilde{g}_{ab} + \frac{1}{2\Lambda} \bar{\nabla}_a \bar{\nabla}_b \delta^2 \tilde{g}$ , which is formally traceless by the field equation (5.60). It is not hard to show that  $\bar{\nabla}^a \delta^2 \tilde{g}'_{ab} = \frac{\kappa}{2} \bar{\nabla}_a (\delta\phi)^2$ , and also that  $\frac{1}{2\Lambda} \bar{\nabla}_a \bar{\nabla}_b \delta^2 \tilde{g}$  satisfies the full field equations (5.58) in the transverse gauge of described by equation (5.57). It is therefore consistent to demand tracelessness in addition to transversality in this case.

The demand for tracelessness  $\delta^2 \tilde{g}'_a{}^a$  can be understood in terms of  $\xi_\ell^{(T)}$ , via the linearized transformation law  $\delta \tilde{g}_{ab} = \delta g_{ab} + \mathcal{L}_{\xi(T)} \bar{g}_{ab}$ , to imply

$$2\bar{\nabla}^\ell \xi_\ell^{(T)} = -\delta^2 \tilde{g}, \quad (5.61)$$

which must be imposed *in addition to* conditions equations (5.59). I will call simultaneous solutions to equations (5.61) and (5.59)  $\xi_\ell^{(Tr)}$ . Choosing this traceless, transverse, linear gauge fixing, i.e.

<sup>6</sup>It is important to note here that I am not claiming that the stress energy of a massless minimally coupled scalar field has no trace. Rather, the metric on the left hand side has nicely absorbed these terms given the gauge choice of equation (5.57).

setting  $\xi_\ell = \xi_\ell^{(Tr)}$ , casts the equations (5.58) in the simple form

$$(\bar{\nabla}^c \bar{\nabla}_c - \frac{2\Lambda}{3}) \delta^2 \tilde{g}_{ab} = -2\kappa \delta\phi \bar{\nabla}_a \bar{\nabla}_b \delta\phi \quad (5.62)$$

However, the procedure of gauge fixing is not done because there are obvious residual coordinate freedoms left in the homogeneous solutions of the new, traceless, analogue to condition (5.59). I.e., any solution  $\xi_\ell^{(Tr)}$  can be modified by the addition of some harmonic function of the operator  $(\bar{\nabla}^b \bar{\nabla}_b + \Lambda)$ , and this harmonic function is known as a *residual gauge transformation*  $\xi_\alpha^{(R)}$ . It represents the corresponding residual gauge freedom consistent with equations (5.61) and (5.59). I now show that one can pick this harmonic function such that  $\delta^2 g_{0i} = 0$ , i.e. such that the shift perturbations are zero. The residual gauge functions  $\xi_\alpha^{(R)}$  by definition obey the ‘homogenous’ equation

$$(\bar{\square} + \Lambda) \xi_\alpha^{(R)} = 0, \quad (5.63)$$

which by a well-posedness theorem for wave equations in globally hyperbolic spacetimes (see Wald in [3]), has a *well-posed initial value formulation on  $\Sigma_t$* . This means that if I give equation (5.63) arbitrary smooth initial data  $(\xi_a^{(R)}, n^a \bar{\nabla}_a \xi_a^{(R)})$  on a hypersurface  $\Sigma_t$  of the closed slicing, where  $n^a$  is a normal to  $\Sigma_t$ , then there exists a *unique* solution<sup>7</sup> of (5.63). The solution is also causal in the sense that if I vary the initial data it can only affect the solution in the domain of dependence of that variation. Therefore one must provide equation (5.63) with smooth initial data which will allow one to set  $\delta g' = 0$ ,  $\delta g'_{0i} = 0$  while *preserving* the main gauge conditions (5.58). It can also be shown that these choices are preserved through their evolution, however in the following I only explicitly demonstrate the procedure for formulating the initial value problem.

### 5.3.1.1 Fixing the residual degrees of freedom

Setting the shift  $\delta^2 g'_{0i} = 0$ , I find

$$-\delta^2 g_{0i} = \bar{\nabla}_i \xi_0^{(R)} + \bar{\nabla}_0 \xi_i^{(R)}, \quad (5.64)$$

and similarly setting  $\delta^2 g' = 0$ , I obtain

$$-\delta^2 g = 2\bar{\nabla}_a \xi_{(R)}^a = 2\left(\bar{\nabla}_0 \xi_{(R)}^0 + \bar{\nabla}_i \xi_{(R)}^i\right) \quad (5.65)$$

Equations (5.63) are four second order equations for the four components of  $\xi_{(R)}^a$ , so I need *eight* initial conditions to fully specify any vector  $\xi_{(R)}^a$ . Equations (5.64) and (5.65) are four of them. I get the other four by specifying the ‘time derivatives’ of  $\delta g$  and  $\delta g_{0i}$  as determined directly from

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<sup>7</sup>Note the crucial fact that  $[\bar{\nabla}_a, (\bar{\square} + \Lambda)] \xi_{(R)}^a = +\Lambda \bar{\nabla}_a \xi_{(R)}^a$ , which implies that  $(\bar{\square} + 2\Lambda) \bar{\nabla}_a \xi_{(R)}^a = 0$  hold iff  $(\bar{\square} + \Lambda) \xi_{(R)}^a = 0$ , i.e. the residual coordinate transformations cannot undo transverse tracelessness. This is an accident of the maximally symmetric background, and doesn’t hold in general.

conditions (5.57) and (5.59). Thus,

$$\bar{\nabla}_0 \delta^2 g = 2 (\bar{\nabla}^0 \delta^2 g_{00} + \bar{\nabla}^i \delta^2 g_{0i} - \kappa \delta \phi \bar{\nabla}_0 \delta \phi) \quad (5.66)$$

$$\bar{\nabla}^0 \delta^2 g_{0i} = \frac{1}{2} \bar{\nabla}_i (\delta^2 \tilde{g} + \kappa (\delta \phi)^2) - \bar{\nabla}^j \delta^2 g_{ji}, \quad (5.67)$$

while conditions (5.58) with  $\delta^2 g' = 0$ ,  $\delta^2 g'_{0i} = 0$  imply

$$\bar{\nabla}^a \delta^2 g'_{a0} = \bar{\nabla}^i \delta^2 g'_{i0} + \bar{\nabla}^0 \delta^2 g'_{00} = \frac{\kappa \bar{\nabla}_0}{2} (\delta \phi)^2 \rightarrow \bar{\nabla}^0 \delta^2 g'_{00} = \frac{\kappa \bar{\nabla}_0}{2} (\delta \phi)^2 \quad (5.68)$$

$$\bar{\nabla}^a \delta^2 g'_{aj} = \bar{\nabla}^i \delta^2 g'_{ij} + \bar{\nabla}^0 \delta^2 g'_{0j} = \frac{\kappa \bar{\nabla}_j}{2} (\delta \phi)^2 \rightarrow \bar{\nabla}^i \delta^2 g'_{ij} = \frac{\kappa \bar{\nabla}_j}{2} (\delta \phi)^2, \quad (5.69)$$

which one may more explicitly rewrite, using  $\bar{\nabla}^0 \delta^2 g'_{00} = \frac{\kappa \bar{\nabla}_0}{2} (\delta \phi)^2 = \bar{\nabla}^0 \left[ \delta^2 g_{00} + 2 \bar{\nabla}_0 \xi_0^{(R)} \right]$  and  $\bar{\nabla}^i \delta^2 g'_{ij} = \frac{\kappa \bar{\nabla}_j}{2} (\delta \phi)^2 = \bar{\nabla}^i \left[ \delta^2 g_{ij} + \bar{\nabla}_i \xi_j^{(R)} + \bar{\nabla}_j \xi_i^{(R)} \right]$ , to the form

$$\bar{\nabla}^0 \delta^2 g_{00} = 2 \bar{\nabla}^i \bar{\nabla}_i \xi_0^{(R)} - \Lambda \xi_0^{(R)} + \frac{\kappa \bar{\nabla}_0}{2} (\delta \phi)^2, \quad (5.70)$$

$$\bar{\nabla}^i \delta^2 g_{ij} = -\bar{\nabla}^i \bar{\nabla}_j \xi_i^{(R)} - \bar{\nabla}^i \bar{\nabla}_i \xi_j^{(R)} + \frac{\kappa \bar{\nabla}_j}{2} (\delta \phi)^2. \quad (5.71)$$

In equation (5.71) I have used (5.63) to eliminate higher time derivatives of  $\xi_{(R)}^a$ . Putting it all together I finally obtain the four additional conditions needed to supplement the four equations (5.64) and (5.65), namely

$$\begin{aligned} \bar{\nabla}_0 \delta^2 g &= 2 \left[ \frac{\kappa}{2} \bar{\nabla}_0 (\delta \phi)^2 - 2(\bar{\Delta} + \Lambda) \xi_0^{(R)} - \frac{\kappa}{2} \bar{\nabla}_0 (\delta \tilde{\phi})^2 \right] + 2 \bar{\nabla}^i (\bar{\nabla}_0 \xi_i^{(R)} + \bar{\nabla}_i \xi_0^{(R)}) \\ &= -2 \left[ (2 \bar{\nabla}^j \bar{\nabla}_j + 2\Lambda) \xi_0^{(R)} + \bar{\nabla}^i \bar{\nabla}_0 \xi_i^{(R)} \right] \end{aligned} \quad (5.72)$$

$$\bar{\nabla}_0 \delta^2 g_{0i} = \left( -\bar{\nabla}_i \bar{\nabla}_0 \xi_{(R)}^0 + (\bar{\nabla}^j \bar{\nabla}_j + \Lambda) \xi_i^{(R)} + \frac{\kappa}{2} \bar{\nabla}_i (\delta \tilde{\phi})^2 - \frac{\kappa}{2} \bar{\nabla}_i (\delta \phi)^2 \right) \quad (5.73)$$

In summary, the set of equations

$$\left. \begin{aligned} \delta^2 g &= -2 \left( \bar{\nabla}_0 \xi_{(R)}^0 + \bar{\nabla}_i \xi_{(R)}^i \right) \\ \bar{\nabla}_0 \delta^2 g &= -2 \left[ (2 \bar{\nabla}^j \bar{\nabla}_j + 2\Lambda) \xi_0^{(R)} - \bar{\nabla}^i \bar{\nabla}_0 \xi_i^{(R)} \right] \\ \delta^2 g_{0i} &= -\bar{\nabla}_i \xi_{(R)}^0 - \bar{\nabla}_0 \xi_i^{(R)} \\ \bar{\nabla}_0 \delta^2 g_{0i} &= \left( -\bar{\nabla}_i \bar{\nabla}_0 \xi_{(R)}^0 + (\bar{\nabla}^j \bar{\nabla}_j + \Lambda) \xi_i^{(R)} \right) \end{aligned} \right\}$$

are *eight* initial value equations<sup>8</sup> for the four components of the vector  $\xi_{(R)}^a$  and its four time derivatives  $\bar{\nabla}_0 \xi_{(R)}^a$  at some initial slice  $\Sigma_t$  such that if  $\xi_{(R)}^a$  is the generator of a linear gauge transformation, then  $\delta g' = 0$ ,  $\delta g'_{0i} = 0$ , and  $\bar{\nabla}^a \delta g'_{ab} = \frac{\kappa \bar{\nabla}_b}{2} (\delta \phi)^2$ . If I define  $\xi_{(R)}^a$  as the solution of equation (5.63) with these initial values, as determined by the above set of equations, then it turns out such a gauge transformation can be made throughout *all* of spacetime. This is so since I am guaranteed a solution to equation (5.63) with this data by Theorem 10.1.2 in Wald [3], and

<sup>8</sup>Which do not involve  $\delta \phi$  at all, as required by its gauge-invariance to first order.

furthermore these conditions are preserved through evolution. Further coordinate conditions are not possible.

