(1) PHYSICAL PROBLEM

(2) MATHEMATICAL PROBLEM

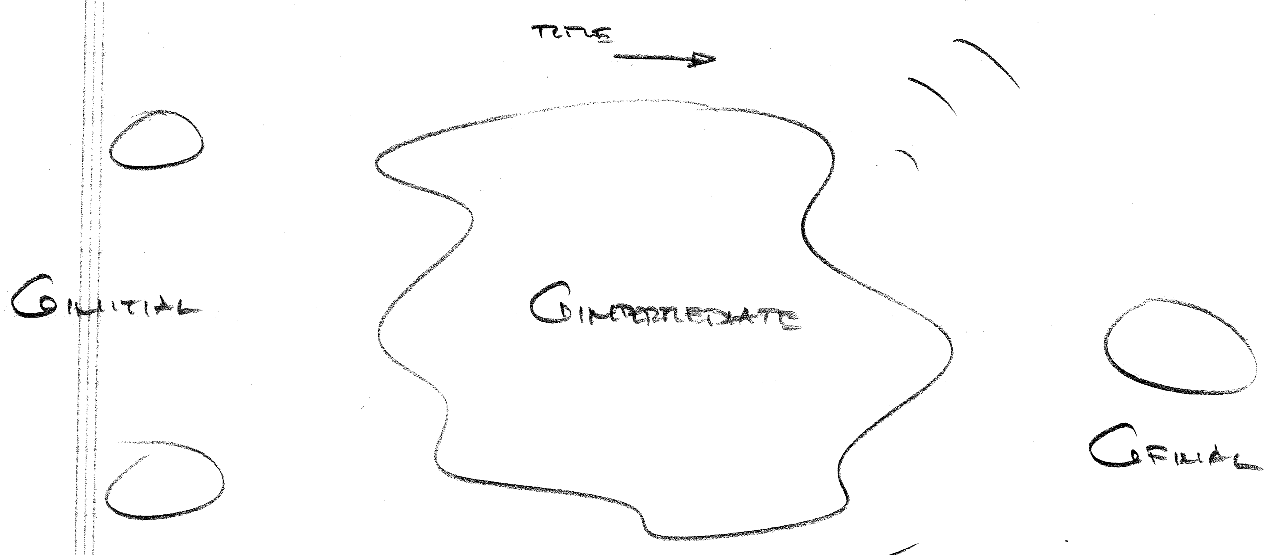
- "AD HOC" APPROACHES
- YORK / Ó MURCHADHA / LICHNEROWICZ (YOL) "CONFORMAL / SPIN DECOMPOSITION" APPROACH

PHYSICAL PROBLEM

- SPACETIME IS, BUT WE WANT TO CONSTRUCT IT VIA EVOLUTION OF THE GEOMETRY AT A PARTICULAR INSTANT OF TIME

- "GIGO" PROPERTY DEFINITELY APPLIES HERE

- FOLLOWING PICTURE IS IMPLICIT IN MOST COMPUTATIONAL STUDIES OF EINSTEIN'S EQUATIONS



- INITIAL & FINAL CONFIGURATIONS (IDEALLY) "WELL UNDERSTOOD"; EITHER "ANALYTICALLY" (CLOSED FORM, HIGH ACCURACY NUMERICAL SOLN)

OF AS PARTICULAR LIMIT (WEAK SELF-GRAVITY, SLOW MOTION, NEARLY-NEWTONIAN, ...)

- INTERMEDIATE CONFIGURATIONS TYPICALLY CHARACTERIZED BY NON-TRIVIAL DYNAMICS (TIME-DEPENDENCE) AND STRONG FIELD INTERACTIONS => DIRECT SOL^N of EOM (SIMULATION) TENDS TO PROVIDE MAX. PAYOFF

- VIS A VIS GIGO, HOW DO WE SET UP REALISTIC INITIAL DATA?

