Notes on Computational Relativistic Hydrodynamics

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1. Introduction

The aim of these notes is to describe the high resolution shock capturing scheme used to numerically solve hydrodynamic equations in the framework of general relativity. The presentation takes a practical approach, i.e. no attempt of mathematical rigor is being made. We start with a brief summary of some relevant results from GR. Later we derive the relativistic hydro equations. The second part deals with numerical approaches for solving the equations. In this notes we focus on the hydro part of the solver, i.e., we assume the background geometry is known at each time step.

2. GR background

The fundamental entity in GR is the spacetime. Spacetime is a collection of *events*. Each event is a point of the spacetime. Each event (point) is labeled by an n-tuple of numbers — one time coordinate and $n-1$ spatial coordinates. In what follows we assume $n = 4$. The *metric* of spacetime describes the intrinsic geometry of the spacetime, i.e., it determines the distances and angles between pairs of neighboring points. A practical way of defining the metric is to write a metric element

$$ds^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu. \tag{2.1}$$

This defines the distance (squared) between points with coordinates x^μ and $x^\mu + dx^\mu$. Note that the metric in general depends on the coordinate x^μ. For example the Minkowski metric is the metric of a flat spacetime of special relativity written in Cartesian coordiantes

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{2.2}$$

If the distance (interval) between two events is *negative* we say that the interval is *timelike*. For $ds^2 < 0$ the interval is called to be *spacelike*. If $ds^2 = 0$ the interval is called to be *null*. A 1-dimensional curve $x^\mu(\lambda)$ in spacetime describes a series of events. If all the neighbouring points on a curve are separated by timelike interval then the curve is called timelike and it can describe a history of a physical particle (also called a worldline). The elapsed proper time "felt" by the particle is the square root of the negative of the total distance between two events A and B. Proper time τ serves as a useful parametrization of the worldline. The 4-velocity u^μ is defined as

$$u^\mu = \frac{dx^\mu(\tau)}{d\tau} \ . \tag{2.3}$$

In special relativity the components of 4-velocity are

$$u^\mu = (u^0, u^1, u^2, u^3) = (W, W\vec{v}) \ , \tag{2.4}$$

where W is the Lorentz factor defined as

$$W = \frac{1}{\sqrt{1-\vec{v}^2}} \ . \tag{2.5}$$

In the above $\vec{v}$ is the standard 3-velocity.

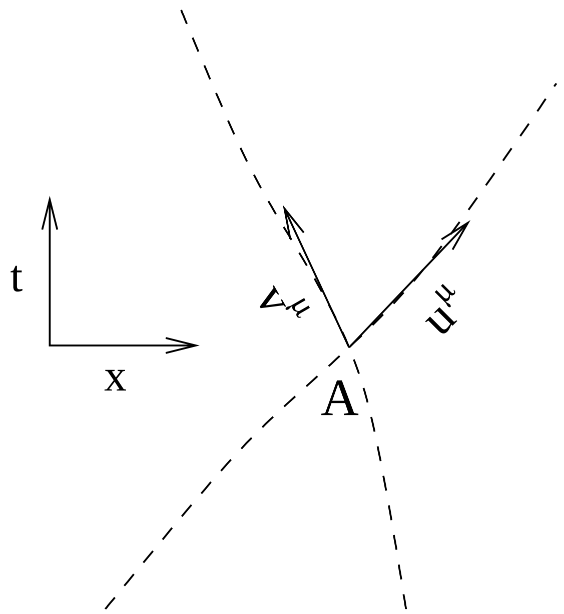

Figure 1. Worldlines of two particles meeting at point A.

Imagine two particles with worldlines that meet at a particular point A (see fig 1). At A the two particles have 4-velocity u_μ and z^μ, respectively. Their product $u_\mu z^\mu$ is an invariant and thus can be evaluated in an arbitrary reference frame. In particular in the Lorentz reference frame comoving with u^μ we have

$$u'_\mu = (-1, 0, 0, 0) \tag{2.6}$$

and therefore

$$u_\mu z^\mu = u'_\mu z'^\mu = -z'^0 = -W \ . \tag{2.7}$$

Therefore the Lorentz factor of the particle moving with z^μ with respect to a particle moving with u^μ can be calculated as

$$W = -u_\mu z^\mu \ . \tag{2.8}$$

2.1. The $3+1$ *decomposition*

The $3+1$ decompositon is a way of slicing a spacetime into spatial slices (hypersurfaces) parametrized by a parameter usually called time t. This parameter t does not need to have any direct correspondence to a physical time (whatever it is).

The spacetime metric $g_{\mu\nu}$ can be written in a particular way

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \ . \tag{2.9}$$

In a component form

$$\left\| \begin{array}{cc} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{array} \right\| = \left\| \begin{array}{cc} \beta_s\beta^s - \alpha^2 & \beta_j \\ \beta_i & \gamma_{ij} \end{array} \right\| . \tag{2.10}$$

It is useful to label the spatial part of the metric by a different symbol

$$\gamma_{ij} = g_{ij} \qquad\qquad i, j = 1, 2, 3 \ . \tag{2.11}$$

The inverse metric has the form

$$\left\| \begin{array}{cc} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{array} \right\| = \left\| \begin{array}{cc} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{array} \right\| . \tag{2.12}$$

The γ^{ij} is the inverse of γ_{ij}, i.e.

$$\gamma_{ik}\gamma^{kj} = \delta_i^j \ . \tag{2.13}$$

In general the indices of spatial 3-vectors are raised and lowered by γ. In particular

$$\beta_i = \gamma_{ij}\beta^j \ . \tag{2.14}$$

Later we will also use the decomposition of the determinant of the 4-metric

$$\sqrt{-g} = \alpha\sqrt{\gamma} \ . \tag{2.15}$$

Figure 2 helps to understand the meaning of the *lapse* parameter α and the *shift* parameters β^i. It shows two infinitesimaly close spacelike hypersurfaces $\Sigma(t)$ and $\Sigma(t+dt)$. The 4-vector n^μ is the unit normal vector to the hypersurface $\Sigma(t)$. Its covariant and contravariant components are

$$n_\mu = (-\alpha, 0, 0, 0), \qquad n^\mu = \left(\frac{1}{\alpha}, \frac{-\beta^i}{\alpha}\right) \ . \tag{2.16}$$

The 4-vector u^μ is the 4-velocity of some particle. In the Eulerian formulation of hydrodynamics we use Eulerian reference frame. This reference frame consists of observers moving perpendiculary to the hypersurfaces $\Sigma(t)$ with clocks showing proper time. In other words, the Eulerian frame consists of the basis vectors (μ, ∂_i).

Let us now translate the 4-velocities from the coordinate reference frame (S) to the Eulerian reference frame (S'). Table (2.21) shows the coordinates of the three points P, Q, R in the two coordinate systems.