Given this entire procedure of classical gauge-fixing, it is possible to count the number of physical degrees of freedom that are left in the physical modes of the gravitational field and scalar field to linear order. Indeed, there is one scalar mode associated with  $\delta\phi$ , one scalar mode associated with the lapse fluctuation  $\delta^2\tilde{g}'_{00}$ , and two polarization degrees of freedom associated with the TT part of  $\delta^2\tilde{g}'_{ij}$ . The two scalar fluctuations are related to each other via the Einstein constraint equations. Thus in this gauge fixing there are *three* degrees of freedom in the perturbation problem to first order, which in this notation is second order in  $\epsilon$ .

However, for the case of *quantum* fluctuations expressions such as the ‘‘Wick monomials’’  $(\delta\phi)^2$  and  $\delta\phi\bar{\nabla}_a\bar{\nabla}_b\delta\phi$  are formally infinite. One typically approaches this problem via some technique of *renormalization*, which basically is some kind of ‘subtraction’ mechanism which is applied to get rid of the formal infinities. It may well be that the renormalization scheme introduces some ambiguities which do not allow all of the conditions required above to hold. Until recently the general procedure for how to handle renormalization ambiguities in general curved spacetimes was not well-understood, and in the next section I briefly describe how I use some of the recent advances in this understanding to justify, among other things, the above classical gauge fixing in the quantum case.

### 5.3.2 Potential quantum anomalies in the gauge-fixing

The theory of a linear quantum field in flat spacetime is well-formulated, and apart from some critical differences stemming from the absence of a preferred vacuum state/ global inertial coordinates, so is the same theory on a curved spacetime<sup>9</sup>[54]. However, nonlinear interacting fields even in Minkowski spacetime generally have to be regulated in some way and indeed the renormalization ambiguities which arise in such a procedure are well specified as renormalized coupling *constants* which appear order by order in perturbation theory. Until recently, a much larger renormalization ambiguity appeared in the curved spacetime case because instead of coupling constants there appeared coupling *functions*, whose dependence on the spacetime point is wholly arbitrary. In a series of recent advances Hollands and Wald showed (in [55], [51], [56], and [50] ) that the imposition of certain key requirements (namely, ‘locality’ and ‘covariance’) can reduce the renormalization ambiguity in curved spacetime to that of Minkowski spacetime together with that of some additional parameters associated with the couplings of the quantum field to curvature.

Among the many other things they proved was how to handle *quantum anomalies* in curved spacetime if one insisted on ‘locality’ and ‘covariance’. These ‘anomalies’ arise as one of the most amazing aspects of quantum field theory, which is that certain relations involving the field equations which are true in the classical theory *cannot* be satisfied in the quantum theory. A familiar example is the trace anomaly for the stress-energy of a conformally invariant field. Another example, which

<sup>9</sup>Technically, a ‘globally hyperbolic’ spacetime. This term (from PDE theory) simply means that the spacetime is predictable in the sense that there exists a Cauchy hypersurface through which every causal worldline must pass. This precludes closed timelike curves, anti-telephones, etc.

they prove, is that for a massless, free quantum field  $\delta\phi$  satisfying the linear equation of motion

$$\bar{\square} \delta\phi = 0 \quad (5.74)$$

it is **not** in general possible to *also* satisfy the nonlinear conditions

$$\delta\phi \bar{\square} \delta\phi = 0 \quad (5.75)$$

$$(\bar{\nabla}_b \delta\phi) \bar{\square} \delta\phi = 0 \quad (5.76)$$

In other words they show that attempts to impose additional, seemingly reasonable, conditions in addition to the equation of motion (5.74) directly leads one to ‘anomalies’ if one *also* insists on maintaining the conditions of locality and covariance<sup>10</sup>. More specifically, they established a set of conditions involving local curvature scalars which one would have to satisfy in order to avoid these sorts of anomalies, and they showed that for general spacetimes these conditions cannot be satisfied.

For our purposes, it is important to note that equations (5.74) - (5.76) represent the masslessness condition on the fluctuation  $\delta\phi$ , the tracelessness condition on  $\delta^2 \tilde{g}_{ab}$  via equation (5.60), and stress-energy conservation for Klein-Gordon  $T_{ab}$  respectively. In other words, if it is true that it is not possible to impose equations (5.74) - (5.76) simultaneously, then one must abandon either stress-energy conservation, tracelessness, or the equation of motion. Insisting on stress-energy conservation and the equation of motion leaves one no choice but to drop the tracelessness condition. However, turns out that the Hollands and Wald (HW) anomaly requirement does not hold for perturbation theory about the maximally symmetric de Sitter spacetime because, e.g., all covariant derivatives of the curvature are zero. In other words, the conditions they derived which forbid the simultaneous satisfaction of the auxilliary conditions (5.75) and (5.76) with the equation of motion may be actually satisfied in de Sitter.

Indeed, if we simply assume a renormalization prescription which satisfies locality and covariance in the sense of HW, then our Klein-Gordon stress-energy may be written exclusively in terms of the ‘Wick monomials’  $\Psi \equiv (\delta\phi)^2$ ,  $\Psi_{ab} \equiv \delta\phi \bar{\nabla}_a \bar{\nabla}_b \delta\phi$  (see HW in [56], section 3.2):

$$T_{ab} = \frac{1}{2} \bar{\nabla}_a \bar{\nabla}_b \Psi - \Psi_{ab} - \frac{\bar{g}_{ab}}{4} \bar{\square} \Psi, \quad (5.77)$$

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<sup>10</sup>Roughly, these terms mean that the field  $\phi$  is constructed in a manner independent of globally defined structures (like a preferred vacuum state) and non-covariant structures like a preferred coordinate system. This is to be distinguished from the ‘locality’ of fields which commute for spacelike separations.

which, using the fact that  $\bar{g}^{\ell a}[\bar{\nabla}_\ell, \bar{\nabla}_b]\bar{\nabla}_a\Psi = \bar{R}_{\ell b}\bar{\nabla}^\ell\Psi$ , implies that<sup>11</sup>

$$T_a^a = \frac{1}{2}\bar{\square}\Psi - \Psi_a^a - \bar{\square}\Psi, \quad (5.78)$$

$$\Psi_a^a = \delta\phi\bar{\square}\delta\phi, \quad (5.79)$$

$$\bar{\nabla}^a T_{ab} = \frac{1}{2}R_{\ell b}\bar{\nabla}^\ell\Psi + \frac{1}{4}\bar{\nabla}_b\bar{\square}\Psi - \bar{\nabla}^a\Psi_{ab} = \bar{\nabla}_b\delta\phi\bar{\square}\delta\phi \quad (5.80)$$

which are essentially equations (5.75) and (5.76). It turns out that one can calculate these quantities using a so-called Hadamard normal ordering prescription (see [56] and reference [17] therein for details) and I simply quote the result:

$$\delta\phi\bar{\square}\delta\phi = Q, \quad (5.81)$$

$$\bar{\nabla}_b\delta\phi\bar{\square}\delta\phi = \frac{1}{3}\bar{\nabla}_a Q, \quad (5.82)$$

where  $Q$  is a nonvanishing local curvature scalar. The main point is that if one wants to ensure that the left hand sides of equations (5.80) and (5.81) vanish, then one must redefine  $\Psi$  and  $\Psi_{ab}$  in a manner consistent with ‘locality’ and ‘covariance’. As HW have proven in [51], the freedom one has in doing this is actually fairly restrictive and amounts to the transformations

$$\Psi \rightarrow \Psi + C \quad (5.83)$$

$$\Psi_{ab} \rightarrow \Psi_{ab} + C_{ab}, \quad (5.84)$$

where  $C$  is any scalar constructed out of the metric, curvature, and derivatives of the curvature with dimension  $[length]^{-2}$  and  $C_{ab}$  is any symmetric tensor that is similarly constructed, with dimension  $[length]^{-4}$ . Therefore, if we try to use the available freedom given by equations (5.83) and (5.84) to make the left hand sides of equations (5.81) and (5.82) vanish, we obtain the conditions, using equations (5.78) and (5.80),

$$\frac{1}{2}R_{\ell b}\bar{\nabla}^\ell(\Psi + C) + \frac{1}{4}\bar{\nabla}_b\bar{\square}(\Psi + C) - \bar{\nabla}^a(\Psi_{ab} + C_{ab}) = 0 \quad (5.85)$$

$$(C_a^a + \Psi_a^a) = 0 \quad (5.86)$$

which are equivalent to<sup>12</sup>

$$\frac{1}{2}R_{\ell b}\bar{\nabla}^\ell C + \frac{1}{4}\bar{\nabla}_b\bar{\square}C - \bar{\nabla}^a C_{ab} = -\frac{1}{3}\bar{\nabla}_b Q \quad (5.87)$$

$$C_a^a = -Q. \quad (5.88)$$

Now, as we noted in the introduction of this Chapter, in the maximally symmetric de Sitter

<sup>11</sup>I am here freely using the Leibniz condition, i.e. asserting that  $\bar{\nabla}_a\Psi = 2\delta\phi\bar{\nabla}_a\delta\phi$ . This condition forms part of the locality and covariance requirements of HW, so that in a direct sense the question of whether or not anomalies exist is also a question of whether or not the Leibniz condition can be imposed along with the equation of motion  $\bar{\square}\delta\phi = 0$ .

