Although the solution given by a finite element method is by definition continuous, on irregular meshes this continuous frunction is insufficient to properly calculate numerical derivatives. We will consider this problem from two angles. First we lay down some notation,

$$L[u] = 0 \tag{1}$$

is the second order, partial differential equation that we want a numerical solution to. The solution is two dimensional,

$$u : \Omega \to \mathbb{R} \tag{2}$$

$$\Omega \subset \mathbb{R}^2. \tag{3}$$

Then we formulate the problem in its weak formulation,

$$L[u, v] = 0, \tag{4}$$

where v is any continuous function with $v|_{\partial\Omega} = 0$. Then by choosing a number of nodes in Ω

$$N = \{x_i | i = 1, 2, ..., n_N, x \in \Omega\} \tag{5}$$

and connect them into a partitioning triangulation of Ω, that is $x_i(K_j)$ is the i^{th} vertex (up to three) of the triangle K_j, the partition is then given by the union of elements

$$\begin{aligned} K_i = \{x \in \Omega| \quad & l_1 x_1(K_i) + l_2 x_2(K_i) + l_3 x_3(K_i) = x, \\ & l_1 + l_2 + l_3 = 1,\ l_1, l_2, l_3 > 0, x_j(K_i) \in N\}, \end{aligned} \tag{6}$$

with the additional stipulations that

$$K_i \cap K_j = \begin{cases} 0 & \text{if } i \neq j; \\ K_i & \text{if } i = j, \end{cases} \tag{7}$$

$$\bigcup_{i=1}^{n_K} K_i = \Omega, \tag{8}$$

and

$$x_i \notin K_j \text{ for } i = 1, 2, ...n_N \text{ unless } x_i = x_k(K_j) \text{ for } k \in [1, 2, 3]. \tag{9}$$

Then we consider spaces

$$H^1(\Omega_h) = \{u : \Omega \to \mathbb{R}|\ u|_{K_i} \in \mathbb{P}_1\} \tag{10}$$

$$H_0^1(\Omega_h) = \{u : \Omega \to \mathbb{R}|\ u|_{K_i} \in \mathbb{P}_1,\ u|_{\partial\Omega} = 0\}, \tag{11}$$

then define the finite element solution as the function $u_h \in H^1(\Omega_h)$ such that, for all $v \in H_0^1(\Omega_h)$ we have that

$$L(u_h, v) = 0. \tag{12}$$

We assume that the finite element method described is second order, and that

$$\left(\int_\Omega (u-u_h)^2\right)^{\frac{1}{2}} < Ch^2. \tag{13}$$

Thus we can generally write that

$$u_h = u + h^2 e(h). \tag{14}$$

Now we consider the standard second order, centred difference operator for the approximation of the Laplacian

$$\triangle_H = \triangle + H^2 \mathcal{L}(H), \tag{15}$$

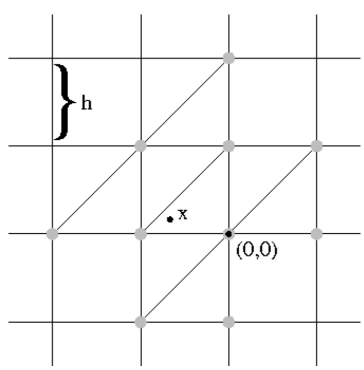
where $\mathcal{L}$ is a higher order differential operator. Applying this to to u_h and considering the error, compared to the desired result

$$\triangle_H u_h - \triangle u = H^2 \mathcal{L}(H)[u] + h^2 \triangle_H e(h). \tag{16}$$

It is necessary that $H > h$, because if $H \lesssim h$, then u_h is piecewise linear on the scale of the finite difference operator and will spuriously vanish. On the necessary scales the error function $e(h)$ is not smooth and thus the finite difference approximation of the derivative is ill posed. The denominator necesarily contains a expression quadratic in the stepsize, so

$$\triangle_H u_h - \triangle u = H^2 \mathcal{L}(H)[u] + O(1)\frac{h^2}{H^2}, \tag{17}$$

and the result is in general an $O(H^{-2})$ quantity, or at best $O(1)$ for an equal scaling.



Now we consider a regular triangular mesh with spacing h over a rectangular domain Ω and a smooth function $u : \Omega \to \mathbb{R}$ with $u \in C^\infty$. We consider calculating an approximation of the Laplacian at a point, x, located inside of an element. We define the linear interpolant of u as the function which is identical to its smooth analog at the nodes of the mesh,

$$\mathbb{I}_1 u(x) = u(x) \text{ for } x \in N \tag{18}$$

and is linear inside of the elements

$$\mathbb{I}_1 u(x) \in H^1(\Omega_h). \tag{19}$$

To calculate the Laplacian we consider the second order, centred finite difference formula

$$\begin{aligned}\triangle_H[u](x) = & \frac{u(x+(0,H)) - 2u(x) + u(x-(0,H))}{H^2} \\ & + \frac{u(x+(H,0)) - 2u(x) + u(x-(H,0))}{H^2}.\end{aligned} \tag{20}$$

