 ADAPTIVE MESH REFINEMENT

KEY REFERENCE: "ADAPTIVE MESH REFINEMENT FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS" BERGER & OLIGER, J. COMP. PHYS 53 484-512 (1984)

MOTIVATION: STRONG-FIELD, DYNAMIC GR PROBLEMS, LIKE MANY OTHERS IN PHYSICS, TYPICALLY EXHIBIT LARGE RANGE OF SCALES, L, ON WHICH FLUID, DYN. VARS ARE VARYING

EXAMPLES

- TYPE II CRITICAL BEHAVIOUR
- BH-BH INTERACTIONS
  - NEAR BH'S $L < M$ ($L \ll M$?)
  - "WAVE ZONE" $L \gg M$

- ASSUME FINITE-DIFFERENCE SOL^{N} (DISCUSSION ∥'S FOR OTHER DISCRETIZATION TECHNIQUES), THEN TYPICALLY F.D. MESH CHARACTERIZED BY SINGLE SCALE h.

- NEED $h \ll L$ FOR ACCURACY

- UNIFORM GRID $\Rightarrow$ $h \ll L_{MIN}$, BUT THIS CAN EASILY LEAD TO WASTEFUL / PROHIBITIVELY EXPENSIVE CALCULATIONS

- KEY IDEA: MAKE

$$h = h(x^{\mu})$$

I.E. ALLOW MESH SCALE TO VARY SPATIO-TEMPORALLY (MESH REFINEMENT) AND IN ACCORD WITH "SOLUTION FEATURES" (ADAPTIVE) => AMR = ADAPTIVE MESH REFINEMENT

BERGER-OLIGER ALGORITHM (MINIMAL IMPLEMENTATION)

KEY FEATURES

- USE NESTED UNIFORM GRIDS, I.E. MULTIPLE LEVELS, ℓ, OF DISCRETIZATION, h_ℓ, $\ell = 1, \ldots, \ell_{max}$ RELATED BY

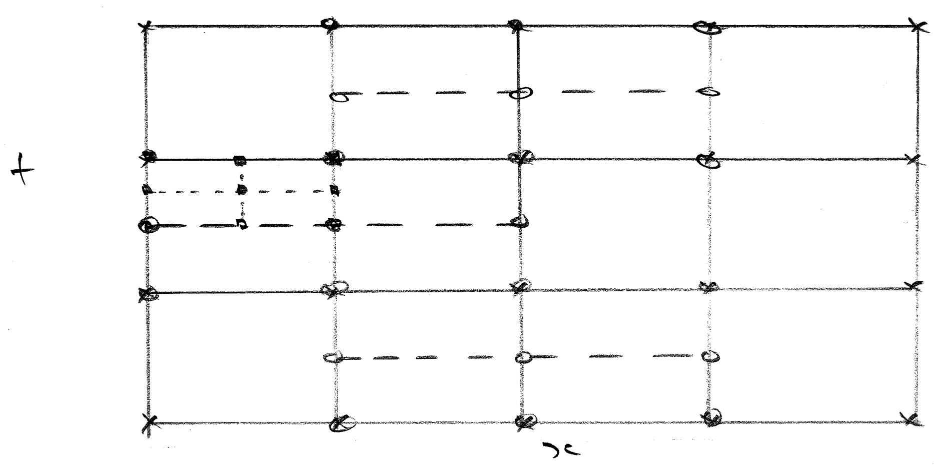
$$h_\ell = \rho\, h_{\ell+1}$$

ρ: INTEGER-VALUED REFINEMENT FACTOR (TYPICALLY CONSTANT - E.G. 2, 3, 4 ... - BUT DOESN'T HAVE TO BE)

- USE LOCAL TRUNCATION ERROR ESTIMATES (BASED ON RICHARDSON EXPANSIONS) TO DECIDE WHERE / WHEN TO ADD / DELETE REFINEMENTS