<sup>12</sup>I note that these equations are different than equations (75) and (76) of HW in [56].



spacetime the curvature is completely determined by the Ricci scalar, which is a constant ( $= 4\Lambda$ ). Using the formalism of HW this implies that the most general form of  $C$  can be  $C = \alpha R$ ,  $\alpha \in \mathfrak{R}$ , since  $R$  is the only natural quantity with dimensions  $[length]^{-2}$  for a massless, minimally coupled, scalar field. Putting this form in for  $C$  we obtain the equations

$$-\bar{\nabla}^a C_{ab} = -\frac{1}{3}\bar{\nabla}_b Q \quad (5.89)$$

$$C_a^a = -Q, \quad (5.90)$$

which is simple enough to guess the solution  $C_{ab} = ng_{ab}Q + \beta g_{ab}R^2$ ,  $n, \beta \in \mathfrak{R}$ . Putting this in, it is simple to obtain the conditions<sup>13</sup>

$$n = \frac{1}{3} \quad (5.91)$$

$$Q = -\frac{12}{7}\beta R^2 \quad (5.92)$$

In [51] and references therein it is shown that  $Q$  can only be a function of curvature invariants and their derivatives, which for maximally symmetric spacetimes reduces to a linear combination of  $R^2$  terms. Therefore it is always possible to pick a particular real value of  $\beta$  to satisfy equation (5.92). Equation (5.91) shows that picking  $n = \frac{1}{3}$  satisfies equation (5.89) as well, so that in total we can satisfy (5.85) and (5.86) *simultaneously* in the maximally symmetric de Sitter spacetime.

The simultaneous satisfaction of equations (5.85) and (5.86) ensures that we can impose the conditions  $\bar{\nabla}_b \delta\phi \bar{\square} \delta\phi = 0$  and  $\delta\phi \bar{\square} \delta\phi = 0$  in addition to the equation of motion  $\bar{\square} \delta\phi$  while *also* insisting that the quantum field  $\delta\phi$  is local and covariant in the sense of HW. Since the former two conditions are equivalent to stress-energy conservation and the tracelessness condition of Section 5.3.1. (as crucially utilized by equation (5.60)) respectively, then the above demonstrates that there exist no quantum anomalies with respect to those conditions and the equation of motion  $\bar{\square} \delta\phi = 0$  using the formalism developed by HW. This is equivalent to the claim that there are no quantum anomalies in our gauge fixing.

## 5.4 Quantization and the LS conditions

Given the gauge conditions for the quantum fluctuations do not have anomalies, one can now tackle the question of what further effect the LS conditions which arise in de Sitter may have. Recalling the structure of the (nonlinear) LS conditions as laid out in Chapter 2-4, their formulation requires a Hamiltonian formalism. This can complicate the analysis of any anomalies which may arise out of the LS conditions, however in the following analysis this concern is allayed by transforming re-expressing the canonical variables into terms which are components of covariant objects and beyond this the issue is not treated in detail, though it in general is a concern.

Although implicit in the above discussion of anomalies, the quantum nature of the fluctuations  $\delta^2 g_{ab}$  and  $\delta\phi$  has not been developed in detail yet. I do this now, with an eye towards formulating

<sup>13</sup>Note that the parameter  $\alpha$  does not appear and hence is left undetermined by the conditions (5.91) and (5.92). This implies that an additional condition can consistently be made, as discussed in the next section.

the linearization stability conditions (LS) discussed in Chapters 2 - 3. I know in advance that these latter LS conditions will be relevant because as discussed in Section 5.2.3 I have chosen a compact slicing of de Sitter (in the sense that the  $S^3$  hypersurfaces  $\Sigma_t$  are compact), which has ten isometries (six rotations and four boosts). Therefore there are ten LS conditions associated with these background symmetries.

### 5.4.1 Smearred quantum constraints

Assuming the Hamiltonian formulation of Sections 2.3.1 onwards and using the slicing described by equations (5.43), we characterize the gravitational field in terms of some initial three-geometry  $h_{ij}$ , and its conjugate momentum  $\pi^{ij}$ , of a compact spacelike surface  $\Sigma_t = S^3$ . Linearizing these quantities using the notation of Chapter 2 and 3 and promoting them to Hermitian operators, their commutation relations are given by

$$\begin{aligned} [\delta^2 h_{ab}(x), \delta^2 \pi^{cd}(x')] &= i\sqrt{|\bar{h}|} [\delta_a^c \delta_b^d + \delta_a^d \delta_b^c] \delta(\chi' - \chi) \delta(\theta' - \theta) \delta(\phi' - \phi) \\ &\equiv i\sqrt{|\bar{h}|} [\delta_a^c \delta_b^d + \delta_a^d \delta_b^c] \delta^3(x - x'), \end{aligned} \quad (5.93)$$

and similarly for the matter fluctuations  $\delta\phi, \delta\pi_\phi$  as described in Section 2.3.1.1. The classical metric fluctuations are not free on  $S^3$ , of course, but satisfy the Hamiltonian and momentum constraints order by order in  $\epsilon$ . Bearing in mind that I have defined the leading order gravitational fluctuations to occur at *second* order in  $\epsilon$ , the second order Hamiltonian and momentum constraints are respectively (cf. equations (2.78-2.82) )

$$\begin{aligned} \delta^2 \mathcal{H}_\perp &= \delta^2 \left\{ \sqrt{|\bar{h}|} \left( G_{abcd} \pi^{ab} \pi^{cd} - {}^{(3)}R \right) - 2\kappa\rho \right\}, \\ &= -\frac{1}{4\sqrt{|\bar{h}|}} \delta^2 h \left( 2\bar{\pi}^{\ell m} \bar{\pi}_{\ell m} - \bar{\pi}^2 \right) + \frac{1}{2|\bar{h}|} \left[ \delta^2 h_{ad} \bar{h}_{bc} + \bar{h}_{ad} \delta^2 h_{bc} + \delta^2 h_{ac} \bar{h}_{bd} + \delta^2 h_{bd} \bar{h}_{ac} \right. \\ &\quad \left. - \delta^2 h_{ab} \bar{h}_{cd} - \delta^2 h_{cd} \bar{h}_{ab} \right] \bar{\pi}^{ab} \bar{\pi}^{cd} + 2\sqrt{|\bar{h}|} \bar{G}_{abcd} \bar{\pi}^{cd} \delta^2 \pi^{ab} \\ &\quad + \sqrt{|\bar{h}|} \left( {}^{(3)}\bar{G}^{ab} \delta^2 h_{ab} - \sqrt{|\bar{h}|} (\bar{D}^m \bar{D}^a - \bar{h}^{ma} \bar{\Delta}) \delta^2 h_{ma} \right) \\ &\quad - 2\kappa \left[ \frac{\sqrt{|\bar{h}|}}{2} V(\bar{\phi}) \delta^2 h + \frac{(\delta\pi_\phi)^2}{2\sqrt{|\bar{h}|}} + \sqrt{|\bar{h}|} \frac{1}{2} \bar{h}^{ij} \delta\phi_{,i} \delta\phi_{,j} \right] = 0 \end{aligned} \quad (5.94)$$

and

$$\begin{aligned} \delta^2 \mathcal{H}^i &= -2D_j \delta^2 \pi^{ij} - \kappa J^i \\ &= -2D_j \delta^2 \pi^{ij} - \kappa \delta\pi_\phi \partial^i \delta\phi = 0. \end{aligned} \quad (5.95)$$

Following the development around equation (2.110), I will write these constraints as smeared over a general vector field  $X^a = (X_\perp, X^i)$ , whose normal and tangential components (with respect to

$\Sigma_t$ ) are  $X_\perp$ ,  $X^i$  respectively<sup>14</sup>. I.e., I *project* the above constraints along  $X^a$  via

$$P(X; \epsilon) = \int_{\Sigma_t} \{X_\perp \mathcal{H}_\perp(\epsilon) + X_i \mathcal{H}^i(\epsilon)\} d^3x \quad (5.96)$$

Turning back to quantum considerations, one lets the operators  $\delta^2 h_{ab}, \delta^2 \pi^{ab}$  act on vectors  $|\Psi\rangle$  of some Hilbert space. According to the standard Dirac quantization procedure, these vectors are in general said to define *physical states* whenever they belong to a subset of the Hilbert space defined by

$$P(X; \epsilon) |\Psi(\epsilon)\rangle = 0. \quad (5.97)$$

These smeared relations are operator equations which have to be regularized via suitable renormalization procedure order by order, starting from  $\epsilon^2$ . However, it will be extremely useful to push the classical analysis somewhat farther to gain better insight into the structure of equation (5.97) at second order in  $\epsilon$ .

#### 5.4.2 The classical smeared, non-vacuum, constraints; LS conditions

As remarked above, the lowest order, nontrivial approximation of equation (5.97) is at  $\epsilon^2$ , and classically it can be calculated by writing down

$$\begin{aligned} \delta^2 P(X) &= \int_{\Sigma_t} \left( X_\perp \overrightarrow{\left( \frac{\delta^2 \mathcal{H}_\perp}{\delta^2 u_\ell} \right)} (\delta^2 u_\ell) + X_i \overrightarrow{\left( \frac{\delta^2 \mathcal{H}^i}{\delta^2 u_\ell} \right)} (\delta^2 u_\ell) \right) d^3x \\ &\quad - 2\kappa \int_{\Sigma_t} \left( X_\perp \overrightarrow{\left( \frac{\delta^2 \rho}{\delta u_i \delta u_j} \right)} (\delta u_i \delta u_j) + X_\ell \overrightarrow{\left( \frac{\delta^2 J^\ell}{\delta u_i \delta u_j} \right)} (\delta u_i \delta u_j) \right) d^3x, \end{aligned} \quad (5.98)$$

where the  $\overrightarrow{(\ )}$  terms represent *matrices of classical differential operators* acting on the phase space variables at order  $\epsilon^2$ , i.e.  $\delta^2 u_a = (\delta^2 h_{ab}, \delta^2 \pi^{ab})$  and  $\delta u_i = (\delta \phi, \delta \pi_\phi)$  following the notation of Chapter 2. One may cast  $\delta^2 P(X)$  in a different form via a by-parts integration (the so-called ‘adjoint form’), which reveals

$$\begin{aligned} \delta^2 P(X) &= \int_{\Sigma_t} \left( X_\perp \overleftarrow{\left( \frac{\delta^2 \mathcal{H}_\perp}{\delta^2 u_\ell} \right)} (\delta^2 u_\ell) + X_i \overleftarrow{\left( \frac{\delta^2 \mathcal{H}^i}{\delta^2 u_\ell} \right)} (\delta^2 u_\ell) \right) d^3x \\ &\quad - 2\kappa \int_{\Sigma_t} \left( X_\perp \overleftarrow{\left( \frac{\delta^2 \rho}{\delta u_i \delta u_j} \right)} (\delta u_i \delta u_j) + \frac{1}{2} X_\ell \overleftarrow{\left( \frac{\delta^2 J^\ell}{\delta u_i \delta u_j} \right)} (\delta u_i \delta u_j) \right) d^3x \\ &\quad + \oint_{\partial \Sigma_t} d^2 \sigma_m \left( \overrightarrow{\delta^2 B^m}_{(G)} + \overrightarrow{\delta^2 B^m}_{(matter)} \right) F(\delta^2 u_\ell, \delta u_i; X^a), \end{aligned} \quad (5.99)$$

where  $F$  is some function of the phase space variables and  $X^a$ . I.e., the ‘direction’ of the operators has now reversed at the cost of introducing the last term, which is a surface term equal whose total contributions  $\delta^2 B$  are identically zero since  $S^3$  has no boundary. After some tedious calculations

<sup>14</sup>I will also refer to the spatial projection of  $X^a$  as  $\vec{X}$ .