→ DEFN WILL DEPEND ON PHYSICAL PROBLEM BEING INVESTIGATED

TYPICAL ISSUE: WILL WANT TO COMPUTE GRAV. RAD. WAVEFORMS WHICH WE WOULD EXPECT TO SEE IN A. FLAT REGION - HOW DO WE KNOW THAT WE'RE NOT PUTTING IN SIG. % OF RADIATION "BY HAND" VIA SPEC. of I.D. ??

- NO GENERAL ANSWERS / MAJOR OPEN RESEARCH ISSUE - MAINLY BECAUSE RELATIVELY LITTLE HAS ACTUALLY BEEN DONE VIA CONSTRUCTIVE APPROACH

- CLEARLY, THERE IS A LOT of PHYSICS INVOLVED ~~HERE~~ - ONE REASON AD HOC SCHEMES LIKELY TO REMAIN USEFUL

MATHEMATICAL PROBLEM

(A) AD HOC APPROACHES

- HISTORICALLY, PART. USEFUL WHEN SYMMETRY ASSUMPTIONS ADOPTED: e.g. SPHERICAL SYMMETRY, CYLINDRICAL SYMMETRY, PLANE SYMMETRY, AXI- (AXIAL) SYMMETRY, TIME REFLECTION ($t \to -t$) SYMMETRY

(1) WRITE DOWN PARTICULAR ANSATZ FOR METRIC COMPATIBLE WITH SYMMETRIES AND COORDINATE CONDITIONS; ASSOCIATED ANSATZ FOR EXTRINSIC CURVATURE

(2) WRITE OUT CONSTRAINT EQN'S FROM ANSATZ

(3) CHOOSE PARTICULAR $\{\gamma_{ij}, K^i{}_j\}$ WHICH ARE TO BE FIXED BY CONSTRAINTS - SPECIFIC FORM OF CONSTRAINTS MAY SUGGEST GOOD CANDIDATES

(4) FREELY SPECIFY UNCONSTRAINED $\{\gamma_{ij}, K^i{}_j\}$, SOLVE CONSTRAINTS FOR REMAINING COMPONENTS

- WILL SEE SPECIFIC EXAMPLES LATER ON (SPHERICAL SYMMETRY)

(B) THE YORK / Ó MURCHADHA / LICHNEROWICZ "CONFORMAL / SPIN-DECOMPOSITION" APPROACH

REFERENCES

- YORK; PIRAN; THE INITIAL VALUE PROBLEM; BEYOND, IN "SPACETIME; GEOMETRY" MATZNER / SHEPLEY eds
- WALD APP. D CONFORMAL TRANSFORMATIONS

MATH. PRELIM. – CONFORMAL TRANSFORMATIONS

- MULTIPLY TENSOR FIELD BY SCALAR FIELD, DENOTED Ω (WALD), ψ (YORK), ϕ (OTHERS), RAISED TO SOME INTEGRAL (; GENERALLY NON-ZERO) POWER q

$$T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \to \tilde{T}^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = \Omega^q T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$$

WITH "CONFORMAL" – IF WE CONFORMALLY TRANSFORM METRIC, E.G., FOLLOWING WALD

DISTINCT SPACETIMES!

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \qquad (M, g) \to (M, \tilde{g})$$

THEN WE "PRESERVE ANGLES BUT NOT SCALES"; CONSIDER, FOR EXAMPLE, THE $\angle$ BETWEEN 2 SPACELIKE VECTORS t^a, v^a

$$t^a v_a = (t^a t_a)^{\frac{1}{2}} (v^b v_b)^{\frac{1}{2}} \cos(\theta)$$

$$\theta = \cos^{-1}\left[\frac{g_{ab} t^a v^b}{(g_{cd} t^c t^d)^{\frac{1}{2}} (g_{ef} v^e v^f)^{\frac{1}{2}}}\right]$$

CLEARLY, θ INVARIANT UNDER $g_{ab} \to \tilde{g}_{ab} = \Omega^p g_{ab}$

TWO MAIN IDEAS

(1) CONFORMAL SCALINGS OF 3-TENSORS

(2) "SPIN DECOMPOSITION" of K^{ij}

(1) CONFORMAL SCALING of 3-METRIC (OTHER 3-TENSORS TO FOLLOW)