- REFINEMENT MADE IN BOTH SPACE AND TIME

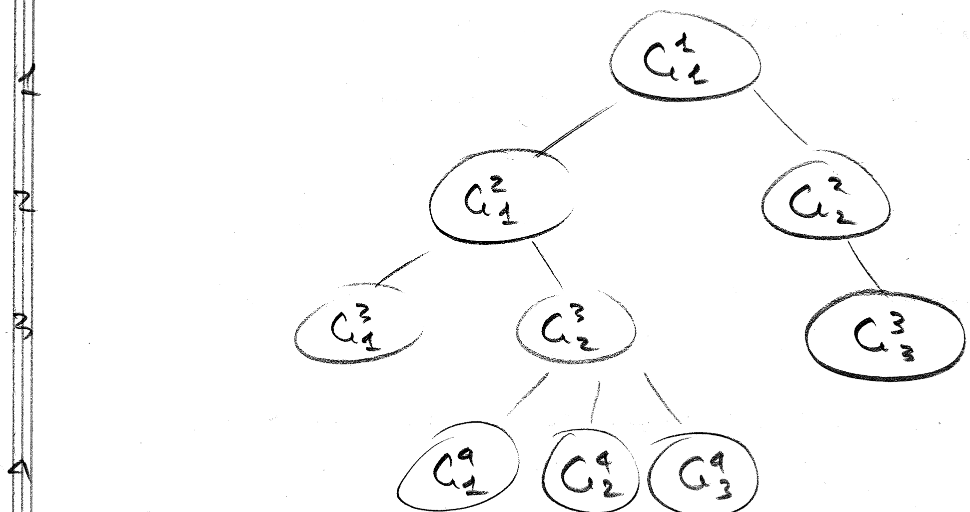
- EXAMPLE GRID STRUCTURE ADMITTED BY BERGER-OLIGER ALG



$\rho = 2$

• AT ANY INSTANT OF TIME, COMPUTATIONAL DOMAIN IS DESCRIBED BY HIERARCHY OF UNIFORM COMPONENT GRIDS ⇒ "TREE" (DIRECTED ACYCLIC GRAPH)

DISCRETIZATION LEVEL



G^ℓ_i — ℓ → LABELS DISCRETIZATION LEVEL; i → LABELS SPECIFIC GRID AT GIVEN LEVEL

• WILL ASSUME THAT EACH G^ℓ_i FOR $\ell > 1$ IS PROPERLY CONTAINED WITHIN A SINGLE $G^{\ell-1}_{i'}$

NOMENCLATURE: $G^{\ell-1}_{i'}$ IS PARENT OF G^ℓ_i

G^ℓ_i IS CHILD OF $G^{\ell-1}_{i'}$

SIBLINGS: GRIDS HAVING THE SAME PARENT

NEIGHBORS: "NEXT" GRID AT SAME LEVEL GIVEN SOME ESSENTIALLY ARBITRARY ORDERING

ADDITIONAL ASSUMPTIONS (MOSTLY TO KEEP ALG. "SIMPLE")

(1) SINGLE, GLOBAL COORDINATE SYSTEM, x^μ, ALL G^l_i "ALIGNED" IN x^μ (NO ROTATED GRIDS AS IN ORIGINAL B/O PAPER)

(2) GRID LIMITS OF G^l_i, $l > 1$ COINCIDE WITH PARENTAL GRID LINES

NOT ALLOWED ALLOWED

(3) G^1_1 COVERS ENTIRE COMPUTATIONAL DOMAIN ($\equiv$ BASE GRID)

LOCAL TRUNCATION ERROR (LTE) ESTIMATION

- CONSIDER 1-D MODEL EQUATION

$$u_t = u_x$$

FOR CONCRETENESS, ASSUME WE HAVE DISCRETIZED TO $O(h^2)$ USING TWO LEVEL EXPLICIT SCHEME ON A UNIFORM MESH ($\Delta x = h$, $\Delta t = \lambda h$)

$$\hat{u}^{n+1} = \hat{G}(h)\, \hat{u}^n$$

"ONE STEP UPDATE OPERATOR AT SCALE h"