it is possible to prove that the only non-zero combinations of terms are (cf. equations (2.105) - (2.106))

$$[\delta^2 h_{ab}] \left( \overrightarrow{\frac{\delta^2 \mathcal{H}^a}{\delta^2 h_{ab}}} \right) (X_a) = \delta^2 h_{ab} \left[ \sqrt{|\bar{h}|} (\hat{L}^{ab} + \bar{M}^{ab}) X_\perp - \mathcal{L}_{\bar{X}} \bar{\pi}^{ab} \right], \quad (5.100)$$

$$[\delta^2 \pi^{ab}] \left( \overrightarrow{\frac{\delta^2 \mathcal{H}^a}{\delta^2 \pi^{ab}}} \right) (X_a) = \delta^2 \pi^{ab} \left[ 2\sqrt{|\bar{h}|} \bar{G}_{abcd} \bar{\pi}^{cd} X_\perp + \mathcal{L}_{\bar{X}} \bar{h}_{ab} \right], \quad (5.101)$$

$$[(\delta\phi)^2] \left( \overrightarrow{\frac{\delta^2 \rho}{(\delta\phi)^2}} \right) (X_\perp) = -(\delta\phi)^2 \left[ \frac{1}{2} \sqrt{|\bar{h}|} \bar{\Delta} X_\perp \right], \quad (5.102)$$

$$[(\delta\pi_\phi)^2] \left( \overrightarrow{\frac{\delta^2 \rho}{(\delta\pi_\phi)^2}} \right) (X_\perp) = \frac{(\delta\pi_\phi)^2}{\sqrt{|\bar{h}|}} X_\perp, \quad (5.103)$$

$$[(\delta\pi_\phi \delta\phi)] \left( \overrightarrow{\frac{\delta^2 J^i}{(\delta\pi_\phi)(\delta\phi)}} \right) (X_i) = \delta\pi_\phi \mathcal{L}_{\bar{X}} \delta\phi, \quad (5.104)$$

where the index  $a$  above ranges over  $0-3$ ,  $\mathcal{L}_{\bar{X}} f_{ab}$  is the usual Lie derivative of a tensor  $f_{ab}$  along the spatial vector  $\bar{X}$ , and  $\bar{\Delta} \equiv \bar{D}^i \bar{D}_i$ . Furthermore, I have used the definitions

$$\begin{aligned} \bar{M}^{ab} &\equiv \frac{1}{|\bar{h}|} \left[ 2\bar{\pi}^a_c \bar{\pi}^{cb} - \bar{\pi} \bar{\pi}^{ab} - \frac{\bar{h}^{ab}}{2} |\bar{h}| \bar{G}_{\ell m n c} \bar{\pi}^{\ell m} \bar{\pi}^{nc} + |\bar{h}|^{(3)} \bar{G}^{ab} \right], \\ \hat{L}^{ab} &\equiv -(\bar{D}^a \bar{D}^b - \bar{h}^{ab} \bar{\Delta}), \end{aligned} \quad (5.105)$$

above.

Equations (5.99)-(5.105) admit an interpretation which, although somewhat technical to arrive at, is simple. Indeed, without the presence of the second order matter which is represented by the  $\rho$  and  $J^i$  terms, the only remaining terms essentially describe a *linearized* gravitational perturbation at order  $\epsilon^2$ . In this case, one may easily show that the form of  $\delta^2 P(X)$  represents the fact that the vacuum constraints generate a diffeomorphism of order  $\epsilon^2$  along the vector  $X^a$ , since they form a first class set of constraints. As discussed in Section 2.2.2., a first class set of constraints  $\phi_i = 0$  by definition satisfies the Poisson bracket identity  $\{\phi_a, \phi_b\} = C_b^a \phi_a$ ,  $C_b^a$  constants, i.e. first class states have mutually vanishing brackets<sup>15</sup>. One may verify that the (adjoint) vacuum constraints are first class by noting that

$$\begin{aligned} \{\delta^2 \mathcal{H}_\perp(V), \delta^2 \mathcal{H}_\perp(M)\} &= \int_{\Sigma_t} \frac{\delta^2 \mathcal{H}_\perp(V)}{\delta^2 h_{ab}} \frac{\delta^2 \mathcal{H}_\perp(M)}{\delta^2 \pi^{ab}} - \frac{\delta^2 \mathcal{H}_\perp(M)}{\delta^2 h_{ab}} \frac{\delta^2 \mathcal{H}_\perp(V)}{\delta^2 \pi^{ab}} d^3 x \\ &= \int_{\Sigma_t} \left[ (\sqrt{|\bar{h}|} \hat{L}^{ab} M) \frac{1}{\sqrt{|\bar{h}|}} (2\bar{\pi}_{ab} - \bar{\pi} \bar{h}_{ab}) V \right. \\ &\quad \left. - (\sqrt{|\bar{h}|} \hat{L}^{ab} V) \frac{1}{\sqrt{|\bar{h}|}} (2\bar{\pi}_{ab} - \bar{\pi} \bar{h}_{ab}) M \right] d^3 x, \end{aligned} \quad (5.106)$$

<sup>15</sup>Actually, even if only  $\det[\{\phi_a, \phi_b\}] = 0$ , then a nontrivial solution of  $\chi^a \{\phi_a, \phi_b\} = 0$  exists, and  $\chi^a \phi_a$  is then clearly of first class.

which, together with the fact

$$\int_{\Sigma_t} [-(\bar{D}^a \bar{D}^b V)(2\bar{\pi}_{ab} - \bar{\pi} \bar{h}_{ab})M] d^3x = \int_{\Sigma_t} [M \bar{D}^b V \bar{D}^a (2\bar{\pi}_{ab} - \bar{\pi} \bar{h}_{ab}) + (2\bar{\pi}_{ab} - \bar{\pi} \bar{h}_{ab})(\bar{D}^b V)(\bar{D}^a M)] d^3x \quad (5.107)$$

implies (since only terms locking  $V$  and  $M$  in derivatives can possibly survive the overall antisymmetry of the Poisson bracket)

$$\{\delta^2 \mathcal{H}_\perp(V), \delta^2 \mathcal{H}_\perp(M)\} = \int_{\Sigma_t} (M \bar{D}^b V - V \bar{D}^b M) 2\bar{D}^a \bar{\pi}_{ab} d^3x \quad (5.108)$$

$$= R^i \delta^2 \mathcal{H}_i, \quad (5.109)$$

where  $R^i \equiv M \bar{D}^i V - V \bar{D}^i M$ . In other words, the bracket of the Hamiltonian constraint  $\delta^2 \mathcal{H}_\perp$  is proportional to a momentum constraint. Similarly, one may prove that the momentum constraints satisfy

$$\{\delta^2 \mathcal{H}_i V^i, \delta^2 \mathcal{H}_j M^j\} = \delta^2 \mathcal{H}_k \mathcal{L}_{\bar{V}} M^k, \quad (5.110)$$

i.e. the momentum constraints close on themselves<sup>16</sup>. The total constraint set therefore closes on itself, i.e. is of first class.

The fact that the vacuum constraints are first-class guarantees, by the so-called Dirac conjecture (which is discussed by Henneaux and Teitelboim in [57]), that one can write their projection along  $X^a$  in the suggestive form

$$\delta^2 P(X) \stackrel{\text{vacuum}}{=} \int_{\Sigma_t} d^3x [(\mathcal{L}_X \bar{g}_{ab}) \delta^2 \pi^{ab} - (\mathcal{L}_X \bar{\pi}^{ab}) \delta^2 h_{ab}]. \quad (5.111)$$

One can verify this in a long and tedious calculation which has already been performed by Moncrief in [19]. Given the form of equation (5.111), it is easy to see that if  $X^a$  happens to be a Killing vector then the linearized gravitational terms at second order will *automatically* drop out of equation (5.99) and leave only the quadratic matter terms<sup>17</sup> I have thusfar ignored. These remainder terms will then form the ten LS conditions given that we choose  $X^a$  to be Killing vectors of the de Sitter background. Indeed, the LS conditions are, using equations (5.102)-(5.104) and (5.99),

$$-2\kappa \int_{\Sigma_t} \left\{ \left( -(\delta\phi)^2 \frac{1}{2} \sqrt{|\bar{h}|} \bar{\Delta} + \frac{(\delta\pi_\phi)^2}{\sqrt{|\bar{h}|}} \right) X_\perp + \frac{1}{2} \delta\pi_\phi \mathcal{L}_{\bar{X}} \delta\phi \right\} d^3x = 0, \quad (5.112)$$

and they clearly form a nontrivial constraint on the solutions to the main classical field equation (5.62). The LS conditions are gauge-invariant and conserved from hypersurface to hypersurface by the arguments given at the end of Chapter 2, in 2.3.2.1, and furthermore there are *ten* of them for each of the Killing isometries of the de Sitter background.

<sup>16</sup>As they do in the full theory, which isn't surprising since they are linear in  $\pi^{ab}$ .

<sup>17</sup>Simply because  $\mathcal{L}_X \bar{g}_{ab} = 0$  and  $\mathcal{L}_X \bar{\pi}^{ab} = 0$ . However, the fact that they drop out enforces linearized gauge-invariance on the quantum level.

One may further simplify the LS conditions (5.112) by making free use of integrations by-parts over densities, the slicing conditions (5.43), and the fact that  $X^a$  is a Killing vector of de Sitter spacetime. Indeed, recalling equation (2.86) in the context of our slicing (which implies zero shift and lapse of 1), it is easy to see that

$$\delta\pi_\phi = \sqrt{|\bar{h}|}\bar{\nabla}_0\delta\phi, \quad (5.113)$$

which when combined with an integration by-parts in this slicing leads to

$$\int_{\Sigma_t} \delta\pi_\phi \mathcal{L}_X \delta\phi d^3x = - \int_{\Sigma_t} \sqrt{|\bar{h}|} \left[ \Psi_{ia} X^i - \frac{1}{2} (\bar{\nabla}_a \Psi) \bar{\nabla}_0 X^0 \right] \bar{n}^a d^3x, \quad (5.114)$$

where I have defined

$$\begin{aligned} \Psi_{ab} &\equiv \delta\phi \bar{\nabla}_a \bar{\nabla}_b \delta\phi \\ \Psi &\equiv (\delta\phi)^2 \end{aligned}$$

Here,  $\bar{n}^a$  is a unit normal to  $\Sigma_t = S^3$ . In a similar way one can prove that

$$\int_{\Sigma_t} \frac{(\delta\pi_\phi)^2}{\sqrt{|\bar{h}|}} X_\perp d^3x = - \int_{\Sigma_t} \sqrt{|\bar{h}|} \left[ \Psi_{ab} X_0 + \frac{1}{2} \bar{\nabla}_a \Psi \bar{\nabla}_b X_0 \right] \bar{n}^a \bar{n}^b d^3x. \quad (5.115)$$

Finally, in simplifying the term whose integrand goes as  $(\delta\phi)^2 \bar{\Delta} X_\perp$ , it is useful to observe that

$$\bar{\nabla}^a [2\bar{\nabla}_{(a} X_{b)}] = -\bar{g}^{am} \bar{R}_{amb}^\ell X_\ell + \bar{\nabla}_b \bar{\nabla}^a X_a + \bar{\nabla}^a \bar{\nabla}_a X_b, \quad (5.116)$$

which implies that  $\bar{\nabla}^a \bar{\nabla}_a X_b = -\bar{R}_b^\ell X_\ell = -\Lambda X_b$  for  $X^a$  satisfying Killing's equation. Given these facts, one may finally show that the LS conditions (5.112) take the form

$$-2\kappa \int_{\Sigma_t} d^3x \sqrt{|\bar{h}|} \left( \Psi_{ab} \bar{n}^a \bar{n}^b X_\perp - \frac{\Lambda}{2} \Psi X_\perp + \frac{1}{2} \Psi_{ia} X^i \bar{n}^a - \frac{1}{4} (\bar{\nabla}_a \Psi) (\bar{\nabla}_i X^i) \bar{n}^a \right) \stackrel{*}{=} 0 \quad (5.117)$$

where Killing's equation has been freely used. It is important to emphasize that this form of the LS conditions, compared to equations (5.112), is less general in the sense that one can still have a compact slicing of de Sitter without demanding that the background metric is diagonal—that is why I have labelled equation (5.117) with an overstar, even though it is consistent with all of the assumptions I have made so far.