	S	S'
P	(t, x^i)	(t', x'^i)
Q	$(t+dt, x^i - \beta^i dt)$	$(t' + \alpha dt, x'^i)$
R	$(t+dt, x^i + dx^i)$	$(t' + \alpha dt, x'^i + \beta^i dt + dx^i)$

(2.17)

The 4-velocity is defined as the vector $\vec{PR}$ divided by rhe proper time τ. In the coordinate reference frame S we get

$$u^\mu = \frac{(dt, dx^i)}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) \ . \tag{2.18}$$

In the Eulerian reference frame S' we get

$$u'^\mu = \frac{(\alpha dt, \beta^i dt + dx^i)}{d\tau} = (\alpha u^0, u^i + \beta^i u^0) = \alpha u^0 \left(1, \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}\right) \ . \tag{2.19}$$

The Lorentz factor as seen from S' is

$$W = -n_\mu u^\mu = \alpha u^0 \ . \tag{2.20}$$

Let us define the *Eulerian 3-velocity* as

$$v^i = \frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha} \ . \tag{2.21}$$

To raise and lower the indices on the Eulerian 3-velocities we use γ_{ij}.

$$v_i = \gamma_{ij}v^j = \gamma_{ij}\left(\frac{u^j}{\alpha u^0} + \frac{\beta^j}{\alpha}\right) = \frac{1}{\alpha u^0}\gamma_{ij}\left(u^j + \beta^j u^0\right) =$$

$$= \frac{1}{\alpha u^0}\left(\gamma_{ij}u^j + \beta_i u^0\right) = \frac{1}{\alpha u^0}\left(g_{i0}\, u^0 + g_{ij}\, u^j\right) = \frac{u_i}{\alpha u^0} \ . \tag{2.22}$$

The Lorentz factor W can be calculated in terms of the spatial metric γ_{ij} and the Eulerian velocities v^i. This can be seen from the basic property of 4-velocity

$$g_{\mu\nu}u^\mu u^\nu = -1 \ . \tag{2.23}$$

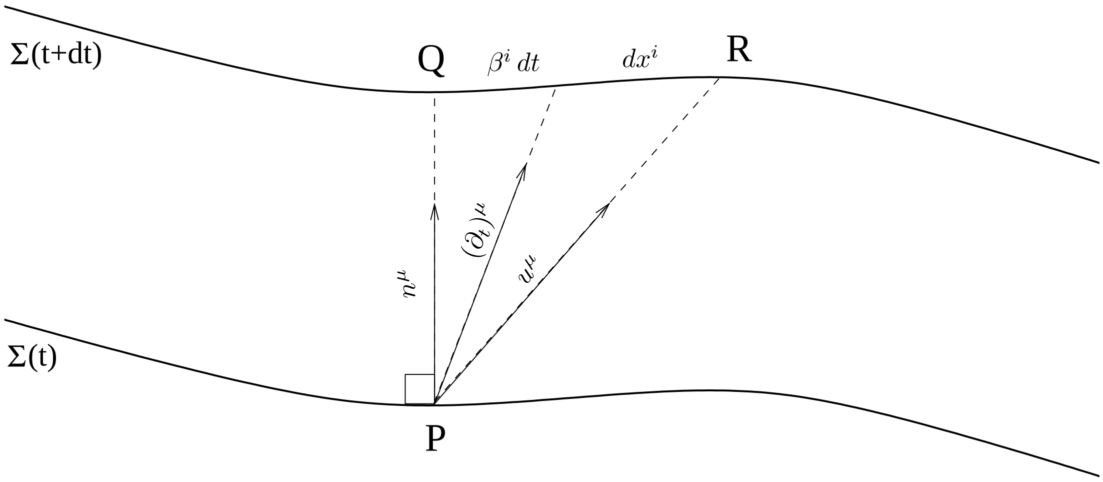

Figure 2. The $3+1$ decomposition.

Using (2.9) we have

$$\begin{aligned} -1 = g_{\mu\nu} u^\mu u^\nu &= -\alpha^2 (u^0)^2 + \gamma_{ij} \left(u^i + \beta^i u^0\right) \left(u^j + \beta^j u^0\right) = \\ &= -\alpha^2 (u^0)^2 \left[1 - \gamma_{ij} \left(\frac{u^i}{\alpha u^0} + \frac{\beta^i}{\alpha}\right) \left(\frac{u^j}{\alpha u^0} + \frac{\beta^j}{\alpha}\right)\right] = \\ &= -\alpha^2 (u^0)^2 \left(1 - \gamma_{ij} v^i v^j\right) \ . \end{aligned} \tag{2.24}$$

Therefore

$$W = \alpha u^0 = \frac{1}{\sqrt{1 - \gamma_{ij} v^i v^j}} \ . \tag{2.25}$$

3. Thermodynamics

A hydrodynamic description relies on the concept of *local thermal equilibrium.* Imagine a box of gas in a thermal bath. After some time the gas will reach the temperature of the thermal bath and the macroscopic properties of the gas in the box will be the same everywhere. The gas in the box is in the *global thermal equilibrium.* Its state is characterized by various *thermodynamic potentials* such as temperature, internal energy, entropy, pressure, etc.. The thermal equilibrium is reached due to internal collisions of its constituents. Imagine now that we get rid of the thermal bath and start squeezing the box slowly. If the time scale needed to reach an equilibrium is much smaller than the time scale of the macroscopic squeezing we can regard the gas in thermal equilibrium all the time. Of course now the macrostate is time dependent and the thermodynamic potentials are functions of time.

Similarly we can imagine a gas that is not in global thermal equilibrium yet each small little volume ΔV located at some coordinate position x is. We say that such system is in *local thermal equilibrium* (LTE). As long as the macroscopic dynamics of the system occurs on longer time scales than the equilibration time scale the system continues to be in LTE. The thermodynamic potentials are then functions of position and time.

In what follows we make several idealizations. We assume the fluid to be a *simple fluid*, i.e., there are no chemical reactions among constituens. Also for the purpose of these lectures we always assume the fluid to be an *ideal fluid.* We also neglect any heat transfer ,i.e., change of the temperature of the fluid is only caused by adiabatic processes. In particular the fluid can reach a stationary state (no time dependence) even if different parts of the fluid have different temperatures. This is clearly unphysical but in many applications the heat transfer has negligible effect on the overal dynamics of the fluid.

In GR thermodynamics all the thermodynamic potential are defined in the *rest frame* of the fluid element at point x^μ.

n baryon number density
ρ_0 rest mass density $\rho_0 = n m_{\mathrm{B}}$ (m_{B} is the average rest mass of the baryons)
ρ total mass-energy density
s entropy per baryon

First law of thermodynamics

$$dE = TdS - PdV \tag{3.1}$$

In the above equation the quantities are related to volume elements with fixed number of baryons N. After substituting

$$E = \frac{E}{V}\frac{V}{N}N = \frac{\rho}{n}N \ , \tag{3.2}$$

$$S = \frac{S}{N}N = sN \ , \tag{3.3}$$

$$S = \frac{V}{N}N = \frac{N}{n} \ , \tag{3.4}$$

into (3.1) we get

$$d\left(\frac{\rho}{n}N\right) = T\left(sN\right) - Pd\left(\frac{N}{n}\right) \ . \tag{3.5}$$

Dividing by N and expanding the derivatives we obtain

$$\frac{d\rho}{n} - \rho\frac{dn}{n^2} = Tds + P\frac{dn}{n^2} \ . \tag{3.6}$$

Multiplying by n we get

$$d\rho = nTds + \frac{\rho + P}{n}dn \ . \tag{3.7}$$

Since ρ is determined by n and s, $d\rho$ is a total differential and thus

$$nT = \left.\frac{\partial\rho}{\partial s}\right|_n \ , \tag{3.8}$$

$$\frac{\rho + P}{n} = \left.\frac{\partial\rho}{\partial n}\right|_s \ . \tag{3.9}$$

3.1. Equation of state

For ideal gas the equation of state reads

$$P = (\Gamma - 1)\rho_0\epsilon \ , \tag{3.10}$$

where ϵ is the internal specific energy density ($\rho_0\epsilon$ is the internal energy density). From (3.1) (after dividing by $m_{\mathrm{B}}N$) we get

$$Tds = d\epsilon + Pd\left(\frac{1}{\rho}\right) = \frac{dP}{\rho_0(\Gamma - 1)} - \frac{P}{\Gamma - 1}\frac{d\rho_0}{\rho_0^2} - \frac{dP}{\rho_0(\Gamma - 1)} - \frac{P}{\rho_0^2}d\rho_0 \ . \tag{3.11}$$

If we assume that s is *constant* then the above equation simplifies

$$\frac{dP}{\rho_0} = \frac{P\Gamma}{\rho_0^2}d\rho_0 \ , \tag{3.12}$$

and can be integrated

$$P = \kappa\rho_0^\Gamma \ . \tag{3.13}$$

The constant of integration κ is in fact a function of s.