L.T.E., τ^n, ASSOC WITH $\hat{G}(h)$ IS

$$\tau^n \equiv \underbrace{u^{n+1} - \hat{G}(h)\, u^n}$$

CONTINUUM SOLN

- GIVEN THAT SCHEME IS $O(h^2)$ ACCURATE (AND CENTRED) HAVE

$$\tau^n = h^3 t_2 + O(h^5)$$

WHERE t_2 IS h-INDEPENDENT WITH SMOOTHNESS COMPARABLE TO u (TYPICALLY $t_2 \sim u_{xxx}$)

- NOW CONSIDER TAKING 2 STEPS ON THE FINE (ORIGINAL) GRID USING $\hat{G}(h)$ AND 1 STEP ON A COARSE GRID (MESH SPACING $2h$) USING $\hat{G}(2h)$ (SAME DIFF. OPERATORS, LARGER MESH SPACING) THEN

$$u^{n+2} - \hat{G}(h)\hat{G}(h)u^n = 2h^3 t_2 + O(h^5)$$

$$u^{n+2} - \hat{G}(2h)u^n = 8h^3 t_2 + O(h^5)$$

SO $$\tau^n = \frac{1}{6}\left(\hat{G}(h)\hat{G}(h) - \hat{G}(2h)\right)u^n + O(h^5)$$

ASSUMING RICHARDSON EXPANDIBILITY:

$$\hat{u}^n = u^n + h^2 e_2^n + O(h^4)$$

$$\boxed{\tau^n \propto \left(\hat{G}(h)\hat{G}(h) - \hat{G}(2h)\right)\hat{u}^n}$$

- PRECISELY SAME PRINCIPLE USED IN "MODERN" ADAPTIVE ODE SOLVERS (SUCH AS LSODA)

BASIC BERGER / OLIGER ALGORITHM: PSEUDO-CODE

ROUTINE TIME_STEP (ℓ)

  IF REGRIDDING TIME AND $\ell < \ell_{max}$ THEN

    REGRID ON ALL LEVELS $\ell' \geq \ell$

  END IF

  FOR EACH G^{ℓ}_{i} (LOOP OVER i)

    ADVANCE SOLUTION ON G^{ℓ}_{i} (BASIC DIFF. EQNS)

    IF NECESSARY, COMPUTE "BOUNDARY VALUES" VIA

      INTERPOLATION IN PARENTAL GRID $G^{\ell-1}_{i'}$

  END FOR

  IF LEVEL $\ell+1$ EXISTS THEN

    DO ρ TIMES

      TIME_STEP ($\ell+1$)

    END DO

  END IF

  IF $\ell > 1$ AND $t_{\ell} = t_{\ell-1}$

    INJECT LEVEL ℓ VALUES INTO PARENTAL GRIDS

    (LEVEL $\ell - 1$)

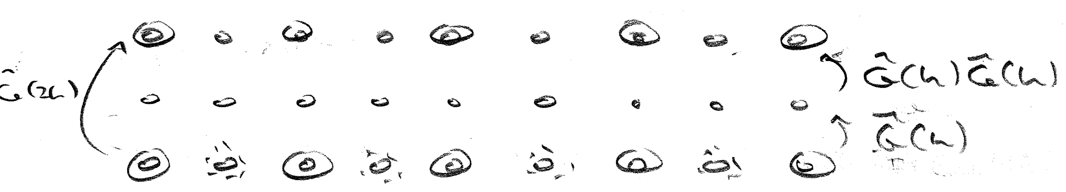
  END IF

END ROUTINE

DETAILS / COMMENTS

(1) REGRIDDING

(A)

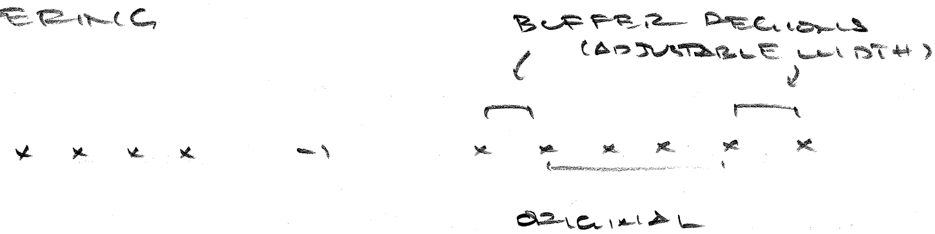


G, VIA INTERPOLATION

LTE ESTIMATION

(B) CLUSTERING (MORE CHALLENGING IN 2,3-D!)

