# Stability of Stripe Solutions for the Gierer-Meinhardt Reaction-diffusion Model

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http://laplace.physics.ubc.ca/People/jhomenuk/JohnHomenuke-ThesisPresentation.pdf

# Outline

**Introduction** Activator-inhibitor systems and stability analysis of stripe solutions for the Gierer-Meinhardt (GM) Model.

**Gierer-Meinhardt System**

**Numerics** Crank-Nicholson and Multigrid schemes.

**Stability Analysis** Determining parameter spaces for which the stripe is unstable.

**Compare Closed-form to Numerical Results**

**Remaining work / Questions**

# Introduction

## Activator-Inhibitor Systems

- Two-component reaction-diffusion systems that form steady state solutions providing positional information.
- Applications to developmental biology.
- Activator undergoes autocatalysis to create local growths.
- Inhibitor suppresses the activator surrounding these centers.
- Both components diffuse. Inhibitor diffuses faster.
- Activator diffusion causes the steady-state.
- Play Animation
- Pattern provides positional information. Play Animation

**Analyzing Stripe Solutions** of the form $\operatorname{sech}^2(x)$ for two kinds of instabilities: Breakup into spots and transverse evolution to zigzag patterns.

# Gierer-Meinhardt System

## Non-dimensionalized Form

$$\Omega = \{\mathbf{X} = (X_1, X_2) : -1 < X_1 < 1, 0 < X_2 < d_0\}, \quad t > 0$$

$$a = a(t, \mathbf{X}), \quad h = h(t, \mathbf{X})$$

$$\frac{\partial a}{\partial t} = \epsilon_0^2 \nabla^2 a - a + \frac{a^p}{h^q}, \quad \tau \frac{\partial h}{\partial t} = D \nabla^2 h - h + \frac{a^r}{\epsilon_0 h^s}$$

$$p > 1, q > 0, r > 1, s \geq 0, \quad \frac{qr}{p-1} - (s+1) > 0$$

$$\partial_n a = \partial_n h = 0, \quad \mathbf{X} \in \partial\Omega$$

# Gierer-Meinhardt System

## Non-dimensionalized Form

$$\Omega = \{\mathbf{X} = (X_1, X_2) : -1 < X_1 < 1, 0 < X_2 < d_0\}, \quad t > 0 \tag{1}$$

$$a = a(t, \mathbf{X}), \quad h = h(t, \mathbf{X}) \tag{2}$$

$$\frac{\partial a}{\partial t} = \epsilon_0^2 \nabla^2 a - a + \frac{a^p}{h^q}, \quad \tau \frac{\partial h}{\partial t} = D\nabla^2 h - h + \frac{a^r}{\epsilon_0 h^s} \tag{3}$$

$$p > 1, q > 0, r > 1, s \geq 0, \quad \frac{qr}{p-1} - (s+1) > 0 \tag{4}$$

$$\partial_n a = \partial_n h = 0, \quad \mathbf{X} \in \partial\Omega \tag{5}$$

## Re-scaled Form

$$\Omega_l = \{\mathbf{x} = (x_1, x_2) : -l < x_1 < l, 0 < x_2 < d\}, \quad t > 0 \tag{6}$$

$$\frac{\partial a}{\partial t} = \epsilon^2 \nabla^2 a - a + \frac{a^p}{h^q}, \quad \tau \frac{\partial h}{\partial t} = \nabla^2 h - h + \frac{a^r}{\epsilon h^s} \tag{7}$$

$$l = 1/\sqrt{D}, \quad X = x/l, \quad d = d_0 l, \quad \epsilon = \epsilon_0 l \tag{8}$$

# Numerics: Finite Differencing

**Domain** Discretize into an array of points.

$$\Omega \to \Omega_h = \{(x_i, y_j) : x_i = (i-1)h, y_j = (j-1)h\}$$

$$t = n\Delta t, \quad \Delta t = \lambda h, \quad 0 < \lambda < 1 \quad \text{(Courant number)}$$

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**Equations** Differential operators $\rightarrow$ Difference Operators via Taylor Series approximations.

