PHY 387M: Relativity Theory I Fall 1997 Homework 1 (revised)
Due: Tuesday, September 16, 12:30pm

Problem 1: Send me (`matt@infeld.ph.utexas.edu`) an e-mail message stating *your* preferred e-mail address. If you have already done so, you're done with this problem!

Problem 2 (Wald 2.2): Prove that any smooth function $F : \mathbf{R}^n \to \mathbf{R}$ can be written in the form

$$F(x) = F(a) + \sum_{\mu=1}^{n} (x^\mu - a^\mu)\, H_\mu\,(x)$$

Note that $\mathbf{R}$ and $\mathbf{R}^n$ denote 1-dimensional and n-dimensional Euclidean space respectively. Hint: For $n = 1$, use the identity

$$F(x) - F(a) = (x - a) \int_0^1 F'[t(x-a) + a]\, dt$$

then prove it for general n by induction.)

Problem 3 (Wald 2.3):
a) Verify that the commutator, defined by

$$[v, w](f) = v[w(f)] - w[v(f)]$$

satisfies the linearity and Leibnitz properties, and hence defines a vector field.

b) Let X, Y, Z be smooth vector fields on a manifold M. Verify that their commutator satisfies the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

c) Let $Y_1, \ldots, Y_n$ be smooth vector fields on an n-dimensional manifold such that at each $p \in M$ they form a basis of the tangent space V_p. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$ by

$$[Y_\alpha, Y_\beta] = \sum_\gamma C^\gamma{}_{\alpha\beta} Y_\gamma$$

Use the Jacobi identity to derive an equation satisfied by $C^\gamma{}_{\alpha\beta}$.

Problem 4 (Wald 2.4):
a) Show that in any coordinate basis, the components of the commutator of two vector fields v and w are given by

$$[v, w]^\mu = \sum_\nu \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right).$$

Problem 4b) Let $Y_1, \ldots, Y_n$ be as in problem 3(c) above. Let $Y^{1^*}, \ldots, Y^{n^*}$ be the dual basis. Show that the components $(Y^{\gamma^*})_\mu$ of Y^{γ^*} in any coordinate basis satisfy

$$\frac{\partial (Y^{\gamma^*})_\mu}{\partial x^\nu} - \frac{\partial (Y^{\gamma^*})_\nu}{\partial x^\mu} = \sum_{\alpha,\,\beta} C^\gamma{}_{\alpha\beta} (Y^{\alpha^*})_\mu (Y^{\beta^*})_\nu$$

(Hint: Contract both sides with $(Y_\sigma)^\mu (Y_\rho)^\nu$).

Problem 5 (Wald 2.7(a)): No longer assigned.

Problem 6 (Wald 2.8(a)): The metric of flat, three-dimensional Euclidean space is

$$ds^2 = dx^2 + dy^2 + dz^2 \ .$$

Show that the metric components $g_{\mu\nu}$ in spherical polar coordinates r, θ, ϕ defined by

$$r = \left(x^2 + y^2 + z^2\right)^{1/2},$$
$$\cos\theta = z/r,$$
$$\tan\phi = y/x$$

are given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2.$$