A Numerical Study of Boson Star Binaries

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Outline

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- ADM / 3+1 Formalism
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Motivation

- Why study compact binaries?
  - One of most promising sources of gravitational waves
  - Terrestrial detectors (like LIGO) were constructed to detect gravitational waves in the frequency range corresponding to a BH-BH and NS-NS collisions
  - Even for these sources the signal strength is expected to be much less than the detectors' noise. An accurate theoretical model of compact binary inspiral can help extract the signal from the noise
  - It is a good laboratory to study the phenomenology of strong gravitational fields

Motivation

- Why boson stars?
  - The study of the entire inspiral of compact objects can be separated in three phases
    - Inspiral phase that can be studied by post-Newtonian methods
    - Plunge and merge phase beginning at the innermost stable circular orbit and demanding numerical relativity
    - Ringdown phase when the merged object (typically a distorted black hole) settles down to equilibrium and can be studied by perturbative techniques
  - Plunge and merge phase is characterized by a strong dynamical gravitational field. In this regime gross features of fluid and boson stars' dynamics may be similar
  - Since the details of the dynamics of the stars (e.g. shocks) tend not to be important gravitationally, boson star binaries may provide some insight into NS binaries

Motivation

- Development of a computational infrastructure for 3D codes
  - 3D numerical relativistic calculations are computationally very expensive
    - Any calculation done using a 3D uniform finite difference method scales as N^4, where N is the number of grid points along one dimension
    - Moore's law asserts that processing speed is roughly doubled every 1.5 yr
    - Then in order to gain in resolution by a factor of $3-4$ for instance we need about 10 yr of processor development
  - Need for more efficient computational techniques
    - Adaptive Mesh Refinement (AMR) allows resolution to vary locally in response to solution features
    - Parallelization - Share the numerical task between many processors
    - This infrastructure is being constructed by Frans Pretorius

Numerical Relativity and its goal

- Einstein Field Equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

  - System of non-linear, time-dependent, partial differential equations
  - No analytic solution except in special cases
  - Solution for most relevant astrophysical scenarios must be constructed numerically
  - Its tensorial nature gives rise to several different formalisms
  - ADM / $3+1$ formalism: slice spacetime in spacelike hypersurfaces; use Einstein equations to evolve in time the 3-geometry of an initial hypersurface in order to construct the spacetime (i.e. the 4-dimensional metric, $g_{\mu\nu}$)

Matter Model: Scalar Field

- A massive complex field is chosen as matter source because it is a simple type of matter that allows a star-like solution and because there will be no problems with shocks, low density regions, ultrarelativistic flows, etc in the evolution of this kind of matter as opposed to fluids

- The matter content is described by the scalar field:

$$\Phi = \phi_1 + i\phi_2 \tag{1}$$

  where ϕ_1 and ϕ_2 are real-valued

- The Lagrangian density associated with this field is given by:

$$L_\Phi = -\frac{1}{8\pi}(g^{ab}\nabla_a\Phi\nabla_b\Phi^* + m^2\Phi\Phi^*) \tag{2}$$

- Extremizing this action with respect to each component of the scalar field, we get the Klein-Gordon equation

$$\Box\phi_A - m^2\phi_A = 0 \qquad A = 1, 2 \tag{3}$$

ADM / 3+1 Formalism

- ADM formalism is a way of writing Einstein eqns as a Cauchy problem or Initial Value formalism

- Manifold with metric $(M, g_{\mu\nu})$ foliated by spacelike hypersurfaces Σ_t

- Coordinates $x^\mu = (t, x^i)$

- Future directed, time unit normal to the hypersurfaces

$$n^\mu = -\alpha \nabla^\mu t$$

  where α is the lapse function

- Shift vector β^μ defined via

$$t^\mu = \alpha n^\mu + \beta^\mu$$
$$\beta^\mu n_\mu = 0$$

ADM / 3+1 Formalism

- Hypersurface metric $\gamma_{\mu\nu}$ induced by $g_{\mu\nu}$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

- Mixed form of $\gamma_{\mu\nu}$ projects into hypersurface

$$\perp^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$$

- Metric compatible covariant derivative in slices

$$D_\mu \equiv \perp^\nu{}_\mu \nabla_\nu$$

$$D_\mu \gamma_{\alpha\beta} = 0$$

ADM / 3+1 Formalism

- 3+1 line element

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

- Extrinsic curvature (second fundamental form)

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$$

- 3+1 form of Einstein's equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ derived by considering various projections of Einstein/Ricci and stress-energy tensors

- Projections of $T_{\mu\nu}$

$$\begin{aligned} \rho &\equiv n^\mu n^\nu T_{\mu\nu} \\ j_\mu &\equiv -\perp^\alpha{}_\mu n^\beta T_{\alpha\beta} \\ S_{\mu\nu} &\equiv \perp^\alpha{}_\mu \perp^\beta{}_\nu T_{\alpha\beta} \end{aligned}$$

