PHY 387M: Relativity Theory I Fall 1997 Homework 2
Due: Tuesday, September 30, 12:30pm

Problem 1 (Wald 2.8b): The spacetime metric of special relativity is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

Find the components, $g_{\mu\nu}$ and $g^{\mu\nu}$, of the metric and inverse metric in "rotating coordinates", defined by

$$\begin{aligned} t' &= t, \\ x' &= \left(x^2 + y^2\right)^{1/2} \cos\left(\phi - \omega t\right), \\ y' &= \left(x^2 + y^2\right)^{1/2} \sin\left(\phi - \omega t\right), \\ z' &= z, \end{aligned}$$

where $\tan\phi = y/x$.

Problem 2 (Wald 3.1): Let property (5) (the "torsion free" condition) be dropped from the definition of derivative operator ∇_a in section 3.1 of Wald.

a) Show that there exists a tensor $T^c{}_{ab}$ (called the *torsion tensor*) such that for all smooth functions, f, we have $\nabla_a\nabla_b f - \nabla_b\nabla_a f = -T^c{}_{ab}\nabla_c f$. (Hint: Repeat the derivation of

$$\nabla_a\nabla_b f = \tilde{\nabla}_a\tilde{\nabla}_b f - C^c{}_{ab}\nabla_c f,$$

letting $\tilde{\nabla}_a$ be a torsion-free derivative operator.)

b) Show that for any smooth vector fields X^a, Y^a we have

$$T^c{}_{ab}X^aY^b = X^a\nabla_a Y^c - Y^a\nabla_a X^c - [X,Y]^c.$$

c) Given a metric, g_{ab}, show that there exists a unique derivative operator ∇_a with torsion $T^c{}_{ab}$ such that $\nabla_c g_{ab} = 0$. Derive the analog of

$$\Gamma^c{}_{ab} = \frac{1}{2}g^{cd}\left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right),$$

expressing this derivative operator in terms of an ordinary derivative ∂_a and $T^c{}_{ab}$.

Problem 3: On a manifold with metric g_{ab}, let ξ^a be a smooth vector field which generates an isometry, i.e. such that

$$\pounds_\xi g_{ab} = 0\,.$$

Further suppose that there is a smooth scalar field, f, such that

$$\xi_a = \nabla_a f,$$

where ∇_a is the torsion-free derivative operator compatible with the metric ($\nabla_a g_{bc} = 0$). Show that

$$\nabla_b\xi_a = 0\,.$$

Problem 4 (Wald 3.6a): The metric of Euclidean $\mathbf{R}^3$ in spherical coordinates is

$$ds^2 = dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right)$$

(see Homework 1, Problem 6). Calculate the Christoffel components $\Gamma^{\sigma}{}_{\mu\nu}$ in this coordinate system.

Problem 5: Let v^a and w^a be smooth vector fields and let $T^{a_1 \cdots a_k}{}_{b_1 \ldots b_l}$ be an arbitrary smooth tensor field. Show that

$$\left(\pounds_v \pounds_w - \pounds_w \pounds_v - \pounds_{[v,w]} \right) T^{a_1 \cdots a_k}{}_{b_1 \ldots b_l} = 0 \, .$$

(Hint: Argue that it suffices to establish the formula for the cases where $T^{a_1 \cdots a_k}{}_{b_1 \ldots b_l}$ is a scalar field or vector field.)