4. The equations of relativistic hydrodynamics

The relativistic hydro equations can be derived from the fundamental conservation laws of nature — the baryon number conservation and the conservation of energy-momentum. The rest mass flux (proportional to the baryon number flux) is defined as

$$J^\mu = \rho_0 u^\mu \ . \tag{4.1}$$

The stress-energy (energy-momentum) tensor of an ideal fluid is

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu} \ . \tag{4.2}$$

In the following derivations we will make use of these useful relations

$$u^0 = \frac{W}{\alpha} \ , \tag{4.3}$$

$$\frac{u^i}{W} = v^i - \frac{\beta^i}{\alpha} \ , \tag{4.4}$$

$$\frac{u_i}{W} = v_i \ . \tag{4.5}$$

The conservative variables are defined as follows (recall (2.25))

$$D = -J^\mu n_\mu = -\rho_0 u^\mu n_\mu = \rho_0 W \ . \tag{4.6}$$

$$S_j = -T^\mu{}_\nu n_\mu \underbrace{(\partial_j)^\nu}_{\delta^\nu_j} = \alpha T^0{}_j = \alpha(\rho + P)u^0 u_j = (\rho + P)W^2 \frac{u_j}{W} =$$

$$= (\rho + P)W^2 v_j \tag{4.7}$$

$$E = T^{\mu\nu} n_\mu n_\nu = \alpha^2 T^{00} = \alpha^2 \left[(\rho + P)u^0 u^0 + P g^{00}\right] =$$

$$= (\rho + P)W^2 - \alpha^2 P \frac{1}{\alpha^2} = (\rho + P)W^2 - P \tag{4.8}$$

In practical numerical calculations we often use a derived conservative variable

$$\tau = E - D \ . \tag{4.9}$$

One can easily show the following relations (exercise)

$$T^{00} = \frac{1}{\alpha^2} E \ , \tag{4.10}$$

$$T^{0i} = \frac{1}{\alpha}(\rho + P)W^2 \left(v^i - \frac{\beta^i}{\alpha}\right) + P\frac{\beta^i}{\alpha^2} \ , \tag{4.11}$$

$$T^0{}_i = \frac{1}{\alpha} S_i \ , \tag{4.12}$$

$$T^i{}_j = S_j \left(v^i - \frac{\beta^i}{\alpha}\right) + P\delta^i_j \ . \tag{4.13}$$

The baryon number conservation can be written in the diffferential form as

$$J^\mu{}_{;\mu} = 0 \ . \tag{4.14}$$

The above can be evaluated as

$$J^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, J^\mu\right)_{,\mu} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, \rho_0 u^\mu\right)_{,\mu} =$$

$$= \frac{1}{\sqrt{-g}} \left(\alpha\sqrt{\gamma}\, \rho_0 \frac{W}{\alpha}\right)_{,0} + \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, \rho_0 W \left(v^i - \frac{\beta^i}{\alpha}\right)\right)_{,i}$$

$$= \frac{1}{\sqrt{-g}} \left(\sqrt{\gamma}\, D\right)_{,\,0} + \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, D \left(v^i - \frac{\beta^i}{\alpha}\right)\right)_{,\,i} = 0 \tag{4.15}$$

The equation (4.15) is the first hydrodynamic equation. Now let us obtain the remaining four. Remember that the Eulerian basis consists of four vectors $\mathbf{e}_\mu$

$$\mathbf{e}_\mu = \{\mathbf{n}, \partial_i\} \ . \tag{4.16}$$

Consider now the expression $(T^\mu{}_\nu (\mathbf{e}_\gamma)^\nu)_{;\mu}$

$$(T^\mu{}_\nu (\mathbf{e}_\gamma)^\nu)_{;\mu} = \underbrace{T^\mu{}_{\nu;\mu}}_{0} (\mathbf{e}_\gamma)^\nu + T^{\mu\nu} (\mathbf{e}_\gamma)_{\nu;\mu} = T^{\mu\nu} \left((\mathbf{e}_\gamma)_{\nu,\mu} - \Gamma^\lambda_{\nu\mu} (\mathbf{e}_\gamma)_\lambda\right) \ . \tag{4.17}$$

By evaluating the above expression we obtain the desired equations. Let us start with the left hand side for the index $\gamma = 0$

$$(T^\mu{}_\nu (\mathbf{e}_0)^\nu)_{;\mu} = (T^{\mu\nu} n_\nu)_{;\mu} = \left(-\alpha T^{\mu 0}\right)_{;\mu}$$

$$= -\frac{1}{\sqrt{-g}} \left\{ \left(\sqrt{-g}\, \alpha T^{00}\right)_{,\,0} + \left(\sqrt{-g}\, \alpha T^{i0}\right)_{,\,i} \right\}$$

$$= -\frac{1}{\sqrt{-g}} \left\{ \left(\sqrt{\gamma}\, \alpha^2 T^{00}\right)_{,\,0} + \left(\sqrt{-g}\, \alpha \left[\frac{1}{\alpha} \underbrace{(\rho + P) W^2}_{E+P} \left(v^i - \frac{\beta^i}{\alpha}\right) + P \frac{\beta^i}{\alpha^2}\right]\right)_{,\,i} \right\} =$$

$$= -\frac{1}{\sqrt{-g}} \left\{ \left(\sqrt{\gamma}\, E\right)_{,\,0} + \left(\sqrt{-g} \left[E \left(v^i - \frac{\beta^i}{\alpha}\right) + P v^i\right]\right)_{,\,i} \right\} \tag{4.18}$$

The right hand side reads

$$T^{\mu\nu} \left((\mathbf{e}_0)_{\nu,\mu} - \Gamma^\lambda_{\nu\mu} (\mathbf{e}_0)_\lambda\right) = -T^{\mu 0} \alpha_{,\mu} + \alpha \Gamma^0_{\nu\mu} T^{\mu\nu} = \alpha \left(-T^{\mu 0} (\ln \alpha)_{,\,\mu} + \Gamma^0_{\mu\nu} T^{\mu\nu}\right) \tag{4.19}$$

Putting all together we get

$$\frac{1}{\sqrt{-g}} \left\{ \frac{\partial}{\partial t} \left(\sqrt{\gamma}\, E\right) + \frac{\partial}{\partial x^i} \left(\sqrt{-g} \left[E \left(v^i - \frac{\beta^i}{\alpha}\right) + P v^i\right]\right) \right\} =$$

$$= \alpha \left(T^{\mu 0} (\ln \alpha)_{,\,\mu} - \Gamma^0_{\mu\nu} T^{\mu\nu}\right) \tag{4.20}$$

For $\gamma = j$ we have

$$(T^\mu{}_\nu (\mathbf{e}_j)^\nu)_{;\mu} = \left(T^\mu{}_j\right)_{;\mu} = \frac{1}{\sqrt{-g}} \left\{ \left(\sqrt{-g}\, T^\mu{}_j\right)_{,\,\mu} \right\} =$$

$$= \frac{1}{\sqrt{-g}} \left\{ \left(\sqrt{-g}\, T^0{}_j\right)_{,\,0} + \left(\sqrt{-g}\, T^i{}_j\right)_{,\,i} \right\} =$$

$$= \frac{1}{\sqrt{-g}} \left\{ \left(\sqrt{-\gamma}\, S_j\right)_{,\,0} + \left(\sqrt{-g} \left[S_j \left(v^i - \frac{\beta^i}{\alpha}\right) + P \delta^i_j\right]\right)_{,\,i} \right\} \tag{4.21}$$

The right hand side reads

$$T^{\mu\nu} \left((\mathbf{e}_j)_{\nu,\mu} - \Gamma^\lambda_{\nu\mu} (\mathbf{e}_j)_\lambda\right) = T^{\mu\nu} \left(\left[g_{\nu\lambda} (\mathbf{e}_j)^\lambda\right]_{,\,\mu} - \Gamma^\lambda_{\nu\mu} g_{\lambda\sigma} (\mathbf{e}_j)^\sigma\right) =$$

$$= T^{\mu\nu} \left(g_{\nu j,\mu} - \Gamma^\lambda_{\nu\mu} g_{\lambda j}\right) \tag{4.22}$$