The primary utility of equation (5.117) is, as the reader has probably guessed by the notation, that it naturally leads to the question of whether or not quantum anomalies exist with respect to it, the conditions (5.85) and (5.86), and the linear equation of motion for  $\delta\phi$ . I treat this in the next section.

### 5.4.3 Quantum anomalies in the LS conditions

If anomalies exist in equation (5.117) then their primary mathematical effect would be to add a source term to its right hand side, which would transform the LS conditions into mere identities. These identities would yield no new information, just as integrating the second order ADM equations of motion for perturbations in Minkowski spacetime (sliced with non-compact hypersurfaces) would simply give flux identities. Therefore it is important to check whether or not anomalies really do exist for our special case of de Sitter<sup>18</sup>, although a-priori there are considerable grounds for optimism in the case of de Sitter since there is still a free parameter in the HW scheme of 5.3.2. (as indicated in Footnote 14).

Returning to the HW formalism, I once again regard the quantity  $\delta\phi$  as a quantum operator and seek to redefine the quantities  $\Psi$  and  $\Psi_{ab}$  in a manner consistent with the HW axioms which define  $\Psi, \Psi_{ab}$  to be local and covariant (once again, see [51] for a proof of uniqueness of  $\Psi, \Psi_{ab}$  up to local curvature terms of the right dimension). This re-definition was already defined in equations (5.83) and (5.84), and for the case of the quantum analogue of equation (5.117) it amounts to writing

$$-2\kappa \int_{\Sigma_t} d^3x \sqrt{|\bar{h}|} \left( (\Psi_{ab} + C_{ab}) \bar{n}^a \bar{n}^b X_\perp - \frac{\Lambda}{2} (\Psi + C) X_\perp + \frac{1}{2} (\Psi_{ia} + C_{ia}) X^i \bar{n}^a - \frac{1}{4} (\bar{\nabla}_a (\Psi + C)) (\bar{\nabla}_i X^i) \bar{n}^a \right) = 0. \quad (5.118)$$

One may go through the procedure of ‘Hadamard normal ordering’ as Moretti does in [58] and, just as equations (5.81) and (5.82) arise, I find that in de Sitter space

$$-2\kappa \int_{\Sigma_t} d^3x \sqrt{|\bar{h}|} \left( \Psi_{ab} \bar{n}^a \bar{n}^b X_\perp - \frac{\Lambda}{2} \Psi X_\perp + \frac{1}{2} \Psi_{ia} X^i \bar{n}^a - \frac{1}{4} (\bar{\nabla}_a \Psi) (\bar{\nabla}_i X^i) \bar{n}^a \right) \propto QV_{S^3}, \quad (5.119)$$

i.e. the left hand side of must be proportional to a nonvanishing curvature scalar times the finite volume of the three-sphere  $V_{S^3}$ . This nonvanishing curvature scalar must in turn be proportional to any other nonvanishing curvature scalar  $Q$  in de Sitter space, owing to maximal symmetry. Therefore, we may finally write the conditions, analagous to equations (5.85) and (5.86), for using the freedom to redefine  $\Psi, \Psi_{ab}$  to unambiguously satisfy equation (5.118):

$$2\kappa \int_{\Sigma_t} d^3x \sqrt{|\bar{h}|} \left( C_{ab} \bar{n}^a \bar{n}^b X_\perp - \frac{\Lambda}{2} C X_\perp + \frac{1}{2} C_{ia} X^i \bar{n}^a - \frac{1}{4} (\bar{\nabla}_a C) (\bar{\nabla}_i X^i) \bar{n}^a \right) = kQV_{S^3}, \quad (5.120)$$

where  $k \in \mathfrak{R}$ . Putting in the particular forms of  $C_{ab}$  and  $C$  computed in Section 5.3.2 and freely making use of by-parts integrations, I obtain a condition involving the numbers  $n, \beta$  and  $\alpha$  which is equivalent to

$$-(n+k)Q + \bar{R}^2 \left[ -\beta - \frac{\alpha}{8} \right] = 0 \quad (5.121)$$

The numbers  $n$  and  $\beta$  were already fixed by equations (5.91) and (5.92) in requiring that the auxilliary conditions  $\bar{\nabla}_i \delta\phi \square \delta\phi = 0$  (conservation of stress-energy) and  $\delta\phi \square \delta\phi = 0$  (tracelessness)

<sup>18</sup>Of course, they will exist in general. Even the gauge-choice will have anomalies, as can be inferred from Section 5.3.2.

hold in addition to  $\bar{\square} \delta\phi = 0$  (the linearized equation of motion). However, the coefficient  $\alpha$  in  $C = \alpha R$  was left free because of the simplicity the equations took in the de Sitter background, i.e. the  $C$  terms did not appear in equations (5.89) and (5.90). Since it *does* appear in equation (5.121), one may always pick a value of  $\alpha$  to satisfy equation (5.121). Therefore, I conclude that the LS conditions (5.117) do *not* exhibit any quantum anomalies given my coordinate conditions, the equations of motion, and the requirements of locality and covariance in the sense of HW. Therefore, they form a nontrivial operator constraint on the quantum states  $|\Psi\rangle$  which the operators  $\delta\phi$  and  $\delta\pi_\phi$  act on. I emphasize that this is the same conclusion that Higuchi reached using completely different methods in [53], however his calculation involved only vacuum gravitational wave (TT) fluctuations and did not consider regularization issues or quantum anomalies as such. However, it is clear that he anticipated the result that even if matter fields were coupled to gravity in a de Sitter background, such as in the scenario presented in this Chapter, one would also obtain a constraint in the quantum case.

It should also be emphasized that the nontrivial nonlinear constraints on  $|\Psi\rangle$  which arise from the LS conditions are over and above the linearized gauge-invariance imposed by the fact that

$$\int_{\Sigma_t} d^3x [(\mathcal{L}_X \bar{g}_{ab}) \delta^2 \pi^{ab} - (\mathcal{L}_X \bar{\pi}^{ab}) \delta^2 h_{ab}] |\Psi\rangle = 0, \quad (5.122)$$

which was proven in the development around equation (5.111). The task of the next section will be to see exactly what kind of additional constraints the LS conditions (5.117) impose on the states  $|\Psi\rangle$ .

#### 5.4.4 Quantum LS conditions and de Sitter invariance of $|\Psi\rangle$

Higuchi in [53] has already shown that the quadratic LS conditions for gravitational fluctuations in vacuum de Sitter demand that all the physical (i.e., gauge-invariant) states in linearized gravity be de Sitter invariant. He did so via an analysis that showed how the de Sitter group transformations of the classical mode functions related with those of the creation and annihilation operators in the quantum theory. In this way he showed that the operators  $\delta^2 P(X)$  generated de Sitter transformations.

However, it turns out that one may also prove this de Sitter invariance condition for our particular backreaction problem in a much more concise manner. One may appeal to the facts proven at the end of Chapter 2, namely equations (2.117) and (2.118) (and the analogous equations for the timelike LS conditions). These relations show that on the classical level

$$\bar{n}^a \bar{\nabla}_a \delta^2 P(X) = 0, \quad (5.123)$$

$$\delta^2 P(X) - \left( \delta^2 \tilde{P}(X) \right) = 0, \quad (5.124)$$

where  $\left( \delta^2 \tilde{P}(X) \right)$  are the LS conditions with their canonical variables transformed along some vector  $\zeta^a$ , i.e.  $\delta^2 \pi^{ij} \rightarrow \delta^2 \pi^{ij} + \mathcal{L}_\zeta \bar{\pi}^{ij}$ , etc. In other words, the LS conditions are gauge-invariant and preserved from slice to slice. Furthermore one may show in a straightforward calculation



resembling that of equation (5.109) that the Poisson bracket of two LS conditions satisfies

$$\{\delta^2 P(X_a), \delta^2 P(X_b)\} = A_{ab}^c \delta^2 P(X_c), \quad (5.125)$$

where the  $A_{ab}^c$  are some trivial structure functions<sup>19</sup>. However, the fact that the Poisson bracket of two LS conditions returns a third to within some structure constants, combined with the fact that the LS conditions are separately conserved (and gauge-invariant), means that they are the so-called Hamiltonian generators of the associated symmetry transformations (see Moncrief in [59] and also Taub in reference therein)  $X^a$ . In other words, their Poisson bracket algebra must be isomorphic to that of the symmetry group represented by  $X^a$ , which I have taken to be Killing vectors of de Sitter. Therefore, the constants  $A_{ab}^c$  are related to the Lie algebra of the Killing fields via

$$[X_a, X_b] = A_{ab}^c X_c, \quad (5.126)$$

and the constraints  $\delta^2 P(X_a)$  must be the generators of de Sitter transformations. This is easy to verify in practice because one may easily derive the Killing vectors of de Sitter spacetime and therefore find the constants  $A_{ab}^c$ . Indeed, the Killing vectors are, in the given coordinate basis,

$$\left. \begin{aligned} X_{(1)}^i &= \partial_\phi \\ X_{(2)}^i &= -\cos(\phi)\partial_\theta + \cot(\theta)\sin(\phi)\partial_\phi \\ X_{(3)}^i &= -\sin(\phi)\partial_\theta - \cot(\theta)\cos(\phi)\partial_\phi \\ X_{(4)}^i &= \cos(\phi)\sin(\theta)\partial_\chi + \cot(\chi)\cos(\theta)\cos(\phi)\partial_\theta - \cot(\chi)\sin(\phi)\csc(\theta)\partial_\phi \\ X_{(5)}^i &= \sin(\phi)\sin(\theta)\partial_\chi + \cot(\chi)\cos(\theta)\sin(\phi)\partial_\theta + \cot(\chi)\cos(\phi)\csc(\theta)\partial_\phi \\ X_{(6)}^i &= \cos(\theta)\partial_\chi - \cot(\sigma)\sin(\theta)\partial_\theta \end{aligned} \right\} \quad (5.127)$$

