PHY329 WAVE EQUATION (note correction) OCT 23, 1996

GENERATING INITIAL DATA: $\hat{u}_j^0$ and $\hat{u}_j^1$

First note a slight change in notation from the original lecture notes. Here, for ease of exposition, $\hat{u}_j^n$ is the quantity which satisfies the difference equation (22.4), while u_j^n satisfies the differential equation (22.2).

We begin by assuming that our difference solution will admit a Richardson expansion

$$\hat{u}(x,t) = u(x,t) + h^2 e_2(x,t) + h^4 e_4(x,t) + \cdots \tag{22.7}$$

and ask what this implies about how accurately we must specify $\hat{u}_j^0$ and $\hat{u}_j^1$. First recall that the problem prescription includes the initial conditions:

$$u\,(x,0) = f\,(x) \qquad\qquad u_t\,(x,0) = g\,(x)$$

Clearly then, we can (and might as well) specify $\hat{u}_j^0$ exactly

$$\hat{u}_j^0 = f_j$$

which implies that $e_2(x,0) = 0$. Then in order for (22.7) to hold, we must have

$$\begin{aligned}\hat{u}_j^1 &\equiv \hat{u}(x_j,\, \triangle t\,) \\ &= u(x_j,\, \triangle t\,) + h^2 e_2(x_j,\, \triangle t\,) + O(h^4)\end{aligned}$$

which implies that

$$\hat{u}(x_j,\, \triangle t\,) - u(x_j,\, \triangle t\,) = h^2 e_2(x_j,\, \triangle t\,) + O(h^4)$$

But

$$\begin{aligned}e_2(x_j,\, \triangle t\,) &= e_2(x_j,0) + \triangle t\, \frac{\partial e_2}{\partial t}(x_j,0) + O(\triangle t^{\,2}) \\ &= \triangle t\, \frac{\partial e_2}{\partial t}(x_j,0) + O(\triangle t^{\,2}) = \lambda h \frac{\partial e_2}{\partial t}(x_j,0) + O(\triangle t^{\,2})\end{aligned}$$

Thus,

$$\hat{u}(x_j,\, \triangle t\,) - u(x_j,\, \triangle t\,) = h^2 \left(\lambda h \frac{\partial e_2}{\partial t}(x_j,0) \right) + \cdots = O(h^3)$$

So we see that we must specify $\hat{u}_j^1$ to $O(h^3)$ accuracy, *not* $O(h^2)$ as was erroneously claimed in the lecture. However, this is readily done by simply Taylor-expanding to higher order:

$$u(x_j,\, \triangle t\,) = u(x_j,0) + \triangle t\; u_t(x_j,0) + \frac{1}{2}\, \triangle t^{\,2}\, u_{tt}(x_j,0) + O(\triangle t^{\,3})$$

Recalling (22.6) where we specified the initial conditions in terms of initially left-moving and right-moving profiles, $l(x)$ and $r(x)$, and their derivatives, $l'(x)$ and $r'(x)$:

$$\begin{aligned}u\,(x,0) &= l\,(x) + r\,(x) \\ u_t\,(x,0) &= l'\,(x) - r'\,(x)\end{aligned}$$

and noting that

$$u_{tt}\,(x,0) = l''\,(x) + r''\,(x)$$

we have for our initialization:

$$\hat{u}_j^0 = l_{\,j} + r_{\,j} \tag{22.7}$$

$$\hat{u}_j^1 = l_{\,j} + r_{\,j} + \triangle t\, \left(l'_{\,j} - r'_{\,j} \right) + \frac{1}{2}\, \triangle t^{\,2} \left(l''_{\,j} + r''_{\,j} \right) \tag{22.8}$$