Static Spherically Symmetric Spacetime

Derivation of Einstein Equations for Vacuum and Perfect Fluid Cases Using grtensor

References: Wald, Sections 6.1 and 6.2
grtensor documentation

> grtensor():

> makeg(schwarztetrad);

```
Makeg 2.0: GRTensor metric/basis entry utility

To quit makeg, type 'exit' at any prompt.

Do you wish to enter a 1) metric [g(dn,dn)],
                       2) line element [ds],
                       3) non-holonomic basis [e1...e4]
, or
                       4) NP tetrad [l,n,m,mbar]?
```

3

Enter coordinates as a LIST (e.g. [r,theta,phi,t]):

[t,r,theta,phi]

```
Would you like to enter 1) covariant components,
                        2) contravariant components, or
                        3) both.
```

1

Enter the covariant components of basis vector '1' as a LIST (e.g. [1,0,0,0]):

[sqrt(f(r)),0,0,0]

Enter the covariant components of basis vector '2' as a LIST (e.g. [1,0,0,0]):

[0,sqrt(h(r)),0,0]

Enter the covariant components of basis vector '3' as a LIST (e.g. [1,0,0,0]):

[0,0,r,0]

Enter the covariant components of basis vector ’4’ as a LIST (e.g. [1,0,0,0]):

[0,0,0,r*sin(theta)]

Is the basis inner product 1) Diagonal, or
2) Symmetric?

1

Enter eta[1,1]

-1

Enter eta[2,2]

1

Enter eta[3,3]

1

Enter eta[4,4]

1

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET (eg. { z, psi }).

Complex quantities [default={}]:

{}

The values you have entered are:

$$Coordinates = [\, t, r, \theta, \phi \,]$$

Basis 1-forms:

$$\omega_1 = \left[\sqrt{\mathrm{f}(r)} \quad 0 \quad 0 \quad 0 \right]$$

$$\omega_2 = \left[0 \quad \sqrt{\mathrm{h}(r)} \quad 0 \quad 0 \right]$$

$$\omega_3 = [\, 0 \quad 0 \quad r \quad 0 \,]$$

$$\omega_4 = [\, 0 \quad 0 \quad 0 \quad r \sin(\theta) \,]$$

$$\textit{Inner product of basis vectors:}$$

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
You may choose to 0) Use the metric WITHOUT saving it,
                  1) Save the metric as it is,
                  2) Re-enter a basis vector,
                  3) Re-enter the inner product,
                  4) Add/change constraints,
                  5) Add a text description, or
                  6) Abandon this metric and return to
Maple.
```

0

$$\textit{Default spacetime} = \textit{schwarztetrad}$$

$$\textit{For the schwarztetrad spacetime:}$$

$$\textit{Coordinates}$$

$$\mathrm{x}(\textit{up})$$

$$x^a = [\, t \quad r \quad \theta \quad \phi \,]$$

Basis inner product

$$\eta(bup, bup)$$

$$\eta^{(a)\ (b)} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basis (covariant components)

$$\mathrm{w1}(dn)$$

$$\omega 1_{a} = \left[\sqrt{\mathrm{f}(r)} \quad 0 \quad 0 \quad 0 \right]$$

$$\mathrm{w2}(dn)$$

$$\omega 2_{a} = \left[0 \quad \sqrt{\mathrm{h}(r)} \quad 0 \quad 0 \right]$$

$$\mathrm{w3}(dn)$$

$$\omega 3_{a} = [\, 0 \quad 0 \quad r \quad 0 \,]$$

$$\mathrm{w4}(dn)$$

$$\omega 4_{a} = [\, 0 \quad 0 \quad 0 \quad r \sin(\theta) \,]$$

`makeg completed.`

Use the alias function to make the r-dependence of some key functions implicit on input and output ...

> alias(f=f(r),h=h(r),m=m(r),Phi=Phi(r),P=P(r),rho=rho(r));

$$I, f, h, m, \Phi, P, \rho$$

Compute and display orthonormal-basis-components of the Einstein tensor (note use of 'bdn' rather than 'dn')

> grcalc(G(bdn,bdn)); grdisplay(_);