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij} \tag{3}$$

$\hat{\gamma}_{ij}$ → BASE 3-METRIC (FREELY SPECIFIED)

γ_{ij} → PHYSICAL 3-METRIC

HERE, AS ELSEWHERE, THE CONFORMAL WEIGHT "4" IS CHOSEN PRIMARILY FOR MATH. CONVENIENCE

NOTE: $\hat{\ }$ → BASE (UNPHYSICAL) TENSOR, INDICES RAISED LOWERED WITH $\hat{\gamma}^{ij}$, $\hat{\gamma}_{ij}$

$$\hat{\gamma}^{ik}\hat{\gamma}_{kj} = \delta^i{}_j = \gamma^{ik}\gamma_{kj}$$

$$\to \quad \gamma^{ij} = \psi^{-4}\hat{\gamma}^{ij} \tag{4}$$

FOR OTHER TENSORS, SPECIFY WEIGHT FOR ONE "INDEX STRUCTURE": E.G.

$$A^{ij} = \psi^{-10}\hat{A}^{ij}$$

WEIGHTS FOR OTHER "INDEX STRUCTURES" THEN FOLLOW FROM (3)-(4)

$$A^i{}_j = \gamma_{jk}A^{ik} = (\psi^4)(\psi^{-10})\hat{\gamma}_{jk}\hat{A}^{ik} = \psi^{-6}\hat{A}^i{}_j$$

SIMILARLY, $A_{ij} = \psi^{-2}\hat{A}_{ij}$

- BUT CAUSAL STRUCTURE IS FIXED BY CONSIDERATION of "SPACETIME ANGLES" ⇒ CONFORMALLY RELATED SPACETIMES HAVE IDENTICAL CAUSAL STRUCTURE

- HOWEVER: USE of CONFORMAL TRANS IN YOL APPROACH HAS NO SUCH FUNDAMENTAL PHYSICAL SIGNIFICANCE; VIEW AS TECHNIQUE TO EXPEDITE MATHEMATICS

- KEY RESULT FROM WALD, APP D

IF $\tilde{g}_{ab} = \Omega^2 g_{ab}$ Ω SMOOTH STRICTLY POS FCN

$$\boxed{\tilde{R} = \Omega^{-2}\left(R - 2(n-1)\nabla^a\nabla_a \ln\Omega - (n-2)(n-1)(\nabla^a \ln\Omega)(\nabla_a \ln\Omega)\right)}$$

(D.9)

END MATH. PRELIMINARIES

- KEY FEATURE of YOL APPROACH: CASTS CONS EQNS

$$\mathcal{R} - K_{ij}K^{ij} + K^2 = 16\pi\rho \qquad (1)$$

$$D_j K^{ij} - D^i K = 8\pi j^i \qquad (2)$$

INTO SET of 4 QUASI-LINEAR (LINEAR IN HIGHEST ORDER (SPATIAL) DERIVS.) COUPLED, ELLIPTIC PDEs FOR 4 "GRAVITATIONAL POTENTIALS" $\{\psi, X^i\}$; FACILITATES THEORETICAL ANALYSIS, COMPUTATIONAL SOL^{N}

2) TRANSVERSE-TRACELESS / LONGITUDINAL / TRACE DECOMP. of K^{ij}

TT (FREE) FREE

DIVERGENCE-FREE

→ LONG. PART GENERATED VIA DIFF. OF VECTOR POTENTIAL X^i WHICH WILL BE FIXED BY MOM C.

NOTE: FIXING γ_{ij} UP TO CONF. TRANS $\equiv$ FIXING $\sqrt{\gamma}$, CAN SHOW THAT $\sqrt{\gamma}$, $K \equiv K^i{}_i$ ARE ESSENTIALLY DYNAMICALLY CONJUGATE; HENCE IT IS "NATURAL" TO NOW VIEW K AS FREELY SPECIFIABLE