$$\partial_t u \rightarrow \Delta_t(u^n) = \frac{u^{n+1} - u^n}{\Delta t} + O(\Delta t)$$

$$\partial_{xx} u \rightarrow \Delta_{xx}(u_i) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + O(h^2)$$

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$$\partial_{xx} u \to \Delta_{xx}(u_i) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + O(h^2) \tag{12}$$

**Crank-Nicholson Scheme** Both sides centered at time index $n + 1/2$.

$$\begin{aligned} \partial_t u &= \nabla^2 u + f(u, v) \\ \frac{1}{\Delta t}\left(u^{n+1} - u^n\right) &= \frac{1}{2}\left(\nabla^2 u^{n+1} + \nabla^2 u^n + f(u^{n+1}, v^{n+1}) + f(u^n, v^n)\right) \end{aligned} \tag{13}$$

$$u^{n+1} - \frac{\Delta t}{2}\left(\nabla^2 u^{n+1} + f(u^{n+1}, v^{n+1})\right) = u^n + \frac{\Delta t}{2}\left(\nabla^2 u^n + f(u^n, v^n)\right) \tag{14}$$

Need to solve this *elliptic* equation at every time step.

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1. Transfer temporary solution to coarser grid.
2. Apply smoothing operator.
3. Transfer solution back to fine grid.

Use as many grids as necessary.

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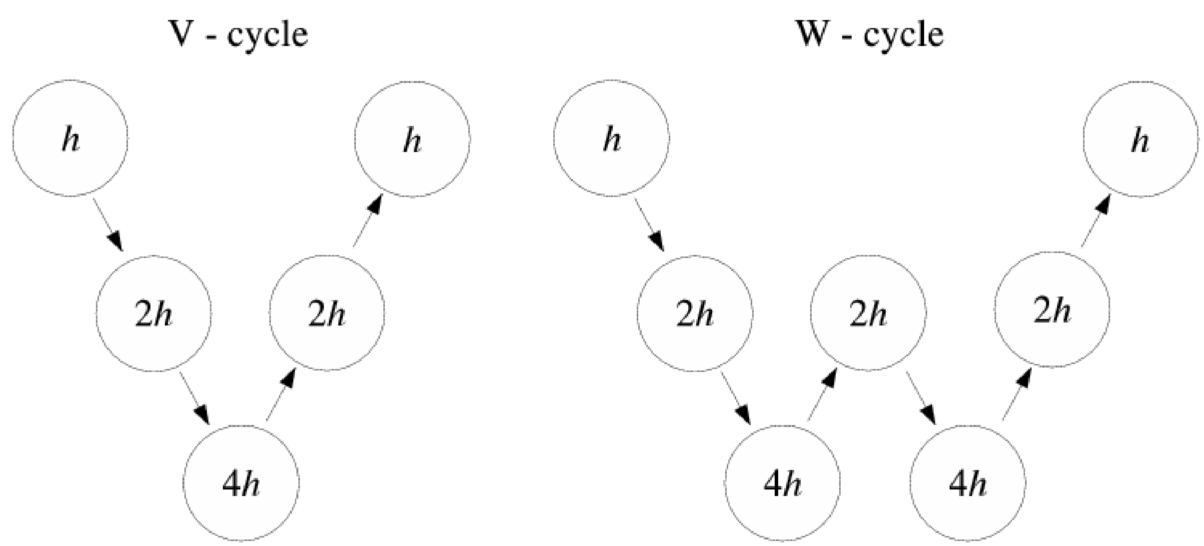
**Schematic of Grid Transfers**



# Numerics: Multigrid

**Restriction Operator $\mathcal{R}$** Transfers the temporary solution to the next coarsest grid of mesh twice the current one using a weighted sum of the surrounding points.

$$\begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

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**Prolongation Operator $\mathcal{P}$** Transfers the temporary solution to the next finest grid using a bilinear interpolation.