(C) BUFFERING



- ALLOWS FOR LONGER INTERVAL BETWEEN REGRIDDING OPERATIONS (PREDICATED ON FINITE SPEED OF SIGNAL PROPAGATION)

(D) MERGING : CLUSTERS SEPARATED BY FEWER THAN SOME SPECIFIED # OF GRID PTS MERGED INTO SINGLE CLUSTER (EFFICIENCY / ACCURACY)

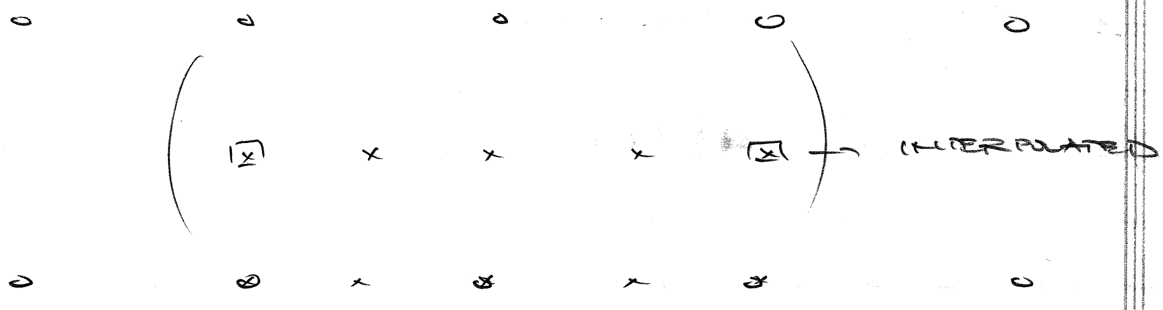
(E) FINAL CLUSTERS ON LEVEL l BECOME LEVEL $l+1$ GRIDS

- REGRIDDING ALWAYS STARTS FROM BOTTOM (FINEST EXTANT LEVEL) TO ENSURE PROPER LEVEL NESTING

(INSERT)

(2) TIME-STEPPING

- COARSE GRIDS ADVANCED FIRST - ENSURES THAT ADVANCED PARENTAL VALUES ARE AVAILABLE TO PROVIDE B.C.'S AT REFINEMENT BOUNDARIES VIA INTERPOLATION



(3) INJECTION OF CHILD VALUES ⇒ PARENTAL GRIDS

- IF THIS IS NOT DONE, COARSE GRID UNKNOWNS WHICH LIE WITHIN REFINED REGIONS (AND THUS BY DEF^N INACCURATE) COULD EVENTUALLY LEAD TO BAD COARSE GRID VALUES NEAR FINE-GRID BOUNDARIES ⇒ BAD FINE GRID B.V'S ⇒ BAD FINE GRID SOL^N

(4) "REGULARIZATION" of L.T.E ESTIMATES

- RICHARDSON-TYPE PROCEDURES EFFECTIVELY ESTIMATING HIGHER DERIVATIVES of SOL^N (pTH ORDER)
- IN FOURIER SPACE, MODE IN SOL^N u WITH FREQ ω WILL RESULT IN SAME MODE IN τ (LTE) BUT WEIGHTED BY ω^p (TYPICALLY ω^2)
- LTE CAN EASILY BECOME DOMINATED BY "NOISE" FROM HIGH FREQUENCY MODES WHICH ARE UNLIKELY TO BE TREATED ACCURATELY IN ANY CASE
- DISSIPATION (KREISS/OLIGER & RELATED) VERY HELPFUL
- NOTE: BAD L.T.E ESTIMATION TENDS TO PRODUCE INEFFICIENCIES RATHER THAN INACCURACIES