Putting all together we get

$$\frac{1}{\sqrt{-g}} \left\{ \frac{\partial}{\partial t} \left(\sqrt{\gamma}\, S_j\right) + \frac{\partial}{\partial x^i} \left(\sqrt{-g} \left[S_j \left(v^i - \frac{\beta^i}{\alpha}\right) + P \delta^i_j\right]\right) \right\} =$$

$$= T^{\mu\nu} \left(g_{\nu j,\mu} - \Gamma^\lambda_{\nu\mu} g_{\lambda j}\right) \tag{4.23}$$

The hydrodynamic equations can be written in a compact form

$$\frac{1}{\sqrt{-g}}\left\{\frac{\partial}{\partial t}\left(\sqrt{\gamma}\,\mathbf{q}\right)+\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,\mathbf{f}^i\right)\right\}=\mathbf{\Sigma}\ . \tag{4.24}$$

The conservative variables $\mathbf{q}$, fluxes $\mathbf{f}^i$, and source terms $\mathbf{\Sigma}$ are defined as follows

$$\mathbf{q}=\left[\begin{array}{c} D \\ S_j \\ \tau \end{array}\right]\ , \tag{4.25}$$

$$\mathbf{f}^i=\left[\begin{array}{c} D\left(v^i-\frac{\beta^1}{\alpha}\right) \\ S_j\left(v^i-\frac{\beta^1}{\alpha}\right)+P\delta^i_j \\ \tau\left(v^i-\frac{\beta^1}{\alpha}\right)+Pv^i \end{array}\right]\ , \tag{4.26}$$

$$\mathbf{\Sigma}=\left[\begin{array}{c} 0 \\ T^{\mu\nu}\left(g_{\nu j,\mu}-\Gamma^{\lambda}_{\nu\mu}g_{\lambda j}\right) \\ \alpha\left(T^{\mu 0}(\ln\alpha)_{,\,\mu}-\Gamma^0_{\mu\nu}T^{\mu\nu}\right) \end{array}\right]\ . \tag{4.27}$$

Note that we have used τ instead of E as our conservative variable. For the Minkowski metric (2.2) the source terms vanish and the determinants $\sqrt{-g}=\sqrt{\gamma}=1$. The primitive variables $\mathbf{u}$ are usually taken to be

$$\mathbf{u}=\left[\begin{array}{c} \rho_0 \\ v^j \\ P \end{array}\right]\ . \tag{4.28}$$

5. Numerical methods for conservative systems

5.1. Conservative systems

In general, conservative systems of equations can be written in the form

$$\frac{\partial\mathbf{q}}{\partial t}+\frac{\partial\mathbf{f}^i(\mathbf{q})}{\partial x^i}=0\ . \tag{5.1}$$

A particular example is a system of linear equations

$$\frac{\partial\mathbf{q}}{\partial t}+\mathbf{A}^{\mathbf{i}}\frac{\partial\mathbf{q}}{\partial x^i}=0\ , \tag{5.2}$$

where $\mathbf{A}^{\mathbf{i}}$ is a matrix with constant coefficients. The flux $\mathbf{f}^i$ is then defined as

$$\mathbf{f}^i=\mathbf{A}^{\mathbf{i}}\mathbf{q}\ . \tag{5.3}$$

To simplify matters consider for now only equations in one spatial dimension

$$\frac{\partial\mathbf{q}}{\partial t}+\mathbf{A}\frac{\partial\mathbf{q}}{\partial x}=0\ . \tag{5.4}$$

The system (5.4) is said to be *hyperbolic* if $\mathbf{A}$ is diagonalizable with real eigenvalues $\{\lambda^1,\ldots,\lambda^m\}$. The matrix $\mathbf{A}$ is diagonalizable if it has a complete set of non-zero (right) eigenvectors $\{\mathbf{r}^1,\ldots,\mathbf{r}^m\}$ that are defined by

$$\mathbf{A}\mathbf{r}^p=\lambda^p\mathbf{r}^p\ . \tag{5.5}$$

The diagonalization is then performed as

$$\mathbf{\Lambda}=\mathbf{R}^{-1}\mathbf{A}\mathbf{R}=\mathrm{diag}\left(\lambda^1,\ldots,\lambda^m\right)\ . \tag{5.6}$$

Several special cases are recognized in the literature. If $\mathbf{A}$ is symmetric it is automatically diagonalizable and the system is said to be *symmetric hyperbolic*. If $\mathbf{A}$ has distinct eigenvlues the system is called *strictly hyperbolic*. If $\mathbf{A}$ has real eigenvalues but is not diagonalizable the system is called *weakly hyperbolic*.

If $\mathbf{A}=\mathbf{A}(t,x)$ the system is still linear but the hyperbolicity must be determined at each point $x$separately. If $\mathbf{A}=\mathbf{A}(\mathbf{q},t,x)$ the system is called *quasilinear* and we can only talk about hyperbolicity at a point $(\mathbf{q},t,x)$.

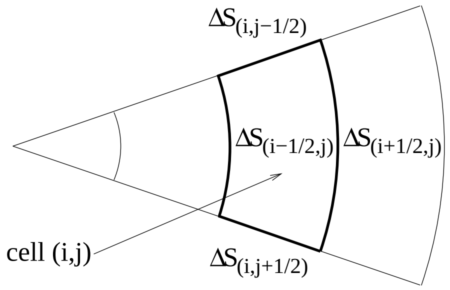

Figure 3. An example of domain decomposition in 2D.

5.2. The Riemann problem

The Riemann problem is to find the solution of a conservative system for discontinuous initial data of the form

$$\mathbf{q}_0(x) = \left\{ \begin{array}{lll} \mathbf{q}^L & \text{if} & x < 0 \\ \mathbf{q}^R & \text{if} & x > 0 \end{array} \right. . \tag{5.7}$$

The discontinuity is typically placed at $x = 0$. The Riemann problem plays a central role in the high resolution shock capturing numerical scheme.

5.3. Method of lines

Let us integrate integrate the equation (5.1) over some volume ΔV

$$\int_{\Delta V} \frac{\partial \mathbf{q}}{\partial t}\, dV + \int_{\Delta V} \frac{\partial \mathbf{f}^i}{\partial x^i}\, dV = \frac{\partial}{\partial t} \int_{\Delta V} \mathbf{q}\, dV + \int_{\partial V} \mathbf{f}^i\, dS_i = 0 \; . \tag{5.8}$$

Equation (5.8) is the integral form of (5.1) and it is in fact more fundamental than (5.1) since it does not assume continuity of the solutions. The numerical methods are based on the integral formulation. The main idea is as follows. First, decompose the computational domain into a cell-like structure. An example is shown in figure 3.