```
Created a definition for    e(bdn,dn,pdn)
Created definition for rot(bdn,bup,bdn)
Calculating e(bup,dn) for schwarztetrad ... Done. (0.02
0000 sec.)
Calculating g(dn,dn) for schwarztetrad ... Done. (0.040
000 sec.)
Calculating detg for schwarztetrad ... Done. (0.000000
sec.)
Calculating g(up,up) for schwarztetrad ... Done. (0.070
000 sec.)
Calculating e(bdn,up) for schwarztetrad ... Done. (0.02
0000 sec.)
Calculating e(bdn,dn,pdn) for schwarztetrad ... Done. (
0.050000 sec.)
Calculating lambda(bdn,bdn,bdn) for schwarztetrad ... D
one. (0.120000 sec.)
Calculating rot(bdn,bdn,bdn) for schwarztetrad ... Done
. (0.030000 sec.)
Calculating rot(bdn,bup,bdn) for schwarztetrad ... Done
. (0.080000 sec.)
Calculating R(bdn,bdn,bdn,bdn) for schwarztetrad ... Do
ne. (0.100000 sec.)
Calculating R(bdn,bdn) for schwarztetrad ... Done. (0.0
40000 sec.)
Calculating Ricciscalar for schwarztetrad ... Done. (0.
020000 sec.)
```

```
Calculating G(bdn,bdn) for schwarztetrad ... Done. (0.0
30000 sec.)
```

$$CPU\ Time\ = .870$$

For the schwarztetrad spacetime:

Covariant Einstein

$$\mathrm{G}(bdn, bdn)$$

$$G_{(a)\ (b)} =$$

$$\left[\frac{\left(\frac{\partial}{\partial r} h\right) r - h + h^2}{h^2\, r^2}, 0, 0, 0 \right]$$

$$\left[0, \frac{r \left(\frac{\partial}{\partial r} f\right) + f - f\, h}{f\, h\, r^2}, 0, 0 \right]$$

$$\left[0, 0, -\frac{1}{4}\left(-2 \left(\frac{\partial}{\partial r} f\right) h\, f + 2 \left(\frac{\partial}{\partial r} h\right) f^2 + r \left(\frac{\partial}{\partial r} f\right)^2 h \right.\right.$$
$$\left.\left. - 2\, r\, h \left(\frac{\partial^2}{\partial r^2} f\right) f + r \left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial r} h\right) f \right) \Big/ (r\, h^2\, f^2), 0 \right]$$

$$\left[0, 0, 0, -\frac{1}{4}\left(-2 \left(\frac{\partial}{\partial r} f\right) h\, f + 2 \left(\frac{\partial}{\partial r} h\right) f^2 + r \left(\frac{\partial}{\partial r} f\right)^2 h \right.\right.$$
$$\left.\left. - 2\, r\, h \left(\frac{\partial^2}{\partial r^2} f\right) f + r \left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial r} h\right) f \right) \Big/ (r\, h^2\, f^2) \right]$$

Save the non-vanishing components in "temporary" vbls. for subsequent manipulation ...

> G00 := grcomponent(G(bdn,bdn) , [t,t]);

$$G00 := \frac{\left(\frac{\partial}{\partial r} h\right) r - h + h^2}{h^2\, r^2}$$

> G11 := grcomponent(G(bdn,bdn) , [r,r]);

$$G11 := \frac{r \left(\frac{\partial}{\partial r} f\right) + f - f\, h}{f\, h\, r^2}$$

> G22 := grcomponent(G(bdn,bdn) , [theta,theta]);

$$G22 := -\frac{1}{4}\left(-2\left(\frac{\partial}{\partial r} f\right) h\, f + 2\left(\frac{\partial}{\partial r} h\right) f^2 + r\left(\frac{\partial}{\partial r} f\right)^2 h - 2\, r\, h\left(\frac{\partial^2}{\partial r^2} f\right) f + r\left(\frac{\partial}{\partial r} f\right)\left(\frac{\partial}{\partial r} h\right) f\right) \Big/ (r\, h^2\, f^2)$$

> G33 := grcomponent(G(bdn,bdn) , [phi,phi]) :

> G22 - G33;

$$0$$

Check Wald's expression (6.2.5) for G00

> WG22 := diff(diff(f,r)/sqrt(f*h),r) / (2*sqrt(f*h)) + diff(f,r) / (2*r*f*h) - diff(h,r) / (2*r*h^2);

$$WG22 := \frac{1}{2}\,\frac{\dfrac{\frac{\partial^2}{\partial r^2} f}{\sqrt{f\,h}} - \dfrac{1}{2}\,\dfrac{\left(\frac{\partial}{\partial r} f\right)\left(\left(\frac{\partial}{\partial r} f\right) h + f\left(\frac{\partial}{\partial r} h\right)\right)}{(f\,h)^{3/2}}}{\sqrt{f\,h}} + \frac{1}{2}\,\frac{\frac{\partial}{\partial r} f}{f\,h\,r} - \frac{1}{2}\,\frac{\frac{\partial}{\partial r} h}{h^2\,r}$$

> simplify(WG22-G22);

$$0$$

A useful combination of Einstein components

> expand(r*h*(G00 + G11));

$$\frac{\frac{\partial}{\partial r} h}{h} + \frac{\frac{\partial}{\partial r} f}{f}$$

Eliminating 'h' from G11

> subs(h = 1/f , r^2 *G11);

$$r\left(\frac{\partial}{\partial r} f\right) + f - 1$$

Einstein and TOV equations for a static perfect fluid star

First define transformed metric variables ...