REGRIDDING

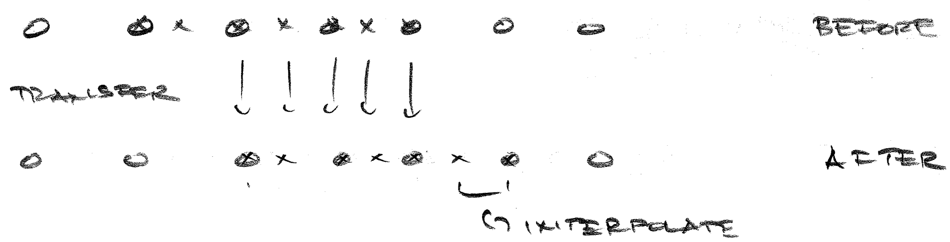
$\in$ CLUSTERS → NEW GRIDS

$\in$ GRID FUNCTIONS → NEW GRIDS NEED INITIALIZATION

GENERALLY COMBO. of

(1) TRANSFERS FROM OLD GRIDS

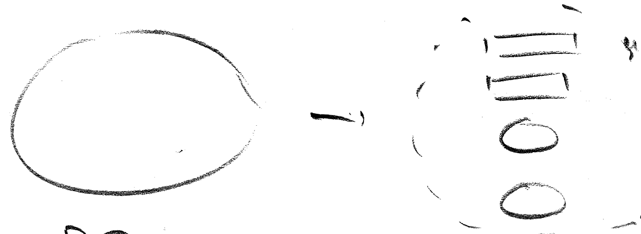
(2) INTERPOLATION FROM PARENTAL GRIDS



DATA STRUCTURING IN FORTRAN

→ OTHER ½ "METHODS" / ROUTINES ACTING ON DS → ACCESS PROVIDED VIA COMMON BLOCKS — WHICH ARE INCLUDED BY ALL SUCH ROUTINES

• DATA STRUCTURES (DATA ½ OF "OBJECT")



N COMPONENTS

COULD BE OTHER D.S

DS

• LABEL SPECIFIC INSTANCE OF DS VIA INTEGER "TAG"

• USE SEPARATE ARRAYS TO MAINTAIN ALL INSTANCES OF SPECIFIC COMPONENTS

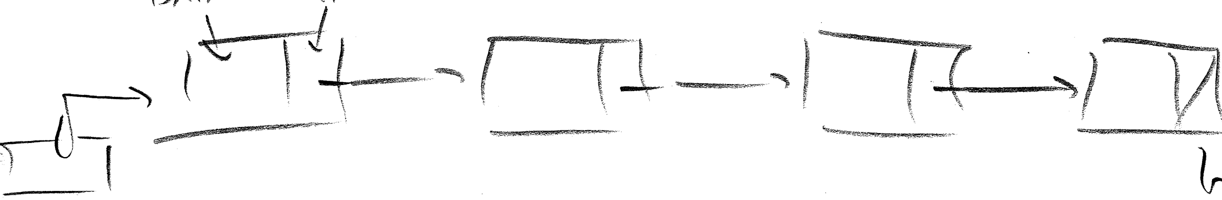
EXAMPLE: DS: GRID, LABELLED BY INTEGER G

GL(G)	GRID G's LEVEL
GPP(G)	GRID G's PARENT
GPCH(G)	GRID G's FIRST CHILD
⋮	

LINKED LIST: NODE: PARTICULAR TYPE OF DS WHICH INCLUDES LINKS / POINTERS TO OTHER INSTANCES OF DS

SIMPLE LINK LIST: NODE := { DATA, LINK TO NEXT NODE

DATA LINK



"NULL POINTER"

POINTER TO HEAD OF LIST.

