PHY 387M: Relativity Theory I   Fall 1997   Homework 4
Due: Thursday, November 6, 12:30pm

Problem 1: Show that

$$\sum_{\mu\nu} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} = \frac{1}{g} \frac{\partial g}{\partial x^\sigma}$$

(Wald, equation (3.4.8)) where $g_{\mu\nu}$ and $g^{\mu\nu}$ are the components of the metric and inverse metric in any coordinate system x^μ, and g is the determinant of the metric.

Problem 2: Consider the stress tensor for a perfect fluid in *flat spacetime:*

$$T_{ab} = \rho\, u_a u_b + P\,(\eta_{ab} + u_a u_b) \tag{2.1}$$

where u^a is the (timelike) 4-velocity of the fluid ($u^a u_a = -1$), and ρ and P are, respectively, the mass-energy density and pressure of the fluid as measured in the fluid's rest frame.

a) Assuming $\rho \geq 0, P \geq 0$ show that the stress tensor satisfies the energy condition

$$T_{ab} v^a v^b \geq 0$$

where v^a is any 4-velocity ($v^a v_a = -1$).

b) By projecting

$$\partial^a T_{ab} = 0$$

parallel and perpendicular to u^a, derive equations (4.2.12) and (4.2.13) of Wald:

$$u^a \partial_a \rho + (\rho + P)\, \partial^a u_a = 0 \tag{4.2.12}$$

$$(P + \rho)\, u^a \partial_a u_b + (\eta_{ab} + u_a u_b)\, \partial^a P = 0 \tag{4.2.13}$$

c) Show that in the non-relativistic limit, $P \ll \rho$, $u^\mu = (1, \vec{v})$, and $v\, \partial P/\partial t \ll |\vec{\nabla} P|$, the above equations become

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{4.2.14}$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} \right) = -\vec{\nabla} P \tag{4.2.15}$$

Problem 3: Consider the 4-acceleration, a^b,

$$a^b = u^a \nabla_a u^b,$$

of an arbitrary observer with 4-velocity u^a ($u^a u_a = -1$). Show that if a^b is non-vanishing then it is a spacelike vector.

Problem 4 (Wald 4.1): Show that Maxwell's equation

$$\nabla^a F_{ab} = -4\pi j_b$$

implies strict charge conservation, $\nabla_a j^a = 0$.

Problem 5 (Wald 4.3) a) Show that in the limit discussed near the end of section 4.4a of Wald—where the lowest order effects of source motion are taken into account—the geodesic equation for a timelike test particle becomes

$$\vec{a} = -\vec{E} - 4\, \vec{v} \times \vec{B}$$

where the gravitational electric and magnetic fields are defined by analogy with the electromagnetic case:

$$F_{ab} = \partial_a A_b - \partial_b A_a \qquad E_a = F_{ab} t^b \qquad B_a = -\frac{1}{2}\epsilon_{ab}{}^{cd} F_{cd} t^b$$

with the vector potential, A_a, defined by

$$A_a \equiv -\frac{1}{4}\bar{\gamma}_{ab} t^b .$$

(Note that t^b is the vector associated with the time coordinate of a global inertial coordinate system of η_{ab}.)

b) Show that the "gravitational electric and magnetic fields" $\vec{E}$ and $\vec{B}$ inside a spherical shell of mass M and radius R (with $M \ll R$) slowly rotating with angular velocity $\vec{\omega}$ are

$$\vec{E} = 0 \qquad \vec{B} = \frac{2}{3}\frac{M}{R}\vec{\omega}$$

Problem 6 (Wald 4.4): Starting with the following equation for the components of the Ricci tensor R_{ab}

$$\begin{aligned} R_{\mu\rho} &= \sum_\nu R_{\mu\nu\rho}{}^\nu \\ &= \sum_\nu \frac{\partial}{\partial x^\nu}\Gamma^\nu{}_{\mu\rho} - \frac{\partial}{\partial x^\mu}\left(\sum_\nu \Gamma^\nu{}_{\nu\rho}\right) + \sum_{\alpha\,,\,\nu}\left(\Gamma^\alpha{}_{\mu\rho}\Gamma^\nu{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\nu{}_{\alpha\mu}\right) \end{aligned}$$

derive the formula for $R^{(2)}_{ab}$

$$\begin{aligned} R^{(2)}_{ab} = \frac{1}{2}\gamma^{cd}\partial_a\partial_b\gamma_{cd} - \gamma^{cd}\partial_c\partial_{(a}\gamma_{b)d} \quad &+ \quad \frac{1}{4}\left(\partial_a\gamma_{cd}\right)\partial_b\gamma^{cd} + \left(\partial^d\gamma^c{}_b\right)\partial_{[d}\gamma_{c]a} + \frac{1}{2}\partial_d\left(\gamma^{dc}\partial_c\gamma_{ab}\right) \\ &- \quad \frac{1}{4}\left(\partial^c\gamma\right)\partial_c\gamma_{ab} - \left(\partial_d\gamma^{cd} - \frac{1}{2}\partial^c\gamma\right)\partial_{(a}\gamma_{b)c} \end{aligned}$$

by substituting $\eta_{ab} + \gamma_{ab}$ for g_{ab} and keeping precisely the terms quadratic in γ_{ab}.