ADM / 3+1 Formalism

- From the point of view of ADM formalism the Hamiltonian formulation of the dynamics of scalar field is more useful

- The conjugate momentum field is defined as

$$\sigma_A \equiv \frac{\delta(\sqrt{-g}L_{\phi_A})}{\delta\dot{\phi}_A} \tag{4}$$

- In terms of these fields, the dynamical equations are given by

$$\begin{aligned} \partial_t\phi_A &= \frac{\alpha^2}{\sqrt{-g}}\sigma_A + \beta^i\partial_i\phi_A && (5) \\ \partial_t\sigma_A &= \partial_i(\beta^i\sigma_A) + \partial_i(\sqrt{-g}\gamma^{ij}\partial_j\phi_A) - \sqrt{-g}m^2\phi_A && (6) \end{aligned}$$

ADM / 3+1 Formalism

- The stress-energy tensor is given by

$$T_{ab} = -2\frac{\delta L_\Phi}{\delta g^{ab}} + g_{ab}L_\Phi \tag{7}$$

- We have the following ADM components of the stress tensor

$$\begin{aligned}
\rho &= n^\mu n^\nu T_{\mu\nu} = \frac{1}{8\pi}\sum_{A=1}^{2}\Big(\frac{\alpha^2}{(-g)}\sigma_A^2 + \gamma^{ij}\partial_i\phi_A\partial_j\phi_A + m^2\phi_A^2\Big) \\
j^i &= -n^\mu T_\mu{}^i = \frac{1}{8\pi}\sum_{A=1}^{2}\Big(-2\frac{\alpha\sigma_A}{\sqrt{-g}}\gamma^{ij}\partial_j\phi_A\Big) \\
S_{ij} &= T_{ij} \\
&= \frac{1}{8\pi}\sum_{A=1}^{2}\Big(2\partial_i\phi_A\partial_j\phi_A + \gamma_{ij}\Big[\frac{\alpha^2\sigma_A^2}{(-g)} - \gamma^{mn}\partial_m\phi_A\partial_n\phi_A - m^2\phi_A^2\Big]\Big)
\end{aligned} \tag{8}$$

ADM / 3+1 Formalism

- Constraint Equations: From $G_{0i} = 8\pi T_{0i}$, which do not contain 2nd time derivatives of the γ_{ij}
- Hamiltonian Constraint

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \tag{9}$$

where R is the 3-dim. Ricci scalar, and $K \equiv K^i{}_i$ is the mean extrinsic curvature.

- Momentum Constraint

$$D_i K^{ij} - D^j K = 8\pi j^i \tag{10}$$

ADM / 3+1 Formalism

- Evolution Equations: From definition of extrinsic curvature, $G_{ij} = 8\pi T_{ij}$, and Ricci's equation.

$$
\begin{aligned}
\mathcal{L}_t \gamma_{ij} &= \mathcal{L}_\beta \gamma_{ij} - 2\alpha K_{ij} && (11)\\
\mathcal{L}_t K_{ij} &= \mathcal{L}_\beta K_{ij} - D_i D_j \alpha + \alpha \left(R_{ij} + K K_{ij} - 2 K_{ik} K^k{}_j \right) - \\
& \quad 8\pi\alpha (S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) && (12)
\end{aligned}
$$

- Constraints / coordinate freedom lead to many options in how discrete solutions of Einstein equations are advanced from one time step to another

- Among them, fully constrained evolution provides more stable discrete schemes. It consists in re-solve the constraint equations at each time step and using the coordinate degree of freedom to eliminate dynamical variables, leaving only two dynamical variables to be evolved in time

Conformal Flat Approximation (CFA)

- Motivation
  - Full 3D Einstein equations are very complex and computationally expensive to solve
  - Heuristic assumption that the dynamical degrees of freedom of the gravitational fields, i.e. the gravitational radiation, play a small role in at least some phases of the strong field interaction of a merging binary
  - Gravitational radiation is small in most systems studied so far
  - CFA effectively eliminates the two dynamical degrees of freedom, simplifies the equations and allows a fully constrained evolution
  - CFA allows us to investigate the same kind of phenomena observed in the full relativistic case, such as the description of compact objects and the dynamics of their interaction; black hole formation; critical phenomena
  - CFA has been used in the past with promising results in certain cases (Wilson-Matthews studies of coalescence of neutron stars; Bruno Rousseau's master's thesis)

Conformal Flat Approximation (CFA)