```
> h := 1/(1 - 2*m/r);
```

$$h := \frac{1}{1 - 2\,\frac{m}{r}}$$

```
> f := exp( 2*Phi );
```

$$f := e^{(2\,\Phi)}$$

Derive the ODE for m(r) from the 00 component of the Einstein equation (aka the Hamiltonian constraint)

```
> dmdr   := simplify( solve( G00=8*pi*rho, diff(m,r) ) );
```

$$dmdr := 4\,\pi\,\rho\,r^2$$

Derive the ODE for Phi(r) from the 11 component of the Einstein equation

```
> dPhidr := simplify( solve( G11=8*pi*P, diff(Phi,r) ) );
```

$$dPhidr := -\frac{m + 4\,\pi\,P\,r^3}{r\,(-r + 2\,m)}$$

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*Derive the ODE for the fluid pressure P(r) (the TOV equation of hydrostatic equilibrium) from the 33 component of the Einstein equation. Calculation is done in stages using the temporary 'G22a' to illustrate technique of continuous simplification of intermediate results. Note in particular the use of an explicit 'exp' argument to simplify in order to effect exp(Phi)*exp(-Phi) -> 1.*

> G22a := simplify(simplify(subs(diff(Phi,r)=dPhidr, G22)),exp);

$$G22a := -\left(8\,\pi\,P\,r^3 - 12\,\pi\,P\,r^2\,m + \left(\frac{\partial}{\partial r}\,m\right)m + 16\,\pi^2\,P^2\,r^5 + 4\,\pi\left(\frac{\partial}{\partial r}\,P\right)r^4 - 8\,\pi\left(\frac{\partial}{\partial r}\,P\right)r^3\,m + 4\,\pi\,P\,r^3\left(\frac{\partial}{\partial r}\,m\right)\right)\Big/\left(r^2\,(-r + 2\,m)\right)$$

> G22a := simplify(subs(diff(m,r)=dmdr,G22a));

$$G22a := -4\,\pi\left(2\,P\,r - 3\,P\,m + \rho\,m + 4\,\pi\,P^2\,r^3 + \left(\frac{\partial}{\partial r}\,P\right)r^2 - 2\left(\frac{\partial}{\partial r}\,P\right)r\,m + 4\,\pi\,P\,r^3\,\rho\right)\Big/(-r + 2\,m)$$

**> dPdr := factor(simplify(solve(G22a = 8*pi*P,diff(P,r))
));**

$$dPdr := \frac{(P+\rho)\,(m+4\,\pi\,P\,r^3)}{r\,(-r+2\,m)}$$