for the spatial rotations, and

$$\left. \begin{aligned} B_{(1)}^a &= \cos(\chi)\partial_t + a(t)\dot{a}(t)\sin(\chi)\partial_\chi \\ B_{(2)}^a &= \sin(\chi)\cos(\theta)\partial_t - a(t)\dot{a}(t)\cos(\chi)\cos(\theta)\partial_\chi + a(t)\dot{a}(t)\sin(\chi)\sin(\theta)\partial_\theta \\ B_{(3)}^a &= \sin(\chi)\sin(\theta)\sin(\phi)\partial_t - a(t)\dot{a}(t)\cos(\chi)\sin(\theta)\sin(\phi)\partial_\chi - a(t)\dot{a}(t)\sin(\chi)\cos(\theta)\sin(\phi)\partial_\theta \\ &\quad - a(t)\dot{a}(t)\sin(\chi)\sin(\theta)\cos(\phi)\partial_\phi \\ B_{(4)}^a &= \sin(\chi)\sin(\theta)\cos(\phi)\partial_t - a(t)\dot{a}(t)\cos(\chi)\sin(\theta)\cos(\phi)\partial_\chi - a(t)\dot{a}(t)\sin(\chi)\cos(\theta)\cos(\phi)\partial_\theta \\ &\quad + a(t)\dot{a}(t)\sin(\chi)\sin(\theta)\sin(\phi)\partial_\phi \end{aligned} \right\} \quad (5.128)$$

for the four boost Killing vectors<sup>20</sup>. Thus, for example,

$$[X_{(1)}^i, X_{(2)}^i] = -X_{(3)}^i, \quad (5.129)$$

and one may repeat this procedure to calculate the constants the  $A_{ab}^c$ . In fact, this procedure is often the easiest to use to *find* all of the linearly independent Killing vectors if only two or more are known.

<sup>19</sup>Therefore the LS conditions form a Lie algebra.

<sup>20</sup>The spatial projection of the boost vectors are actually *conformal* Killing vectors on  $S^3$ , with conformal factor equal to  $a\dot{a}$ .

Returning to the quantum LS conditions, the equivalent condition to (5.125) which one must demand is

$$[\delta^2 P(X_a), \delta^2 P(X_a)] = iA_{ab}^c \delta^2 P(X_c), \quad (5.130)$$

and given that we have proven that there are no quantum anomalies which arise in the LS conditions it is possible to find a ‘normal ordering’ such that this relationship is true. This implies that we may identify the LS conditions  $\delta^2 P(X)|\Psi\rangle = 0$  as equivalent to the demand that the physical states  $|\Psi\rangle$  must be invariant under the de Sitter group  $SO(4,1)$ .

## 5.5 The LS conditions and de Sitter invariant states

An immediate conclusion one can draw from the de Sitter invariance requirement

$$\delta^2 P(X)|\Psi\rangle = 0 \quad (5.131)$$

is that it applies to the *entire spectrum* of states  $|\Psi\rangle$ . In particular, it does not just apply to the vacuum state. This should be contrasted with the limit in which the cosmological constant  $\Lambda$  goes to zero, where one would expect to recover physics in Minkowski spacetime. In the case of flat spacetime, only the vacuum state is invariant under the Poincare group whereas any excited states break this symmetry. It would seem therefore that any dynamics in de Sitter spacetime are highly restricted by this requirement of de Sitter invariant states, and furthermore Higuchi in [53] shows that for the case of vacuum gravitational (TT) fluctuations the only normalizable de Sitter invariant state is the vacuum. One cannot do much with just one allowed state, the vacuum, and it is even harder to see how the flat spacetime limit occurs in this context. Moncrief, Higuchi, and others have described this dearth of dynamics as the ‘apparent rigidity’ of de Sitter spacetime, which may be thought of as the crude ‘remnants’ of general diffeomorphism invariance of an underlying theory of quantum gravity<sup>21</sup>.

It is useful to be somewhat more specific about the vacuum in de Sitter, which is considerably richer than in flat spacetime even for a massive, minimally coupled scalar field. In particular, for this case, there is an infinite family of vacua which are de Sitter invariant which are usually parametrized by one complex parameter  $\alpha$ , and which can have interesting short-distance (UV) and long-distance (IR) behaviour. However, only one unique state is thought to reduce to the Poincare invariant vacuum of Minkowski spacetime in the limit of  $\Lambda \rightarrow 0$  and in particular for the two-point function in the vacuum to be of the ‘Hadamard’ form (i.e. to have the same singularity structure as in Minkowski). This is known as the Bunch-Davies (or ‘Euclidean’) vacuum.

For our massless, minimally coupled case, however, the situation is somewhat more peculiar. For example, in the inflationary context it is usually derived that the mean squared fluctuations of

<sup>21</sup>In this sense one could imagine a tower of LS conditions or similar constraints bearing on higher and higher orders of perturbation theory, in turn demanding ‘finer’ and ‘finer’ invariances. de Sitter invariance is after all a rather crude, ‘bulk motion’ invariance.

$\delta\phi$  grow linearly in time with inflation, i.e

$$\langle \delta\phi^2 \rangle \approx \frac{H^3}{4\pi^2} t. \quad (5.132)$$

However, this expression manifestly violates de Sitter invariance since it is not a function of the geodesic distance  $Z$  defined by equation (5.15), and is instead a function of comoving time alone. Indeed, if one insisted on a de Sitter invariant two-point function then one would quickly find that it is infrared divergent in the massless, minimally coupled limit. One can easily see why this is by recalling the second order matter Lagrangian action for our case, from equation (2.83),

$${}^{(2)}S_M = -\frac{1}{2} \int \sqrt{|\bar{g}|} [\bar{g}^{ab} \delta\phi_{,a} \delta\phi_{,b}] d^4x, \quad (5.133)$$

and observing that it is invariant under the transformation  $\delta\phi \rightarrow \delta\phi + \text{constant}$ . This is simply a zero mode, and the two point function is ill defined because all values of the spatially constant part of  $\delta\phi$  are equally probable in a de Sitter invariant state, just as for an eigenstate of the momentum in the quantum mechanics of a free particle. In fact this observation underlies the claim that Allen proves in [60], which is that there exists no de Sitter invariant vacuum state for the massless, minimally coupled field. For this reason it is often assumed that the symmetry group of the vacuum is smaller, for example that of the  $O(4)$  subgroup of  $SO(4,1)$  which are the spatial rotations on  $S^3$ . In that case the two-point function has no infrared divergences anymore<sup>22</sup>. In other words, one may sacrifice invariance with respect to boosts in order to obtain a sensible expression for the two-point function.

However, in our particular case we have also allowed the gravitational field to fluctuate to leading order in response to the second order, quantum, scalar field fluctuations. As I have shown above, this inevitably leads to a relation like (5.131) which imposes a further invariance requirement on all the physical states of the metric and matter fluctuations. Thus, in the context of equation (5.131) and the discussion leading to it, we know that assuming anything less than full de Sitter invariance would imply violating some LS conditions, which may be problematic. In order to avoid this, one may try to at least formally construct fully de Sitter invariant states for massless fields and grapple with the issue of trying to make nontrivial dynamics with them. It is important to stress that if one does not allow the gravitational field to fluctuate, as in the semiclassical approximation, this issue of imposing de Sitter invariance does not arise at all. I also once again refer the reader to Woodard in [52] for extensive and much more in-depth commentary on the whole issue of whether or not one really wants, or can even attain, de Sitter invariance in related considerations.

To get a sense of how to construct de Sitter invariant states, one may start by quantizing the *massive* minimally coupled field in de Sitter

$$(\square - m^2)\delta\phi = 0 \quad (5.134)$$

as discussed underneath equation (5.93). Expanding  $\delta\phi$  in terms of the orthonormal spherical

<sup>22</sup>And in fact equation (5.132) is derived by assuming  $O(4)$  invariance only.

harmonics on  $S^3$ ,  $Y_{L\ell m}$  one has

$$\delta\phi = \sum_{L\ell m} \chi_{L\ell m} Y_{L\ell m}, \quad (5.135)$$

one recalls that  $\bar{\Delta}Y_{L\ell m} = -L(L+2)Y_{L\ell m}$  for  $L \in \mathcal{Z}^+$ . Putting this expansion into the action (5.133) (with a matter term  $m^2\delta\phi^2$  added) one obtains, using by-parts integration to convert the gradient terms and the orthonormality of the spherical harmonics,

$${}^{(2)}S = \frac{1}{2} \sum_{L\ell m} \int \frac{1}{(H \sin \eta)^2} [(\dot{\chi}_{L\ell m})^2 - \omega_L^2(\eta) \chi_{L\ell m}^2] d\eta, \quad (5.136)$$

where

$$\omega_L^2(\eta) \equiv L(L+2) + \frac{m^2}{(H \sin \eta)^2}. \quad (5.137)$$

I have also defined (for convenience) and used above a ‘conformal time’ quantity  $\eta \equiv \frac{t}{a(t)}$  where  $a(t)$  is the scale factor of the metric given by equation (5.44)<sup>23</sup>. The classical equation of motion is therefore

$$\ddot{\chi}_{L\ell m} - 2 \cot(\eta) \dot{\chi}_{L\ell m} + \omega(\eta)^2 \chi_{L\ell m} = 0, \quad (5.138)$$

where the dots indicate derivatives with respect to conformal time, and from the action one can define the canonical momenta

$$\frac{\partial^{(2)}L}{\partial \dot{\chi}_{L\ell m}} \equiv \pi_{L\ell m} = \frac{\dot{\chi}_{L\ell m}}{(H \sin(\eta))^2}. \quad (5.139)$$

Just as indicated under equation (5.93) we promote these quantities to operators, whose commutation relations are

$$[\chi_{L\ell m}, \pi_{L\ell m}] = i\delta_{L\ell} \delta_{\ell\ell'} \delta_{mm'}, \quad (5.140)$$

which are time dependent in the Heisenberg picture. Expanding them in terms of time independent annihilation and creation operators  $a_{L\ell m}$  and  $a_{L\ell m}^\dagger$ , we write

$$\chi_{L\ell m} = u_{L\ell m} a_{L\ell m} + u_{L\ell m}^* a_{L\ell m}^\dagger \quad (5.141)$$

$$\pi_{L\ell m} = \frac{1}{(H \sin \eta)^2} \left[ \dot{u}_{L\ell m} a_{L\ell m} + \dot{U}_{L\ell m}^* a_{L\ell m}^\dagger \right], \quad (5.142)$$

where the  $u_{L\ell m}$  are solutions of (5.134) which are normalized according to the usual Wronskian condition  $u_{L\ell m} \dot{u}_{L\ell m}^* - u_{L\ell m}^* \dot{u}_{L\ell m} = i(H \sin \eta)^2$ .