- Formalism
  - The CFA prescribes a conformally flat spatial metric at all times
  - Introduce a flat metric f_{ij} as a base / background metric:

$$\gamma_{ij} = \psi^4 f_{ij} \tag{13}$$

    where the conformal factor ψ is a positive scalar function describing the ratio between the scale of distance in the curved space and flat space($f_{ij} \equiv \delta_{ij}$ in cartesian coordinates)
  - In this approximation all of the geometric variables can be computed from the constraints as well as from a specific choice of coordinates
  - Maximum slicing condition is used to fix the time coordinate

$$\begin{aligned} K_i^i &= 0 \\ \partial_t K_i^i &= 0 \end{aligned} \tag{14}$$

Conformal Flat Approximation (CFA)

- Slicing Condition
  - Gives an elliptic equation for the lapse function α

$$\nabla^2 \alpha = \cdots \tag{15}$$

- Hamiltonian Constraint
  - Gives an elliptic equation for the conformal factor ψ

$$\nabla^2 \psi = \cdots \tag{16}$$

- Momentum Constraints
  - Given elliptic equations for the shift vector components β^i

$$\nabla^2 \beta^i = \cdots \tag{17}$$

Conformal Flat Approximation (CFA)

- Then the following set of functions completely characterize the geometry at each time slice

$$\alpha = \alpha(t, \vec{r}), \quad \psi = \psi(t, \vec{r}), \quad \beta^i = \beta^i(t, \vec{r}) \tag{18}$$

  where $\vec{r}$ depends on the coordinate choice for the spatial hypersurface

- The solution of the gravitational system under CFA and maximal slicing condition can be summarized as:
  - Specify initial conditions for the complex scalar field
  - Solve the elliptic equations for the geometric quantities on the initial slice
  - Update the matter field values to the next slice using their equation of motion
  - For the new configuration of matter fields, re-solve the elliptic equations for the geometric variables and again allow the matter fields to react and evolve to the next slice and so on

Conformal Flat Approximation (CFA)

- Counting Argument - full geometrodynamics
  - 4 Kinematical variables:$\{\alpha, \beta^i\}$
  - 6 pairs dynamically conjugate variables:$\{\gamma_{ij}, K_{ij}\}$
  - 4 constraint equations
  - 12 evolution equations
- Geometrodynamics under CFA:
  - CFA reduces γ_{ij} from 6 to 1 variable
  - Demand that CFA holds at all times plus momentum constraints allows us to ignore all K_{ij} evolution equations excepts for $\dot{K}^i_i = 0$
  - End up with a set of 5 geometric fields α, β^i, ψ fixed by elliptic equations
- It is worth noting that CFA is not self consistent in the sense of being derivable from a lagrangian

Previous Work

- Wilson, Matthews, Marronetti, (1996) Phys. Rev. D 54, 1317
  - Study of general relativistic hydrodynamics of a coalescing neutron-star binary system
  - They discuss the evidence that, for a realistic neutron-star equation of state, general relativistic effects may cause the stars to individually collapse into black holes prior to merging
  - Strong fields cause the last stable orbit (ISCO) to occur at a larger separation distance and lower frequency than previously estimated. This is important, since it places the coalescence closer to the maximum sensitivity range of the LIGO detectors and others

- Bruno Rousseau' masters thesis
  - Boson stars studied in axisymmetry under conformally flat approximation have been shown to behave similarly to the spherical solutions of the Einstein-Klein-Gordon equations under small perturbation

Current Project - Coalescence of Boson Stars

- Motivation
  - Wilson-Mathews compression effect results raised a controversy about the validity of the conformal flat approximation
  - In order to decide if CFA is a good approximation to model compact binaries it would be interesting to simulate it using a simpler model
  - Fluid stars and Boson stars have some similarity concerning the way they are modelled, e.g. both can be parametrized by their central density ρ_0 and have qualitatively similar plots of total mass vs ρ_0
  - Then in the strong field regime for the compact binary system the dynamics may not depend sensitively on the details of the model
  - Advantage of using scalar fields: no problems with shocks, evolution done by Klein-Gordon eqn, should not present any stability problem.

Current Project - Coalescence of Boson Stars

- Questions to be addressed
  - Would the individual collapse occur before merging for boson stars as well or it is model dependent?
  - How good is the approximation? How do we test if the results are close to solutions of Einstein equations?
  - Is the individual collapse a spurious result coming from CFA?
  - What is the final result of the merging? Can we compare to results from other techniques?
  - Where is the ISCO? Does this result match to the fluid star ones? Can be at least qualitatively compared?

Current Project - Coalescence of Boson Stars

- Phases of the project
  - The final goal is to run detailed 3D simulations of boson stars in coalescence
  - Before starting the main project, small projects must be done
    - Next 4-6 months
      - IVP - generation of initial data for a boson star in spherical symmetry (1D code)
      - Multigrid techniques for solving the elliptic equations
      - Building a unigrid, serial 3D code compatible/ready for parallelization and AMR implementation
  - Thereafter
    - Modify the code for use of parallel adaptive infrastructure that is being constructed
    - Start investigating collisions