Using barycentric coordinates, we can label the point x in the above figure as

$$x = l_1(0,0) + l_2(-h,0) + l_3(0,h) \tag{21}$$

with

$$l_1 + l_2 + l_3 = 1, \text{ and } l_1, l_2, l_3 > 0. \tag{22}$$

We can use this fact to simply compute the interpolant's value at the chosen points, where we will choose $H = h$ for the finite difference formula,

$$\mathbb{I}_1 u(x) = l_1 u(0,0) + l_2 u(-h,0)_+ l_2 u(0,h), \tag{23}$$

$$\mathbb{I}_1 u(x \pm (h,0)) = l_1 u(\pm h,0) + l_2 u(-h \pm h,0)_+ l_2 u(\pm h,h), \tag{24}$$

$$\mathbb{I}_1 u(x \pm (0,h)) = l_1 u(0,\pm h) + l_2 u(-h,\pm h)_+ l_2 u(0,h \pm h). \tag{25}$$

It is now quite evident that

$$\begin{aligned}\triangle_h[\mathbb{I}_1 u](x) = & \frac{\mathbb{I}_1 u(x+(0,h)) - 2\mathbb{I}_1 u(x) + \mathbb{I}_1 u(x-(0,h))}{h^2} \\ & + \frac{\mathbb{I}_1 u(x+(h,0)) - 2\mathbb{I}_1 u(x) + \mathbb{I}_1 u(x-(h,0))}{h^2}.\end{aligned} \tag{26}$$

leads to

$$\triangle_h[\mathbb{I}_1 u](x) = l_1 \triangle_h [\mathbb{I}_1 u](0,0) + l_2 \triangle_h [\mathbb{I}_1 u](-h,0) + l_3 \triangle_h [\mathbb{I}_1 u](0,h), \tag{27}$$

and because of the exactness of the nodes

$$\triangle_h[\mathbb{I}_1 u](x) = l_1 \triangle_h [u](0,0) + l_2 \triangle_h [u](-h,0) + l_3 \triangle_h [u](0,h) = \mathbb{I}_1[\triangle_h u](x), \tag{28}$$

standard interpolation theory describes this as an $O(h^2)$ approximation for the Laplacian.

We now consider a simple irregular mesh (see figure below), where the only difference is a different orientation of two triangles compared with the previous mesh. Using the same barycentric coordinates, and finite difference formula we arrive at the same equations for the interpolated values except

$$\mathbb{I}_1 u(x - (0,h)) = l_4 u(0,0) + l_5 u(-h,0) + l_6 u(-h,-h) + l_7 u(0,-h). \tag{29}$$

We will assume without loss of generality that $l_4 = 0$. Next, due to choice of the origin of the coordinate system

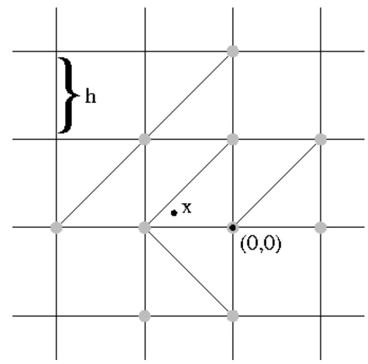
$$x = (-l_2 h, l_3 h), \tag{30}$$

so that we can form the linear equations

$$x - (0,h) = (-l_2 h, l_3 h - h) = (-(l_5 + l_6)h, -(l_6 + l_7)h), \tag{31}$$

which are identical to

$$l_1 = l_7 - l_5, \tag{32}$$
$$l_2 = l_5 + l_6, \tag{33}$$
$$l_3 = -(-1 + l_6 + l_7) = l_5. \tag{34}$$



When we compute the finite difference formulas this time, we can pull out the previous result with parts leftover, that is

$$\triangle_h[\mathbb{I}_1' u](x) = \triangle_h[\mathbb{I}_1 u](x) + e, \tag{35}$$

where e is given by

$$e = \frac{l_5 u(-h,0) + (l_6 - l_2)u(-h,-h) + (l_7 - l_1)u(0,-h) - l_3 u(0,0)}{h^2}, \tag{36}$$

looking at the formulas for l_1, l_2, and l_3 we see that this error term can be expressed as

$$e = \frac{l_5 u(-h,0) - l_5 u(-h,-h) + l_5 u(0,-h) - l_5 u(0,0)}{h^2}. \tag{37}$$

A little knowledge of some of the more rare finite difference formulas recognizes this error term as

$$e = l_5 \frac{\partial^2 u}{\partial x_0 \partial x_1}\bigg|_{(\frac{h}{2},\frac{h}{2})} + O(h^2). \tag{38}$$

In a general irregular mesh there will be contributions from mixed derivatives at many locations.