(NEED TO MAINTAIN ONE PER SEPARATE LINKED LIST)

AD MAINTAINS SEP. L.L.'s FOR

- GRIDS AT EACH LEVEL

```
INTEGER G, L

G = LHEAD(L)
DO WHILE ( G.NE. 0 )
   :
   G = GPNIGH(G)
END DO
```

"NULL POINTER"

- CHILDREN OF EACH GRID

```
INTEGER G, CH

CH = GPCH(G)
DO WHILE ( CH .NE. 0 )
   :
   CH = GPSIB(G)
END DO
```

AND ~~OTHERS~~

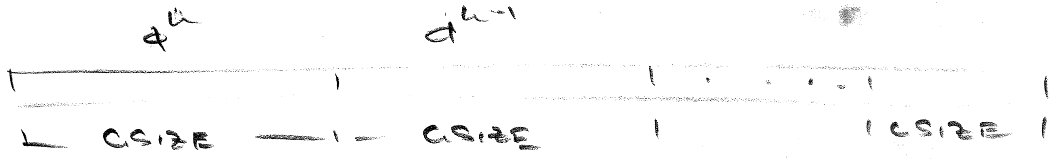
STORAGE FOR GRID FUNCTIONS

- ~~ASSUME~~ NFCN DISTINCT GRID FUNCTIONS (COUNTING MULTIPLE TIME LEVELS SEPARATELY) DEFINED ON EACH GRID, EACH OF SIZE GSIZE.

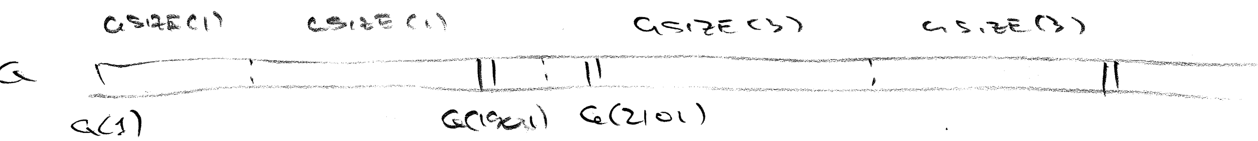
GSIZE = NR (NX ...) 1-D

GSIZE = NX * NY 2-D

- "ALLOCATE" STORAGE FOR ALL GRID FCNS ASSOCIATED WITH G IN ONE CONTIGUOUS BLOCK



- "ALLOCATE" ALL GRID FCN STORAGE FROM SINGLE ARRAY (G)



- NEED TO MAINTAIN "POINTERS" TO GRID FUNCTIONS

INTEGER GPFCN(NFCN, GNMAX)

↳ NOTE TYPO IN COMMENT IN gridsto.inc

GPFCN(1,1) = 1
GPFCN(2,1) = 951
GPFCN(1,2) = 1901
⋮

CONVENIENT TO INTRODUCE SYMBOLIC ALIASES FOR PTRS

AN, ANP1, . . .

AN = GPFCN(1,1)

REAL*8 AN(NR), . . .   CALL LDPTR1(G)

CALL UPDATE_A(AN, NR)   → CALL UPDATE_A(G(AN), . . .
GNR(G))

TRUNCATION ERROR VIA SHADOW HIERARCHY

- ad IMPLEMENTS a priori LTE ESTIMATION
  - AT t^n_ℓ = regridding_time, COMPUTE

$$\tilde{\tau}^n_\ell \quad \left(\tilde{G}^{\ell}\tilde{G}^{\ell} - \tilde{G}^{\ell-1}\right) u^n_\ell$$

BUT THEN "DISCARD" $\hat{G}^{\ell}\hat{G}^{\ell} u^n_\ell$, $\hat{G}^{\ell-1} u^n_\ell$

- LTE ROUTINE DUPLICATES (AND WORSE) CODE IN BASIC UPDATE ROUTINE

- SHADOW HIERARCHY: DO SAME THING BUT a posteriori: - SIMULTANEOUSLY INTEGRATE 2:1 COARSENING of HIERARCHY - AT REGRIDDING TIMES "AUTOMATICALLY" HAVE

$$\hat{G}^{\ell}\hat{G}^{\ell} u^{n-2}_\ell \qquad \hat{G}^{\ell-1} u^{n-2}_\ell$$

ASSUME DIFFERENCE WILL BE ADEQUATE ESTIMATOR FOR τ^n_ℓ

- ADVANTAGES: SINGLE CALL TO UPDATE ROUTINE, REDUCED MEMORY REQUIREMENTS, STRUCTURE of OVERALL ALG CONSIDERABLY SIMPLIFIED