To each cell we assign the average density $\mathbf{q}_{\vec{i}}$ defined as

$$\mathbf{q}_{\vec{i}} = \frac{1}{\Delta V_{\vec{i}}} \int_{\Delta V_{\vec{i}}} \mathbf{q}\, dV \; . \tag{5.9}$$

The index $\vec{i}$ labels each cell and quantities associated with it. For n-dimensional domain it is an n-tuple of integer numbers. There are also quantities associated with the cell boundaries, such as the area of each cell boundary and fluxes through each boundary. Each n-dimensional cell has $2n$ boundaries. Each boundary is labeled by $\vec{i} + \vec{l}$, where $\vec{l}$ is a set of $2n$ n-tuples defined as follows (for $n = 1, 2, 3$).

$$\vec{l} = \left\{ \frac{1}{2}, -\frac{1}{2} \right\} \; , \tag{5.10}$$

$$\vec{l} = \left\{ \left(\frac{1}{2}, 0\right), \left(-\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(0, -\frac{1}{2}\right) \right\} \; , \tag{5.11}$$

$$\vec{l} = \left\{ \left(\frac{1}{2}, 0, 0\right), \left(-\frac{1}{2}, 0, 0\right), \left(0, \frac{1}{2}, 0\right), \left(0, -\frac{1}{2}, 0\right), \left(0, 0, \frac{1}{2}\right), \left(0, 0, -\frac{1}{2}\right) \right\} . \tag{5.12}$$

Our system then becomes a system of ODE's

$$\frac{d\mathbf{q}_{\vec{i}}}{dt} + \frac{1}{\Delta V_{\vec{i}}} \int_{\partial \Delta V_{\vec{i}}} \mathbf{f}^i\, dS_i = 0 \; . \tag{5.13}$$

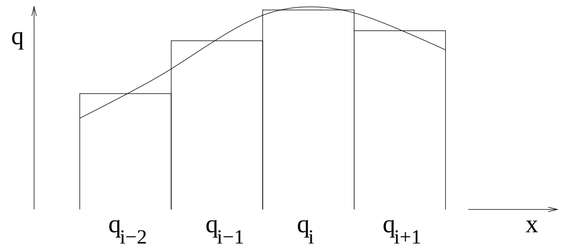

Figure 4. Piecewise constant approximation of a continuous solution. The fluxes between cell interfaces are calculated by solving the Riemann problem at each cell interface.

The above system can be approximated by

$$\frac{d\mathbf{q}_{\vec{i}}}{dt} + \frac{1}{\Delta V_{\vec{i}}} \sum_{\vec{l}} \mathbf{f}_{\vec{i}+\vec{l}} \Delta S_{\vec{i}+\vec{l}} = 0 \ , \tag{5.14}$$

where $\mathbf{f}_{\vec{i}+\vec{l}}$ is the average flux through the area element $\Delta S_{\vec{i}+\vec{l}}$. The sign of the arrea element is negative for $\vec{l}$ containing $-1/2$. The core of the numerical method lies in the procedure of obtaining some approximation of the average flux. This approach to the numerical solution is called the *method of lines.*

5.4. The flux calculations

The key ingredient to the success of the HRSC methods is to calculate the fluxes through the boundaries. Remember that the state of the fluid at each cell is described by the averages $\mathbf{q}_{\vec{i}}$. Thus the (mostly) continuous fields are approximated by a piecewise constant function. Godunov came up with the idea to calculate the fluxes at each computational boundary by solving a Rieman problem at each cell interface. Figure 4 illustrates this schematically.

This method works fine but it is only first order accurate in space (the order of accuracy in time is determined by the integrator used to advance the solution in time). To improve the spatial accuracy of the solver we *reconstruct* the continuous function from the piecewise approximation. This is, basically, a special type of interpolation. Our interpolation shoul handle discontinuous function well, and this is one of the feature is what makes it "special". Also, the reconstructed function should reproduce the correct averages $\mathbf{q}_{\vec{i}}$ at each cells. Standard interpolation techniques cause oscillations around discontinuities that makes them unusable. The methods of lines calculates the fluxes in each direction independently, therefore only one dimensional reconstruction is needed. Also, we reconstruct each variable separately.

In these notes we mention only one example of second order reconstruction, the piecewise linear *minmod* reconstruction method. The idea is to replace the piecewise constant approximation by piecewise linear approximation. To calculate the slope of some variable at the cell labeled by i (since we only need 1D reconstruction i is an integer and thus the arrow is omitted) we first calculate the left and right slopes by

$$s_i^L = \frac{q_i - q_{i-1}}{x_i - x_{i-1}} \ , \tag{5.15}$$

$$s_i^R = \frac{q_{i+1} - q_i}{x_{i+1} - x_i} \ . \tag{5.16}$$

The slope is then calculated by using the minmod function

$$s_i = \mathrm{minmod}(s_i^L, s_i^R) \ . \tag{5.17}$$

The minmod function (sometimes called a *limiter* is defined as follows

$$\mathrm{minmod}(a,b) = \begin{cases} 0 & \text{if} \quad ab \le 0 \\ a & \text{if} \quad |a| \le |b| \quad \text{and} \quad (ab > 0) \\ b & \text{if} \quad |b| < |a| \quad \text{and} \quad (ab > 0) \end{cases} \ . \tag{5.18}$$

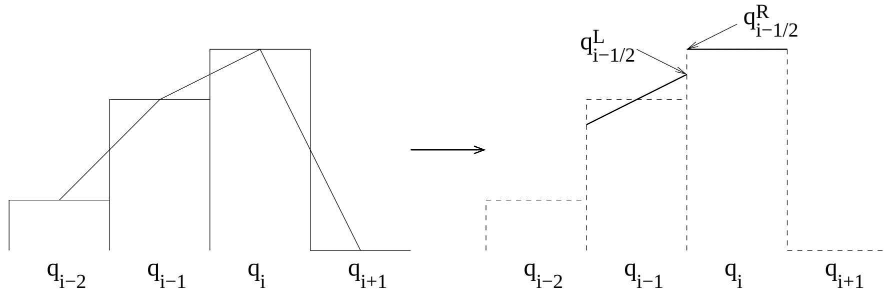

Figure 5. Minmod reconstruction.

Note that at local extrema (or shocks) the minmod gives slope zero, i.e., it is equivalent to the first order Godunov method. This is the price we pay for its the ability to handle discontinuities. Figure 5 illustrates the minmod reconstruction. ‡

Once we calculated the slopes at each cell, we can calculate the reconstructed values of the variables at the left and right cell boundary

$$q_i^L \equiv q_{i-1/2}^R = q_i - s_i(x_i - x_{i-1/2}) \ , \tag{5.19}$$

$$q_i^R \equiv q_{i+1/2}^L = q_i + s_i(x_{i+1/2} - x_i) \ . \tag{5.20}$$

Note the two different notations used for the same quantity. The flux at each cell boundary is then calculated by solving the Rimann problem with the initial conditions given by the reconstructed values at each side of the boundary

$$\mathbf{f}_{i+1/2} = \mathbf{f}^{\text{Riemann}}(\mathbf{q}_{i+1/2}^L, q_{i+1/2}^R) \equiv \mathbf{f}^{\text{Riemann}}(\mathbf{q}_i^R, q_{i+1}^L) \ , \tag{5.21}$$

$$\mathbf{f}_{i-1/2} = \mathbf{f}^{\text{Riemann}}(\mathbf{q}_{i-1/2}^L, q_{i-1/2}^R) \equiv \mathbf{f}^{\text{Riemann}}(\mathbf{q}_{i-1}^R, q_i^L) \ . \tag{5.22}$$

In what follows we explain the methods of calculating the Riemann fluxes. We start with analyzing linear systems and then use our knowledge to obtain a sensible approximations for nonlinear systems. It is possible to calculate the Riemann flux exactly for the equations of relativistic hydrodynamic, but the task is computationally expensive and therefore it is seldom used in practice.

5.5. The advection equation

The advection equation is a prototype example of a conservative system. It is a very simple linear equation

$$\frac{\partial q}{\partial t} + v\frac{\partial q}{\partial x} = 0 \ . \tag{5.23}$$

The equation states that the directional derivative of the function $q(t, x)$ in the direction given by the vector $(1, v)$ (in the t-x plane) vanishes. In other words, $q(t, x)$ is constant along the lines with slope $1/v$. Figure 6 illustrates this.

The solution of the advection equation then consists of a constant shape given by the initial data $q_0(x) \equiv q(0, x)$ moving with velocity v. Thus the formal solution of (5.23) can be written as

$$q(t, x) = q_0(x - vt) \ . \tag{5.24}$$

Note that if $q_0(x)$ is not differentiable then (5.24) solves the integral form of the advection equation.

