1 Equations of Motion

We want to take the first-order equations of motion

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)' \tag{1}$$

$$\dot{\Pi} = \left(\frac{\alpha}{a}\Phi\right)' + \frac{a\alpha}{r^2}W(1-W^2) \tag{2}$$

$$\frac{a'}{a} + \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) = 0 \tag{3}$$

$$\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}(1-W^2)^2\right) = 0 \tag{4}$$

$$\dot{a} = \frac{2\alpha}{r}\Pi\Phi \tag{5}$$

$$\Phi = W' \tag{6}$$

$$\Pi = \frac{a}{\alpha}\dot{W}, \tag{7}$$

and write them as second-order in W. This is achieved somewhat easily by using the formula for $\dot{\Pi}$, so

$$\dot{\left(\frac{a}{\alpha}\dot{W}\right)} = \left(\frac{\alpha}{a}W'\right)' + \frac{a\alpha}{r^2}W(1-W^2) \tag{8}$$

gives the equation for W. However, α and a become even more tricky than before, since we have

$$\frac{a'}{a} + \frac{a^2-1}{2r} - \frac{1}{r}\left(W'^2 + \frac{a^2}{\alpha^2}\dot{W}^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) = 0, \tag{9}$$

$$\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - \frac{1}{r}\left(W'^2 + \frac{a^2}{\alpha^2}\dot{W}^2 - \frac{a^2}{2r^2}(1-W^2)^2\right) = 0. \tag{10}$$

So if we are considering a three-level difference scheme, there are a few problems that we need to consider.

1. The presence of the $(\frac{a}{\alpha})_t$ term in (8) after applying the product rule. Without it we could explicitly find W^{n+1} knowing the n, $n-1$ time levels. But with it, we need to know a_j^{n+1} and α_j^{n+1} to find W_j^{n+1}. This would seem to imply that we need to solve for *alpha* and a iteratively at each time step along with W. I know RNPL solves for W iteratively because we gave it explicit residuals, but will it do the same for α and a? It seems like the answer is no, but then in what order does it solve things in?

2. The Hamiltonian constraint and slicing condition are now coupled together. There are two suggested ways to approach this:

(a) Iterate between the two. With a good guess for α, the equation for a has the exact same form (assuming $\dot{W}$ is somehow known). The equation for α is now more tricky to solve, having an α^{-2} term appearing. Appropriate changes to the slicing condition solver would need to be made.

(b) Solve the two of them simultaneously, step by step. α may be rescaled afterwards to satisfy the boundary conditions. This also requires fairly significant changes to the HC solver.

3. $\dot{W}$ now appears in the afforementioned constraint equations, which may be gotten around by applying the conditions at an in-between time level $n+\frac{1}{2}$, which is probably better than using a differencing like $\dot{W_j^{n+1}} = \frac{W_j^{n+1}-W_j^n}{\Delta t}$.

2 Three-Level Difference Scheme

2.1 Implementation

In evaluating (8), we look to evaluate the quantity $\lambda_j = \left(\frac{a_j}{\alpha_j}\dot{W}_j\right)$ at time levels $n+\frac{1}{2}$, $n-\frac{1}{2}$, so that we approximate

$$\dot{\lambda_j} \approx \frac{\lambda_j^{n+1/2} - \lambda_j^{n-1/2}}{\Delta t}. \tag{11}$$

We'll let $q = \frac{a}{\alpha}$ so $\lambda = q\dot{W}$, and it turns that we're okay if we use

$$\lambda_j^{n+\frac{1}{2}} = q_j^{n+1/2}\frac{W_j^{n+1} - W_j^n}{\Delta t} \tag{12}$$

and similarly at $n - \frac{1}{2}$.

I've worked it out on paper and have shown that using $q_j^{n+1/2} = \frac{q_j^{n+1}+q_j^n}{2}$ gives the $\mathrm{O}(h^2)$ expression for $\dot{\lambda}$ so we have that

$$\dot{\lambda}_j^n = \left[q_j^{n+1/2}(W_j^{n+1} - W_j^n) - q_j^{n-1/2}(W_j^n - W_j^{n-1})\right]/(\Delta t)^2 + O(h^2), \tag{13}$$

where we have the governing equation for W,

$$\dot{\lambda}_j^n = \left[(q_{j+1}^n + q_j^n)(W_{j+1}^n - W_j^n) - (q_j^n + q_{j-1}^n)(W_j^n - W_{j-1}^n)\right]/(2\Delta r)^2 + \frac{a_j^n\alpha_j^n}{r_j^2}W_j^n(1-(W_j^n)^2) + O(h^2). \tag{14}$$

The discretization of $'$ follows exactly like that of $\dot{}$.

2.2 Issues

We have an evolution equation that gives us W_j^{n+1} explicitly, but requires knowing a, α at level $n+1$, or at least $q = \frac{a}{alpha}$ at level $n+\frac{1}{2}$. a and α are determined by spatial integration at whatever time level is chosen. Both a' and α' depend on a, α, W and $\dot{W}$ at the current indices. Because it's unfeasible to get $\dot{W}$ at time level $n+1$, we need to apply the constraint equations at level $n+\frac{1}{2}$, which would get us $q^{n+1/2}$.

If we're just going to iterate solving these three, the equation for a as mentioned has nothing new in the HC solver; at time level $n+\frac{1}{2}$ then $\dot{W} = \frac{W^{n+1}-W^n}{\Delta t}$, you plug it in and it (should) all work as before. The equation for α is more tricky, because we have a $\frac{1}{\alpha^2}$ that we need to deal with, as in eqn (10).

So we'll want to put the slicing condition solver in the form of

$$\ln{(\alpha)}' + f0 + \frac{f2}{\alpha^2} = 0 \tag{15}$$

at some time step, with $f0$ and $f2$ as known functions at the spatial grid points j.

In the previous incarnation of the HC solver we'd have used $f0$ and $f2$ at point $j+\frac{1}{2}$ (by taking the average) and then using $a_j a_{j+1}$ as the approximation for a^2, and $\frac{\ln{(a_{j+1})}-\ln{(a_j)}}{\Delta r}$ as the approximation for $ln(a)' = \frac{a'}{a}$.

So I propose we use $\frac{1}{\alpha_j \alpha_{j+1}}$ as the approximation for $\frac{1}{\alpha^2}$ that we now need.

3 Summary

I need to know what iterations RNPL does and what iterations I explicitly need to do. Do everything look okay? Do the HC / SC solver ideas look right? I hope this is what you were looking for.