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**Relaxation Operator** Newton Gauss-Seidel Method:

1. Nonlinear system $\mathbf{L}(\mathbf{u}) = \mathbf{f}$. (Our case: $\mathbf{L}(\mathbf{u}^{n+1}) = \mathbf{f}(\mathbf{u}^n)$).
2. Compute residual $\mathbf{r} = \mathbf{L}(\mathbf{u}^n) - \mathbf{f}(\mathbf{u}^n)$.
3. Compute Jacobian $J_{ij} = \partial L_{ij}/\partial u$ for $i = 1, \ldots, D$ and $j = 1, \ldots, D$. $D$ is number of components.
4. Solve $\mathbf{J}\,\delta\mathbf{u} = \mathbf{r}$ to get adjustment to solution.
5. Repeat until local convergence is achieved.

# Stability Analysis

## Two Interaction Regimes

- Semi-strong: Diffusion coefficients ${\epsilon_0}^2 \ll 1$ and $D = O(1)$.
- Weak: Diffusion coefficients ${\epsilon_0}^2 \ll 1$ and $D = D_0{\epsilon_0}^2 \ll 1$.

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**Homoclinic Stripes** Solution to steady state cross-section

$$a_{yy} - a + \frac{a^p}{h^q} = 0, D_0 \quad h_{yy} - h + \frac{a^r}{h^s} = 0$$

where $y = \epsilon_0 x_1$. Numerically determine $a(0)$ and $h(0)$. Then use an approximation in full solutions.

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where $y = \epsilon_0 x_1$. Numerically determine $a(0)$ and $h(0)$. Then use an approximation in full solutions.

## Two Types of Instabilities for Homoclinic Stripes

- Varicose (breakup): Even perturbation eigenfunctions.
- Transverse (zigzag): Odd perturbation eigenfunctions.

# Stability Analysis

**Determining the Stability** Use the ansatz

$$a = a_e(x_1/\epsilon_0) + \Phi(x_1/\epsilon_0)e^{\lambda t}\cos(mx_2), \quad h = h_e(x_1/\epsilon_0) + N(x_1/\epsilon_0)e^{\lambda t}\cos(mx_2) \quad (18)$$

in the original GM system. Then we get the eigenvalue problem

$$\Phi_{yy} - (1+\mu)\Phi + \frac{p{a_e}^{p-1}}{{h_e}^{q}}\Phi - \frac{q{a_e}^{p}}{{h_e}^{q+1}}N = \lambda\Phi$$

$$D_0 N_{yy} - (1+D_0\mu)N + \frac{r{a_e}^{r-1}}{{h_e}^{s}}\Phi - \frac{s{a_e}^{r}}{{h_e}^{s+1}}N = \tau\lambda N$$

which can be solved numerically to determine the range of $\mu = {\epsilon_0}^2 m^2$ for which $\Re(\lambda) > 0$.

# Numerical Simulations

## Initial Conditions

$$a(0,x_1,x_2) = a_e(0)\text{sech}^2(x_1/\epsilon_0), \quad h(0,x_1,x_2) = h_e(0)\text{sech}^2(x_1/\epsilon_0) \tag{19}$$

## Solutions (Weak Interaction Regime)

1. $\epsilon_0 = 0.025, \quad D_0 = 15.0, \quad \tau = 0.01, \quad (p,q,r,s) = (2,1,2,0)$
   Expected Behaviour: 8 spots followed by transverse migration.
   Play Animation
2. $\epsilon_0 = 0.025, \quad D_0 = 7.6, \quad \tau = 0.01, \quad (p,q,r,s) = (2,1,2,0)$
   Expected Behaviour: Zigzag pattern.
   Play Animation
3. $\epsilon_0 = 0.025, \quad D_0 = 6.8, \quad \tau = 0.01, \quad (p,q,r,s) = (2,1,2,0)$
   Expected Behaviour: No equilibrium stripe solution. Replication of stripe followed by breakup.
   Play Animation

# Remaining Work / Questions

**Independent Residual Test** Rewriting the program using a different scheme. If solutions are the same within truncation error, then they must be correct.

**Comparison of Numerical to Theorized Results** Behaviour on paper not reproduced precisely in numerical results.

- Different numbers of spots.
- No zigzag instabilities observed yet.

**Semi-strong Regime** Have yet to generate solutions in this regime.