‡ There are other limiters one can use that might give slightly better results in certain situations (MC, superbee) but the minmod appears to be an overall robust choice. Also, if higher accuracy is desired one can use higher order reconstruction methods In the literature, look for essentially non-oscillating (ENO) reconstruction.

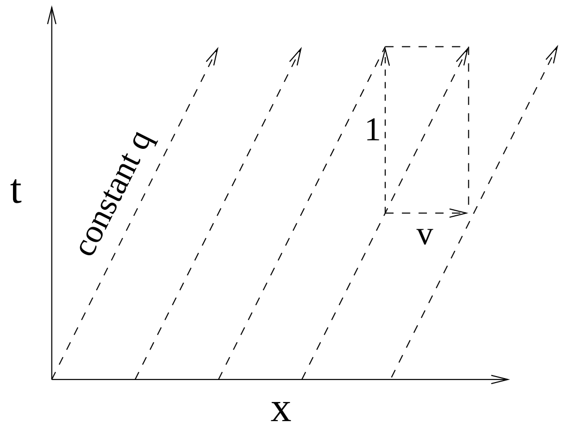

Figure 6. Characteristics of the advection equations.

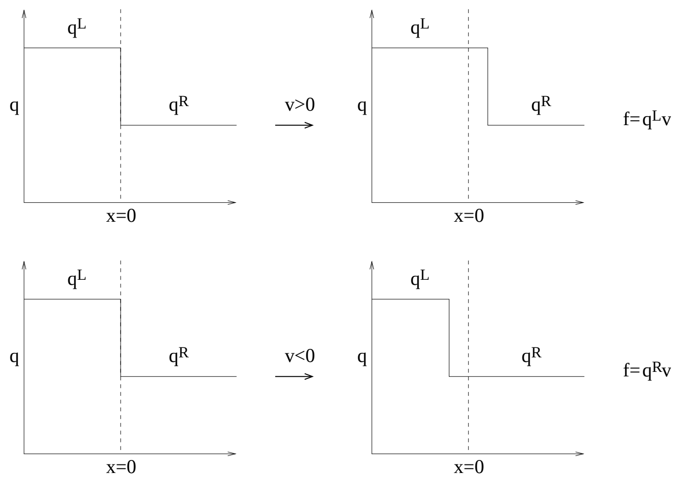

Figure 7. The solution of Riemann problem for the advection equation.

Let us take a look at the Riemann problem. For the numerical method we only need to know the flux at $x = 0$. The solution is shown in figure 7. The flux at $x = 0$ for $t > 0$ depends on the sign of the velocity. We can write the flux in a robust form valid for any sign as

$$f = \frac{1}{2}\left[(q^L v + q^R v) - |v|(q^R - q^L)\right] = \frac{1}{2}\left[(f^L + f^R) - |v|(q^R - q^L)\right] \ . \qquad (5.25)$$

5.6. Systems of linear equations

Let us take a look at system of linear equations

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{q}}{\partial x} = 0 \ . \qquad (5.26)$$

The matrix $\mathbf{A}$ is has constant coefficients and is diagonalizable as defined by (5.8) and (5.9). The matrix $\mathbf{R}$ and $\mathbf{R}^{-1}$ have the following structure

$$R = \left(\mathbf{r}^1, \ldots, \mathbf{r}^m\right) \ , \qquad (5.27)$$

$$\mathbf{R}^{-1} = \begin{pmatrix} \mathbf{l}^1 \\ \vdots \\ \mathbf{l}^m \end{pmatrix} \ , \qquad (5.28)$$

where $\mathbf{l}^j$ are called the left eigenvectors satisfying

$$\mathbf{l}^j \mathbf{A} = \lambda^j \mathbf{l}^j \ . \tag{5.29}$$

The left and right eigenvectors are orthonormal

$$\mathbf{l}^j \mathbf{r}^k = \delta_{jk} \ . \tag{5.30}$$

Note that $\mathbf{r}^j$ are *column* vectors whereas $\mathbf{l}^j$ are *row* vectors.

Using the decomposition (5.8) we can write (5.26) as

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}\frac{\partial \mathbf{q}}{\partial x} = 0 \ . \tag{5.31}$$

Multiplying from the left by $\mathbf{R}^{-1}$ we obtain

$$\frac{\partial\left(\mathbf{R}^{-1}\mathbf{q}\right)}{\partial t} + \mathbf{\Lambda}\frac{\partial\left(\mathbf{R}^{-1}\mathbf{q}\right)}{\partial x} = 0 \ . \tag{5.32}$$

By defining new set of *characteristic* variables $\mathbf{w} = \mathbf{R}^{-1}\mathbf{q}$ we obtain a set of m independent advection equations

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{w}}{\partial x} = 0 \ . \tag{5.33}$$

The solution can be obtained by transforming the initial data into the characteristic variables, finding the solution and transforming back to the conservative variables.

$$w_0^j(x) = \mathbf{l}^j \mathbf{q}_0(x) \ , \tag{5.34}$$

$$\mathbf{q}(t,x) = \sum_{j=1}^{m} w_0^j(x - \lambda^j t)\mathbf{r}^j \ . \tag{5.35}$$

The Riemann problem is defined as usual by the initial data given by (5.10). To find the Riemann flux at $x = 0$ is now easy. In general, flux is defined as

$$\mathbf{f} = \mathbf{A}\mathbf{q} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}\mathbf{q} = \mathbf{R}\mathbf{\Lambda}\mathbf{w} = \sum_{j=1}^{m} w^j \lambda^j \mathbf{r}^j \ . \tag{5.36}$$

The combination $w^j\lambda^j$ is just the flux for each characteristic variable and for the Riemann problem for the advection equation it is given by (5.25). Therefore

$$w^j\lambda^j = \frac{1}{2}\left[(w_L^j\lambda^j + w_R^j\lambda^j) - |\lambda^j|(w_R^j - w_L^j)\right] \ . \tag{5.37}$$

Substituting (5.37) into (5.36) we get

$$\begin{aligned}
&\sum_{j=1}^{m}\left\{\frac{1}{2}\left[(w_L^j\lambda^j + w_R^j\lambda^j) - |\lambda^j|(w_R^j - w_L^j)\right]\right\}\mathbf{r}^j = \\
&= \frac{1}{2}\left(\mathbf{f}^L + \mathbf{f}^R\right) - \frac{1}{2}\sum_{j=1}^{m}|\lambda^j|(w_R^j - w_L^j)\mathbf{r}^j = \frac{1}{2}\left(\mathbf{f}^L + \mathbf{f}^R\right) - \frac{1}{2}\sum_{j=1}^{m}|\lambda^j|\Delta w^j\mathbf{r}^j \ , \\
&= \frac{1}{2}\left(\mathbf{f}^L + \mathbf{f}^R\right) - \frac{1}{2}\sum_{j=1}^{m}|\lambda^j|\Delta w^j\mathbf{r}^j \ ,
\end{aligned} \tag{5.38}$$

where

$$\Delta w^j = \mathbf{l}^j(\mathbf{q}^R - \mathbf{q}^L) \ . \tag{5.39}$$

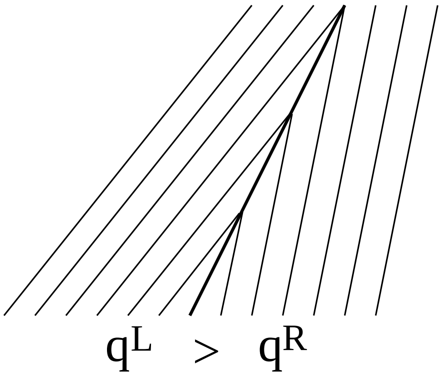

Figure 8. The characteristic structure of Riemann problem for initial conditions with $q^L > q^R$. The solution is a shock.