- INTRODUCE TRACE FREE PART, A^{ij}, of K^{ij}

$$A^{ij} = K^{ij} - \frac{1}{3}\gamma^{ij}K \qquad (5)$$

DON'T DECOMPOSE (TT / LONG) A^{ij} DIRECTLY, RATHER CONF. TRANS, THEN DECOMPOSE $\hat{A}^{ij}$; THUS TAKE

$$A^{ij} = \psi^{-10}\hat{A}^{ij} \qquad (6)$$

WHY -10? SINCE THEN WE HAVE

$$D_j A^{ij} = \psi^{-10}\hat{D}_j \hat{A}^{ij} \qquad (7)$$

WHERE $\hat{D}_i$ IS THE NATURAL DERIV. OP ON THE BASE-SPACE, I.E. $\hat{D}_i\hat{\gamma}_{jk} = \hat{D}_i\hat{\gamma}^{jk} = 0$

- NOW, ON $\mathbb{R}^3$, ANY REGULAR, TRACELESS SYMMETRIC TENSOR $\hat{A}^{ij}$ CAN BE DECOMPOSED AS

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + (\hat{\ell} w)^{ij} \tag{8}$$

WHERE THE TT-PART ("CO-EXACT" PIECE), $\hat{A}^{ij}_{TT}$ SATISFIES (BY DEF^{N})

$$\hat{D}_i \hat{A}^{ij} = 0 \tag{9}$$

AND $\hat{\ell}$ IS A SYMMETRIC, TRACE-FREE DERIV. OP

$$(\hat{\ell} w)^{ij} = \hat{D}^i w^j + \hat{D}^j w^i - \frac{2}{3} \hat{\gamma}^{ij} \hat{D}_k w^k \tag{10}$$

IS INCONVENIENT TO GIVE FREELY. SPEC. PART of $\hat{A}^{ij}$ IN TERMS OF <u>TRANSVERSE</u> TENSOR – THUS, WE REVERSE-DECOMPOSE $\hat{A}^{ij}_{TT}$ AS

$$\hat{A}^{ij}_{TT} = \hat{T}^{ij} - (\hat{\ell} V)^{ij} \tag{11}$$

WHERE, OTHER THAN NEEDING TO BE SYMMETRIC & TRACE-FREE, $\hat{T}^{ij}$ IS FREELY SPECIFIABLE, AND V^i IS ANOTHER VECTOR FIELD THEN (11) → (8) YIELDS

$$\hat{A}^{ij} = \hat{T}^{ij} + (\hat{\ell} w)^{ij} - (\hat{\ell} V)^{ij}$$

$$= \hat{T}^{ij} + (\hat{\ell} X)^{ij} \tag{12}$$

WHERE $\quad X^i = w^i - V^i \tag{13}$

• <u>CONFORMAL WEIGHTS FOR ρ, j^i</u>

YORK SUGGESTS

$$\rho = \psi^{-5} \hat{\rho} \tag{14}$$

$$j^i = \psi^{-10} \hat{j}^i \tag{15}$$

MAIN RATIONALE: CAN "BUILD-IN" WEAK ENERGY CONDITION, I.E.

IF $\hat{\rho} \geq (\hat{j}^i \hat{j}_i)^{\frac{1}{2}}$ THEN $\rho \geq (j^i j_i)^{\frac{1}{2}}$

- CAN NOW ASSEMBLE ABOVE RESULTS TO REWRITE THE CONSTRAINTS; START WITH MOM. CONSTRAINT

$$D_j K^{ij} - D^i K = 8\pi j^i \qquad (2)$$

USING (5) SOLVED FOR K^{ij}

$$D_j A^{ij} + \frac{1}{3}\gamma^{ij} D_j K - D^i K = 8\pi j^i$$

USING (7) AND (15) AND NOTING THAT $\hat{D}_i K = D_i K$

$$\psi^{-10}\hat{D}_j \hat{A}^{ij} - \frac{2}{3}\psi^{-4}\hat{\gamma}^{ij}\hat{D}_j K = 8\pi \psi^{-10}\hat{j}^i$$

$$\to \hat{D}_j \hat{A}^{ij} - \frac{2}{3}\psi^6 \hat{D}^i K = 8\pi \hat{j}^i \qquad (16)$$