The vacuum state  $|0\rangle$  is the state for which  $a_{L\ell m} = 0$ ,  $\forall L, \ell, m$ . Its precise definition in a

<sup>23</sup>The metric as expressed in terms of this conformal time is given by  $ds^2 = \frac{1}{(H \sin \eta)^2} (-d\eta^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2))$ .

curved spacetime is of course ambiguous, however we can follow the flat spacetime case closely and demand that it be invariant under the de Sitter group. As I have said above, we wish only to focus on the Bunch-Davies vacuum, for which the mode functions may be verified to be (see Birrel and Davies in [61] for details)

$$u_{L\ell m} = A_L(\sin \eta)^{(3/2)} \left[ P_\nu^\lambda(-\cos(\eta)) - \frac{2i}{\pi} Q_\nu^\lambda(-\cos \eta) \right], \quad (5.143)$$

where  $P$  and  $Q$  above are Legendre polynomials,  $\lambda \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$  and  $\nu \equiv L + \frac{1}{2}$ , and  $A_L = \sqrt{\pi} 2H e^{i\pi\lambda} 2\sqrt{\frac{\Gamma(L-\lambda+3/2)}{\Gamma(L+\lambda+3/2)}}$ . It is clear that this state is de Sitter invariant since one may calculate the symmetric two-point function (or, the 'Wightman' function), which is given by

$$\begin{aligned} W(x, x') &\equiv \langle 0 | \delta\phi(x)\delta\phi(x') + \delta\phi(x')\delta\phi(x) | 0 \rangle \\ &= \sum_{L\ell m} [u_{L\ell m}(\eta)u_{L\ell m}^*(\eta')Y_{L\ell m}(\chi, \theta, \phi)Y_{L\ell m}^*(\chi', \theta', \phi')] \end{aligned} \quad (5.144)$$

$$u_{L\ell m}(\eta')u_{L\ell m}^*(\eta)Y_{L\ell m}(\chi', \theta', \phi')Y_{L\ell m}^*(\chi, \theta, \phi), \quad (5.145)$$

using Maple and expression (5.143) along with the spherical harmonics. The answer, which generalizes (5.132) to the case of a de Sitter invariant vacuum, is

$$W = \frac{H^2}{(4\pi)^2} \Gamma\left(\frac{3}{2} + \lambda\right)\Gamma\left(\frac{3}{2} - \lambda\right)F\left(\frac{3}{2} + \lambda, \frac{3}{2} - \lambda; \frac{1+Z}{2}\right) = W(Z), \quad (5.146)$$

where  $Z$  is defined in equation (5.15) as  $H^2\zeta^a(x)\zeta_a(x')$  and  $F$  is the hypergeometric function<sup>24</sup>. It is clear that this expression has a problem for  $m = 0$  since the gamma function has a pole whenever  $\lambda = \frac{3}{2}$  (see [60] for more details), however I will ignore this problem for now.

Given the above, I will follow Higuchi in [53] and call a state of the form

$$a_{L_1\ell_1 m_1}^\dagger a_{L_2\ell_2 m_2}^\dagger \dots a_{L_n\ell_n m_n}^\dagger |0\rangle,$$

an ' $n$ -particle state' even though there is no globally timelike Killing vector in de Sitter which could justify this phrase. I will also call a finite linear combination of such  $n$ -particle states  $|\psi\rangle$ . The general idea is to take  $|\psi\rangle$  and *smear it over the group volume of  $SO(4,1)$*  using the volume measure we have already calculated in equation (5.47). More specifically, first one may define  $U(M)$ , with  $M$  defined in equation (5.45), as the unitary operator associated with de Sitter transformations of the creation and annihilation operators (which Higuchi has already calculated in [53]). Then one can take  $dV$  as the group volume element from equation (5.47). Acting on  $|\psi\rangle$  with these de Sitter transformations and integrating over each such 'action', one may formally define the 'state'

<sup>24</sup>Type '?' hypergeom' without the quotes in the GUI of any recent version of Maple to see the particular conventions Maple uses in this expression of the hypergeometric function.

$|\Psi\rangle$  via

$$\begin{aligned} |\Psi\rangle &\equiv \int dV_M U(M) |\psi\rangle \\ &= \int_0^\infty d\beta \sinh^3(\beta) B_i(\beta) \int d\chi d\theta d\phi \sin^2(\chi) \sin(\theta) R_{\chi\theta}(\chi_1) R_{\theta\phi}(\theta_2) R_{\chi\phi}(\phi_3) |\psi\rangle \end{aligned} \quad (5.147)$$

where by  $R_{ab}(\alpha)$  I mean a rotation between the axes  $a$  and  $b$  by an amount  $\alpha$ . Similarly,  $B_i(\beta)$  denotes a boost along the basis vector  $e_{[i]}^\mu$  by an amount  $\beta$ .

It is at least formally clear that such a construction for  $|\Psi\rangle$  (which Higuchi in [53] was the first to construct) is automatically de Sitter invariant because

$$U(N)|\Psi\rangle = \int dV_M U(NM) |\psi\rangle = \int dV_M U(M) |\psi\rangle = |\Psi\rangle, \quad (5.148)$$

where the invariance of the group volume under a de Sitter transformation has been used. It is clear from these definitions and manipulations, however, that the infinite group volume of  $\text{SO}(4,1)$  enters in uncomfortable ways, so that  $|\Psi\rangle$  is not normalizable. The infinities arise because of the integrations over the entire infinite group volume and make it difficult to make practical use of this smearing scheme. It is interesting, however, that one may create and use exactly the same sort of scheme for our current problem, which has a scalar degree of freedom arising from the nonlinear matter contributions in addition to the usual gravitational wave (TT) waves considered by Higuchi.

### 5.5.1 Massless de Sitter invariant vacua

I turn away from this line of thought here and instead consider more closely the notion that the infrared divergence in the symmetric two-point function indicates that de Sitter invariance is broken for a massless, minimally, coupled scalar field<sup>25</sup>. Actually, the precise statement Allen in [60] proved was that one *can* construct a de Sitter invariant vacuum for this case, but it cannot be used to create a space of states in the usual ‘Fock’ sense of applying creation operators.

An entirely analogous situation occurs with the simple harmonic oscillator, where the expressions for  $x(t)$  and  $p(t)$  in terms of the creation and annihilation operators make no sense in the limit of the frequency  $\omega \rightarrow 0$  because the Hamiltonian spectrum becomes continuous. In that case one quantizes in terms of the canonical position and momentum operators, which are simply

$$x(t) = A + Bt \quad (5.149)$$

$$p(t) = B, \quad (5.150)$$

where  $[A, B] = i$ . It is clear that  $A$  and  $B$  are constants of the motion in this case. By analogy this suggests that one may take the zero mode associated with (5.134) when  $m = 0$  whenever  $L = 0$ , and simply define some new constants of the motion, instead of creation and annihilation

<sup>25</sup>The primary technical difficulty is the presence of the spatially homogeneous zero mode, for which an expansion in terms of creation and annihilation operators is well-known to be problematic.

operators, via

$$\chi_{000} = [f_{h1}\tilde{A} + f_{h2}\tilde{B}], \quad (5.151)$$

$$\pi_{000} = \frac{1}{(H \sin \eta)^2}(\dot{f}_{h1}\tilde{A} + \dot{f}_{h2}\tilde{B}), \quad (5.152)$$

where  $f_{h1}$  and  $f_{h2}$  are of (5.138) for  $L = 0 = m$ , i.e. homogeneous solutions satisfying the Wronskian condition. Taking these definitions seriously, then the commutation relations (5.140) between 'the rest of' the  $\chi, \pi$  solutions imply simply

$$[\tilde{A}, \tilde{B}] = i \quad (5.153)$$

In this sense one can then define a vacuum via the requirements that

$$\tilde{B}|0\rangle = 0 \quad (5.154)$$

$$a_{L\ell m}|0\rangle = 0 \quad \forall L > 0, \quad (5.155)$$

and it is easy to see that the usual freedom one has in picking a vacuum state is distributed between picking a homogeneous function  $f_{hi}$  plus that of picking an inhomogeneous function, i.e.  $u_{L\ell m}$  for  $L \neq 0$ .

Drawing on the work of Section 5.2.2, one can set  $B$  (as defined by equation (5.35) to be represented by the de Sitter Casimir invariant computed in (5.35) because it is the natural generalization of a classical constant of the motion for de Sitter spacetime. In any case this appears to be a good starting point for developing a sensible de Sitter invariant vacuum state (for a massless, minimally coupled field) in which it is possible to calculate expectation values of physical observables. In fact, after this work was completed it was pointed out to me that precisely such a programme was already carried<sup>26</sup> out some 12 years ago by Kirsten and Garrigas in [62]. In this work they advocated taking the zero mode more seriously using basically the same lines of thought, and actually extended the result (5.132) by computing the dispersion of massless  $\delta\phi$  in their constructed de Sitter invariant 'vacuum' state. However, they did not actually calculate the de Sitter invariant constants of motion which are required by the analysis. These amount to the Casimir invariants of the de Sitter group. In this sense their relation (38) is less general than (5.154) above, which shows exactly what the structure of  $\tilde{B}$  (their  $P$ ) must be in order to be a constant of the motion in de Sitter. As emphasized in Section 5.2.2, the constants of the motion in de Sitter lead to fundamentally different conservation laws and notions of de Sitter invariant energy and momentum, and in this sense the above equations combined with the analysis of Section 5.2.2 are more general than their given framework. This is especially important given the claim in [62] that  $P$  should be observable, as the only observers which are possible in this theory are de Sitter invariant, for whom, e.g., the centre of mass must be constant in space or total angular momentum must be zero.

<sup>26</sup>I thank A.O. Barvinsky for pointing this out to me.

## 5.6 Summary

I have shown that the leading order gravitational response to nonlinear scalar field fluctuations in de Sitter spacetime can be reduced to solving for three physical degrees of freedom, namely a lone scalar fluctuation and two transverse-traceless fluctuations. These fluctuations obey, given the closed slicing I have chosen, initial value constraints which are the perturbed Einstein constraint equations. If one demands that these equations be satisfied to leading order, then one must also satisfy a nontrivial global LS condition for each of the ten background Killing vectors of de Sitter spacetime. These LS conditions only involve the nonlinear scalar field fluctuations because, as is demonstrated, the gravitational terms drop out. On the classical level these LS conditions possess a Poisson bracket algebra which is isomorphic to the Lie algebra of the Killing vectors in de Sitter space. Since the LS conditions are proven to be conserved and gauge invariant in Chapter 2, this shows that they are the generators of the de Sitter transformations. It is also shown that if one insists on the nonlinear quantum scalar field perturbations being local and covariant in the sense of Hollands and Wald, then one can use the freedom to redefine these nonlinear perturbations to render all quantum anomalies associated with the imposition of tracelessness, the equations of motion, and the LS conditions to zero. This implies that the quantum version of the LS conditions impose strict de Sitter invariance on the quantum states of the nonlinear matter and linear gravitational fluctuations. For our massless fields this is highly problematic, as discussed, and a sketch is presented to construct a de Sitter invariant spectrum of states using the Casimir invariants of the de Sitter group. This is advocated over  $SO(4,1)$  group-smearing ordinary members of the  $O(4)$  invariant Fock space.