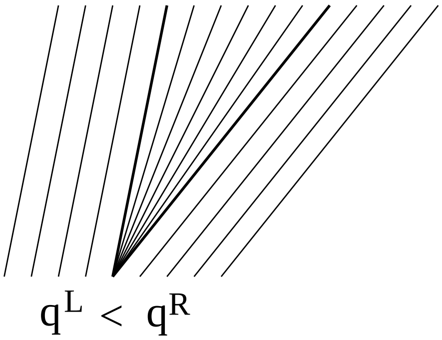

Figure 9. The characteristic structure of Riemann problem for initial conditions with $q^L < q^R$.The solution is a rarefaction fan.

5.7. Burger's equation

The Burger's equation is a prototype of a scalar nonlinear equation. It illustrates some of the differences between linear and nonlinear system. The equation has the form

$$\frac{\partial q}{\partial t} + \frac{\partial\left(\frac{1}{2}q^2\right)}{\partial x} = 0\ . \tag{5.40}$$

For differentiable functions it can be recast into an equivalent form

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial x} = 0\ . \tag{5.41}$$

This equation resembles an advection equation with the advection velocity $v = q$.

The solution of the Riemann problem depends on the initial conditions. For $q^L > q^R$ the solution is a propagating shock. The characteristic structure is shown in figure 8. The slope of the characteristic lines equals $1/q$ and thus for $q^L > q^R$ they have the tendency to meet. This behavior indicates the presence of a shock. The shock propagation speed can be calculated using the Rankine-Hugoniot jump conditions (see later).

For $q^L > q^R$ the solution is qualitatively different. It is called a rarefaction fan. The name "rarefaction fan" is derived from the shape of the characteristic lines as can be seen from figure 9.

The analytic form of the rarefaction fan can be derived by assuming that the solution is self-similar, i.e., $q(t, x) = \bar{q}(x/t)$. This is often true for the Riemann problem solutions. Consider the equation of the form

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0\ . \tag{5.42}$$

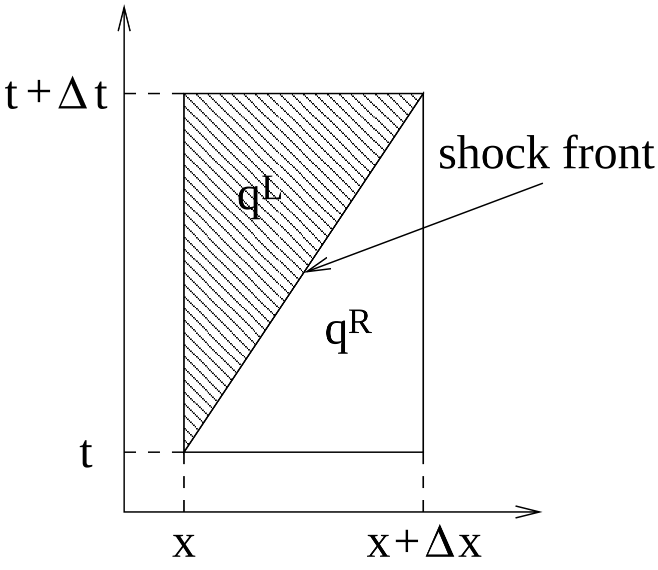

Figure 10. Integration domain for the derivation of the Rankine-Hugoniot jump conditions.

In addition, we also assume that $f''(q) > 0$. In terms of $\bar{q}$ we get

$$-q' \frac{x}{t^2} + f'(\bar{q}) q' \frac{x}{t^2} = 0 \ . \tag{5.43}$$

After simplification we get the final form

$$f'(\bar{q}) = \frac{x}{t} \ . \tag{5.44}$$

The rarefaction solution to the Riemann problem for (5.42) then can be written as

$$q(t,x) = \left\{ \begin{array}{ccc} q^L & \text{if} & \frac{x}{t} \leq f'(q^L) \\ \bar{q}(\frac{x}{t}) & \text{if} & f'(q^L) < \frac{x}{t} < f'(q^R) \\ q^R & \text{if} & \frac{x}{t} \geq f'(q^R) \end{array} \right. \tag{5.45}$$

In case of Burger's equation $f'(\bar{q}) = \bar{q}$ and thus

$$q(t,x) = \bar{q}(x/t) = \frac{x}{t} \ . \tag{5.46}$$

Let us get back to the question of shock speed propagation. It can be calculated by integrating the equation (5.42) over an area as shown in figure 10

$$\frac{\partial q}{\partial t'} + \frac{\partial f(q)}{\partial x'} = 0 \quad \Big/ \int_t^{t+\Delta t} dt' \int_x^{x+\Delta x} dx' \ . \tag{5.47}$$

We obtain

$$\int_x^{x+\Delta x} \left[(q(t+\Delta t, x') - q(t,x') \right] dx' = \int_t^{t+\Delta t} \left[(f(t',x) - f(t', x+\Delta x) \right] dt' \ . \tag{5.48}$$

From the figure it is clear that

$$q(t+\Delta t, x') = q^L \ , \tag{5.49}$$

$$q(t,x') = q^R \ , \tag{5.50}$$

$$f(t', x+\Delta x) = f(q^R) \ , \tag{5.51}$$

$$f(t',x) = f(q^L) \ . \tag{5.52}$$

The equation then reduces to

$$(q^L - q^R)\Delta x = \left[f(q^L) - f(q^R)\right] \Delta t \ . \tag{5.53}$$

The shock speed $v_s = \Delta x/\Delta t$ and thus

$$(q^L - q^R)v_s = \left[f(q^L) - f(q^R)\right] \tag{5.54}$$

or

$$v_s = \frac{f(q^L) - f(q^R)}{q^L - q^R} \ . \tag{5.55}$$

The shock solution of (5.42) has the form

$$q(t,x) = \left\{ \begin{array}{ll} q^L & \text{if} \quad \frac{x}{t} < v_s \\ q^R & \text{if} \quad \frac{x}{t} > v_s \end{array} \right. \tag{5.56}$$

For Burger's equation $v_s = (q^L + q^R)/2$. If q is a vector the above analysis is still valid up to the final division step. The equations (5.53) (or (5.54)) are called the Rankine-Hugoniot jump conditions. Note that for linear systems $\mathbf{f} = \mathbf{A}\mathbf{q}$ and (5.53) transforms into the form

$$\mathbf{A}(\mathbf{q}^L - \mathbf{q}^R) = v_s(\mathbf{q}^L - \mathbf{q}^R) \ . \tag{5.57}$$

In order to satisfy the above equation $(\mathbf{q}^L - \mathbf{q}^R)$ must be an eigenvector of $\mathbf{A}$, and therefore the shock must propagates with the characteristic speed associated with the eigenvector.

5.8. Nonlinear conservative systems

The hydrodynamic equations form a system of nonlinear hyperbolic equations. The core of the numerical method is to calculate the Riemann flux. If we can solve the Riemann problem analytically then we can use the analytic solution. However, in many cases the analytic solution is not known or is computationally expensive (recall that we have to calculate the Riemann flux for every cell interface).

The idea then is to locally approximate the nonlinear system by a linear system for which we know how to calculate the Riemann flux. The linear system has the form

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}(\mathbf{q}^L, \mathbf{q}^R)\frac{\partial \mathbf{q}}{\partial x} = 0 \ . \tag{5.58}$$

The matrix $\mathbf{A}(\mathbf{q}^L, \mathbf{q}^R)$ must satisfy the following conditions:

(i)
$$\mathbf{A}(\mathbf{q}^L, \mathbf{q}^R)(\mathbf{q}^R - \mathbf{q}^L) = \mathbf{f}^R - \mathbf{f}^L \tag{5.59}$$

(ii) $\mathbf{A}(\mathbf{q}^L, \mathbf{q}^R)$ is diagonalizable with real eigenvalues

(iii)
$$\mathbf{A}(\mathbf{q}^L, \mathbf{q}^R) \longrightarrow \frac{\partial \mathbf{f}}{\partial \mathbf{q}}(\mathbf{q}) \quad \text{as} \quad \mathbf{q}^L, \mathbf{q}^L \to \mathbf{q} \tag{5.60}$$

The equation (5.59) is the Rankine-Hugoniot jump condition that enforces the proper shock speeds for shock solutions. The second condition guarantees hyperbolicity. The third condition guarantees that the linear problem will tend to nonlinear one in smooth regions.