- NOW, DEFINE A "VECTOR ELLIPTIC OPERATOR", $\hat{\Delta}_\ell$, VIA

$$(\hat{\Delta}_\ell w)^i \equiv \hat{D}_j (\hat{\ell} w)^{ij}$$

$$= \hat{D}_j (\hat{D}^i w^j + \hat{D}^j w^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_k w^k) \qquad (17)$$

THEN, USING (12), WE HAVE

$$\hat{D}_j \bar{T}^{ij} + (\hat{\Delta}_\ell X)^i - \frac{2}{3}\psi^6 \tilde{D}^i K = 8\pi \tilde{j}^i$$

OR

$$\boxed{(\hat{\Delta}_\ell X)^i = 8\pi \hat{j}^i - \hat{D}_j \bar{T}^{ij} + \frac{2}{3}\psi^6 \tilde{D}^i K} \quad (18)$$

∘ THUS THE MOM. CONS. BECOMES THE "VECTOR ELLIPTIC" EQUATION FOR THE "VECTOR POTENTIAL", X^i

HAMILTONIAN CONSTRAINT

$$R - K_{ij}K^{ij} + K^2 = 16\pi\rho \quad (1)$$

(1) FROM WALD (D.9) WITH $n = 3$, $\Omega = \psi^2$ (EXERCISE)

$$R = \psi^{-4}\hat{R} - 8\psi^{-5}\hat{\Delta}\psi \quad (19)$$

WHERE $\hat{\Delta} \equiv \hat{D}^i\hat{D}_i$ IS THE USUAL LAPLACIAN ON THE BASE 3-SPACE

(2) $-K_{ij}K^{ij} + K^2$

$$= -(A_{ij} + \tfrac{1}{3}\gamma_{ij}K)(A^{ij} + \tfrac{1}{3}\gamma^{ij}K) + K^2$$

$$= -A_{ij}A^{ij} - \tfrac{1}{3}K^2 + K^2 = -A_{ij}A^{ij} + \tfrac{2}{3}K^2$$

FROM (6) $A^{ij} = \psi^{-10}\hat{A}^{ij}$ $\quad A_{ij} = \psi^{-2}\hat{A}_{ij}$

$$= -\psi^{-12}\hat{A}_{ij}\hat{A}^{ij} + \tfrac{2}{3}K^2$$

(3) $\hat{A}^{ij} = \hat{T}^{ij} + (\hat{\ell}X)^{ij}$ (12)

(4) $\rho = \psi^{-8}\hat{\rho}$ (14)

- PLUG ABOVE INTO (*); SOLVE FOR $-8\hat{\Delta}\psi$, FIND

$$\boxed{-8\hat{\Delta}\psi = -\hat{R}\psi - \frac{2}{3}K^2\psi^{-5} + (\hat{T}_{ij} + (\hat{\ell}X)_{ij})(\hat{T}^{ij} + (\hat{\ell}X)^{ij})\psi^{-7} + 16\pi\hat{\rho}\psi^{-3}} \tag{20}$$

- THUS, THE HAM CONS BECOMES AN ELLIPTIC EQUATION FOR THE CONFORMAL FACTOR / SCALAR POTENTIAL, ψ

SUMMARY OF YOL INITIAL-DATA PROCEDURE

1) FREELY SPECIFY BASE QUANTITIES (12 + 4 TOTAL)

$$\{\hat{\gamma}_{ij},\ K,\ \hat{T}^{ij},\ \hat{\rho},\ \hat{j}^i\}$$

↳ SYMMETRIC ; TRACELESS

2) SOLVE CONSTRAINTS (18), (20) FOR POTENTIALS

$$\{\psi,\ X^i\}$$

3) CONSTRUCT PHYSICAL INITIAL DATA

$$\gamma_{ij} = \psi^4\hat{\gamma}_{ij} \tag{21}$$

$$K^{ij} = \psi^{-10}(\hat{T}^{ij} + (\hat{\ell}X)^{ij}) + \frac{1}{3}\psi^{-4}\hat{\gamma}^{ij}K \tag{22}$$

$$\rho = \psi^{-8}\hat{\rho} \tag{23}$$

$$j^i = \psi^{-10}\hat{j}^i \tag{24}$$

CANONICAL (I.E. COMMON) SIMPLIFICATIONS (PART. FOR B.H. WORK)