## CHAPTER 6

### SUMMARY AND CONCLUSIONS

#### 6.1 Outline

In this Chapter I conclude and summarize the three projects which comprise this thesis. I also offer some thoughts on possible follow-up work to be left for the future.

##### 6.1.1 Linearization instabilities in Einstein static

There are perhaps two noteworthy features of this analysis, namely that of an extension of the linearization stability analysis proper and of the proof that linearization stabilities can have a direct physical consequence in a linearized perturbative analysis. In order to summarize how this extended the linearization stability analysis, I will place my work in the context of preceding work. Linearization instabilities in the Einstein equations were initially discovered by Fischer and Marsden in [9] and formulated under rather restrictive conditions, which included the demand of extremal slicing in the background (namely, that the trace of the extrinsic curvature  $K_{ab}$  be constant) and also a restriction to vacuum background spacetimes. Moncrief in [19] and [5] greatly strengthened these results by doing away with the extremal slicing requirement and emphasizing the direct role of background isometries in linearization instabilities, primarily because he used the more physically motivated ADM formalism [17] rather than the implicit function theorem to express the results. However, he retained the vacuum restriction because of the general intuition that another matter degree of freedom will in general allow one to 'soak up the instability' (since a freely specifiable stress-energy allows any metric to solve the linearized Einstein equations). As shown much later by Kastor and Traschen in [63] (following closely the earlier intuition of Taub concerning conserved quantities in perturbation theory), if one restricts the matter perturbations and background then a similar linearization instability can arise. In [1] I show that such a situation *does* indeed arise in Einstein static, where the background matter must have enough pressure to render the speed of sound of the perturbations sufficient to stabilize inhomogeneous perturbations. The linearization instability associated with the background timelike Killing vector in Einstein static implies that the negative-norm spatially homogeneous modes, which are famously (exponentially) unstable, must be present in addition to any neutrally stable inhomogeneous modes. This calculation forms the first concrete example of a linearization instability implying a dynamical instability to linear order. It is useful to point out that I have not considered the LS conditions associated with the spacelike Killing vectors of Einstein static. I do not expect any further strong constraints from them.

The main conclusion one can draw is that linearization instabilities are not simply an academic oddity, but can provide considerable insight into global questions about the modes of the linearized perturbation. By global I mean that the LS conditions are expressed as integrals of modes over a

closed space, so that if one wishes to delete one class of modes or include only special ones, it may well be that these global conditions can be violated. A violation of the LS conditions means that one has not yet really solved the linearized constraint equations, so that either spurious solutions exist or not all of them have been found or included in the analysis. Given that much of what we know about cosmology is from perturbation theory about highly symmetric backgrounds, these features are important to keep in mind.

### 6.1.2 Backreactions in slow-roll

The question of whether or not higher order terms can become important in slow-roll inflation is crucial to understanding currently favoured mechanisms of structure formation. In the fourth Chapter, using [2], I looked at this question for the case of nonlinear super-Hubble modes back-reacting on suitably defined notions of energy density and pressure. The fact that the calculation conceptually reduced to determining how scalars (which appear in the energy density and pressure) behave is no accident, given the relative ease with which one can handle coordinate issues in scalars compared to tensors such as the metric. This will remain the case until some sensible methods are created to compute an averaged spacetime metric, and this is an active area of research today in the context of the so-called cosmological averaging problem (see Coley in [64]).

Nevertheless, the behaviour of scalars, and in particular the eigenvalues of the total stress energy of the fluctuations can potentially say quite a lot about the behaviour of the metric in general, as one can see in the analysis of Chapter 4. When I compute the dispersion of the second order energy density and compare it to the analagous dispersion at linear order, the surprising conclusion is that the second order terms dominate. One could immediately argue that this result may be specific to the many coordinate choices made at first and second order, however there are actually grounds for optimism, as I argue underneath equation (4.70), as it seems that one can go to any other coordinate system and the dominating terms will still go as sufficiently large inverse powers of  $\beta$  as to dominate the linearized result. A rigorous proof of this result, or its falsity, will be a focus of my future research. Another way to state the coordinate issue is to ask if one can find an exhaustive coordinate system in which all the higher order effects are rendered subdominant<sup>1</sup>. Although one can certainly find a coordinate system where  $\delta^2\rho_{IR}$  is zero, the metric as cast in this coordinate system will be very complicated and almost certainly be dominated by second and higher order terms (much like the metric of the surface of a raging river would be if one chose a coordinate system in which the density of the water were constant). Furthermore, it is not possible to set  $\delta^2\rho_{IR}$  and  $\delta^2 p_{IR}$  simultaneously to zero. On physical grounds it is also very hard to ignore the compelling fact that these higher order effects are *cumulative* in the k-modes and that as the IR (superhorizon) phase space rapidly grows, with more and more modes exiting the horizon during inflation, their relative amplitude only grows.

The fact that the *dispersion* of  $\delta^2\rho_{IR}$  dominates that of  $\delta\rho$  combines with the fact that the quantum *average*  $\langle \delta\rho \rangle = 0$  to allow for a simple constraint on the slow-roll parameter (since there are no  $\langle \sqrt{(\delta^2\rho_{IR})^2} \rangle < \langle \delta\rho \rangle$  cross-terms), equation (4.113). This inequality provides a simple window of slow-roll parameters for the background which in turn lead to strong or weak

<sup>1</sup>I am indebted to W.G. Unruh for emphasizing this to me.

relative second order effects. The inequality is certainly violated for Planck-scale initial values of the inflaton (which is a common assumption in chaotic inflation models among others) given the standard number of  $N \sim 70$  e-foldings of inflation. In other words, the inequality asserts that for such an initial value of the inflaton and for a standard duration of inflation, the slow-roll condition is precisely the condition for higher order terms to dominate. If this radical possibility is actually true, then one is forced to accept that the linearized approximation to slowly-rolling spacetimes is intrinsically ill-posed. It may well also be that the inequality (4.113) can be weakened substantially by more realistic assumptions, including coordinate conditions which render higher order effects subdominant as mentioned above, however the calculation at least seems to warrant a closer look at the consistency of the slow-roll approximation in this sense. After all, the whole premise of fluctuations in inflation is based on the assumption of a *linear* evolution of squeezed Gaussian vacua during slow-roll.

### 6.1.3 Backreactions in de Sitter

Given the summary of Chapter 5, I would like to comment on the exclusive use of invariant states to describe dynamics in de Sitter. Moncrief notes in [59] that in order to obtain a reasonable interpretation of invariant physical states one needs to regard the observer as an intrinsic part of the quantum system rather than something external. The work in Chapter 5 suggests that even in the leading order backreaction problem of de Sitter this issue can crop up, and indeed one may regard the requirement of this invariance as a relic of the full diffeomorphism invariant theory underlying a fully quantized theory of gravity. In this sense the backreaction problem outlined in Chapter 5 provides a relatively tractable playing ground for looking at much more fundamental issues. This should be sharply contrasted with the analagous analysis carried out in the semiclassical approximation, in which the background geometry is not allowed to fluctuate, for in that case the whole issue of de Sitter invariance is not forced on the level of the LS conditions. It is also important to note that the LS conditions ensure that if one were to examine the next order of perturbative corrections, they would be integrable and in that sense well-posed.

The construction of sensible, de Sitter, invariant states remains a big challenge. The small beginnings described at the end of Chapter 5 must be extended and much thought still remains to be put in to extending the de Sitter invariant vacua for the massless fields of this backreaction problem. This project is made all the more urgent by the result that quantum anomalies cannot ruin the invariance requirement, for even if there was something wrong in the calculation one can always choose to abandon the tracelessness gauge-condition and retain the combination of stress-energy conservation, equation of motion, and the LS conditions. This would simply give one another scalar mode in the problem, but nothing would fundamentally change. The robustness of this result therefore ensures that the invariance requirement is very much relevant even at leading order in the backreactions. In some sense this is the main contribution of this Chapter to the young and exciting study of backreactions in the early universe.

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## APPENDIX A

### Second order Scalar Constraints

The solutions to the first and second order scalar constraints, which I use right above equation (4.53) to eliminate the matter fluctuations (and their first time derivatives) to first and second order in  $\epsilon$ , are

$$\mathcal{F} = \frac{H}{2\kappa\beta} \left( 6\dot{\mathcal{Q}} + 18H^2t\psi^2 + 18Ht\psi\dot{\psi} - 9\psi\dot{\psi} - 6\mathcal{A}H + 6H\psi^2 \right) - \frac{81H^3k^2}{\kappa^2a^2\beta^3} (\dot{\psi} + H\psi)\psi \quad (\text{A.1})$$

$$\begin{aligned} \dot{\mathcal{F}} = & \frac{\beta}{6} (9\psi^2Ht^2 - \mathcal{A}t) + \frac{3H}{4\kappa a^2\beta} \left( -36k^2\psi^2Ht + 36a^2H^3t\psi^2 - 12k^2\mathcal{Q} - 12a^2H^2\psi^2 \right. \\ & \left. + 36a^2H^2t\psi\dot{\psi} - 3a^2(\dot{\psi})^2 - 18a^2H\psi\dot{\psi} + 12a^2H^2\psi^2 + 45k^2\psi^2 \right) \\ & - \frac{81H^3k}{2\kappa^2a^4\beta^3} \left( -3k^2\psi^2 + 6a^2Hk\psi\dot{\psi} + a^2k(\dot{\psi})^2 + a^2H^2k\psi^2 + 2Ha^2k\psi\dot{\psi} + 6a^2H^2k\psi^2 \right) \end{aligned} \quad (\text{A.2})$$

$$\Phi = \frac{3H}{\kappa\beta} (\dot{\psi} + H\psi) \quad (\text{A.3})$$

$$\dot{\Phi} = t\beta\psi - \frac{9Hk^2\psi}{\kappa a^2\beta} \quad (\text{A.4})$$

Recall here that the times ‘t’ above come from the slowly decaying background scalar field:

$$\bar{\phi}(t) = \phi_0 - \frac{\beta t}{3H} \quad (\text{A.5})$$

In the second order solutions above I have not substituted in particular value of the lapse (equation (4.50)) given by the gauge conditions (equations (4.48) - (4.49)), however I have imposed the conditions (4.51) and (4.52) in order to simplify the expressions. Also, I have not carried out the implied integrals over  $k$  using equation (4.60), however this is done straightforwardly in the main thesis. In this thesis the above constraints are only used to eliminate the matter fluctuations in favour of the metric fluctuations.