0) VACUUM → $\rho = \hat{\rho} = 0 \quad j^i = \hat{j}^i = 0$

1) FLAT BASE-GEOMETRY

$$\hat{\gamma}_{ij} = f_{ij} \quad \rightarrow \quad \hat{R} = 0$$

AND ELLIPTIC OPS $\hat{\Delta}$, $\hat{\Delta}_\ell$ TAKE ON RELATIVELY SIMPLE FORM

2) MAXIMAL INITIAL SLICE

$$K = 0$$

3) "MINIMAL RADIATION" CONDITION

$$\hat{T}^{ij} = 0$$

THEN CONSTRAINTS REDUCE TO

$$\boxed{\begin{aligned} \hat{\Delta}\psi &= -\tfrac{1}{8}(\hat{\ell}X)^{ij}(\hat{\ell}X)_{ij}\,\psi^{-7} \\ &= -\tfrac{1}{8}\hat{A}^{ij}\hat{A}_{ij}\,\psi^{-7} \qquad (25) \\ \hat{\Delta}_\ell X^i &= 0 \qquad (26) \end{aligned}}$$

NOTE: MOM. CONS COMPLETELY DECOUPLED FROM HAM. CONS; CAN FIRST SOLVE (26) (OFTEN "ANALYTICALLY"), THEN SOLVE (25) (USUALLY NUMERICALLY) FOR ψ.

EXAMPLE: (CONSISTENCY DEMONSTRATION)

- CONSIDER THE SCHWARZSCHILD SPACETIME AND RECALL FROM 387M HW 5.1 (WALD PROB 6.1b) THAT IN "ISOTROPIC COORDINATES" WE HAVE

$$ds^2 = -\frac{\left(1 - \frac{M}{2r}\right)^2}{\left(1+\frac{M}{2r}\right)^2} dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right)$$

OBSERVE:

(a) VACUUM SOL^{N}: $\rho = j^i = 0$

(b) $K^i{}_j = K^{ij} = 0 \rightarrow K = 0,\ \hat{A}^{ij} = 0,\ X^i = 0$

RECALL $K_{ij} = \frac{1}{2\alpha}\left(-\partial_t \gamma_{ij} + D_i \beta_j + D_j \beta_i\right)$

(c) $\gamma_{ij} = \psi^4 \hat{\gamma}_{ij} = \psi^4 f_{ij}$

WHERE $\psi = 1 + \frac{M}{2r}$

(A) MOMENTUM CONSTRAINTS ARE TRIVIALLY SATISFIED (AS THEY ARE ON ANY HYPERSURFACE (IN ANY SPACETIME) ON WHICH $K^{ij} = 0$: NOMENCLATURE "$t = 0$ IS A MOMENT OF TIME SYMMETRY")

$$D_j K^{ij} - D^i K = 8\pi j^i$$

(B) HAMILTONIAN CONSTRAINT

$$\hat{\Delta}\psi = -\frac{1}{8}\hat{A}^{ij}\hat{A}_{ij}\psi^{-7} \Rightarrow \hat{\Delta}\psi = 0$$

AND $\psi = 1 + \frac{M}{2r}$ <u>DOES</u> SATISFY $\hat{\Delta}\psi = 0$ WHERE

$\hat{\Delta}$ IS THE FLAT-SPACE LAPLACIAN

- CAN ALSO SHOW THAT $M = E$ WHERE E IS THE ADM MASS DEFINED PREVIOUSLY

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \oint D^j (h^i{}_j - \delta^i{}_j h) \, d^2S_i$$

WHERE $h_{ij} \equiv \gamma_{ij} - f_{ij}$

- WILL RETURN TO MORE INTERESTING (NON-TRIVIAL) EXAMPLES, TIME PERMITTING, LATER ON IN THE COURSE

SEE. COOK et al PRD, 47, 1471-1490 (1993)
AND REFERENCES CITED THEREIN