In many practical applications $\mathbf{A}(\mathbf{q}^L, \mathbf{q}^R)$ is taken to be

$$\mathbf{A}(\mathbf{q}^L, \mathbf{q}^R) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\left(\frac{1}{2}\left[\mathbf{q}^L + \mathbf{q}^R\right]\right) \ . \tag{5.61}$$

This matrix satisfies conditions (ii) and (iii) but in general not (i). The Roe flux at a cell boundary $\vec{i} + \vec{l}$ is then calculated by the formula

$$\mathbf{f}^{\text{Roe}}_{\vec{i}+\vec{l}} = \frac{1}{2}\left[\mathbf{f}(\mathbf{q}^L) + \mathbf{f}(\mathbf{q}^R)\right] - \frac{1}{2}\sum_{j=1}^{m} |\lambda^j| \Delta w^j \mathbf{r}^j \ . \tag{5.62}$$

Note that $\mathbf{f}(\mathbf{q})$ is the flux given by (4.26).

The Roe approximation is not the only one used, other successful approximations do exist. Since the Roe approximation is based on linear systems it can not calculate the flux correctly for rarefactions and it can give incorrect results in certain cases. If that is the case further care must be taken to ameliorate the problem.

5.9. Boundary conditions

In practical applications all calculations occur on a finite domain so we must pay attention to boundary conditions. Sometimes boundary conditions are implemented not at physical boundaries but to assure regularity conditions at coordinate boundaries, such as the $r = 0$ point in spherical coordinates.

In order to calculate fluxes at each cell boundary, we need to know the values of k neighbouring cells, where k is the order of the reconstruction. Of course a boundary cell has no neighbors in a particular direction. The boundary conditions tell us how to create a "fake" neighboring cells so we can use our formulas as usual. These fake cells are sometimes called the *ghost cells*. They may or may not be physically implemented.

There are three basic boundary conditions we recognize. These are *reflective*, *ingoing* and *outgoing* boundary conditions. We describe how to implement each of them and illustrate it for $k = 2$.

Reflective boundary conditions are implemented for physical boundaries representing *solid walls* or at centers of symmetry for enforcing regularity conditions. Figure 11 shows how to implement reflective boundary conditions at a boundary for even functions. For solid wall even functions are all the fluid variables except velocity perpendicular to the wall. Figure 12 shows how to implement reflective boundary conditions at a boundary for odd functions. For solid wall odd function is the velocity perpendicular to the wall. This forces the perpendicular velocity to vanish at the wall (or nearly vanish).

Ingoing boundary conditions are used if we know the values of the fluid variables at the boundary at all times. Figure 13 shows how to implement ingoing boundary conditions. The values of fluid variables at the ghost cells $\mathbf{q}_{g1}$ and $\mathbf{q}_{g2}$ are kept at prescribed values.

Outgoing boundary conditions are used when we want the fluid to pass through physical boundaries with the least amount of (artificial) reflection from the boundary. Figure 14 shows how to implement outgoing boundary conditions. In the simplest approach the values of the fluid variables of the boundary cell $\mathbf{q}_1$ are copied into the ghost cells. Sometimes linear extrapolation for some of the fluid variables can give better results.

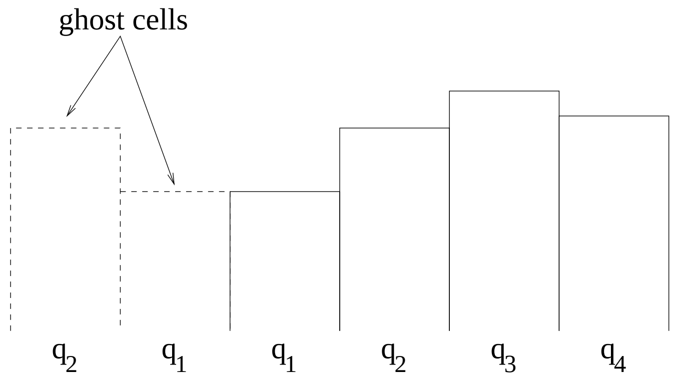

Figure 11. Implementation of reflective boundary conditions for even functions.

5.10. Conservative to primitive variable transformation

For the numerical method to work we must know both the conservative and primitive variables at each computational cell. The transformation from primitive to conservative variables is straightforward by using (4.6)–(4.9). The inverse transformation is more complicated and for relativistic fluid it involves a numerical solution of one algebraic equation. There is more than one way to do this but we describe one that solves for the pressure P.

Let us express the primitive variables in terms of the conservative ones and the pressure.

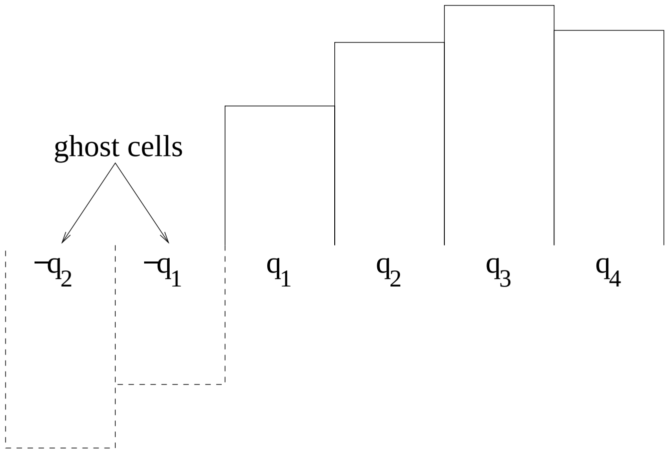

Figure 12. Implementation of reflective boundary conditions for odd functions.

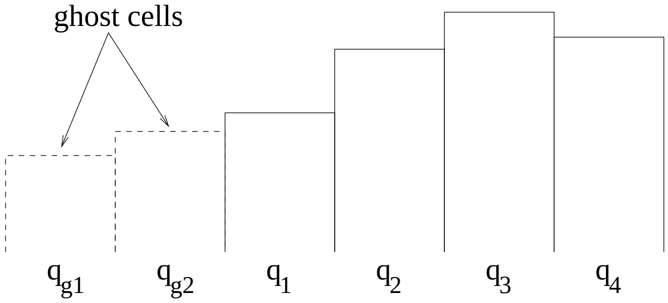

Figure 13. Implementation of ingoing boundary conditions. q_{g1} and q_{g2} are prescribed values.

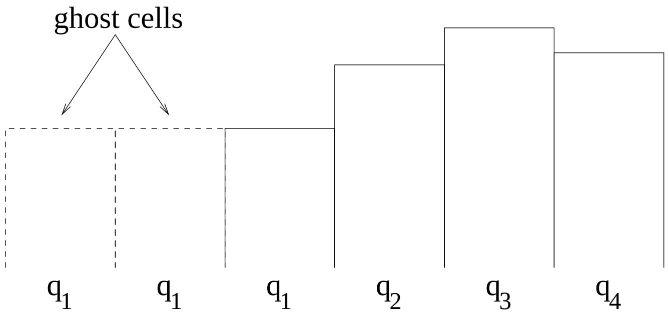

Figure 14. Implementation of outgoing boundary conditions.

In what follows we only indicate the dependence on the pressure since this is the only unknown. First we obtain

$$v_i(P) = \frac{S_i}{\tau + P + D} \ . \tag{5.63}$$

From that we can obtain the Lorentz factor W as a function of P, and consequently

$$\rho_0(P) = \frac{D}{W(P)} \ . \tag{5.64}$$

We can calculate $\epsilon = \epsilon(P)$ from the expression for τ as follows. First,

$$\rho \equiv \rho_0(1+\epsilon) = \frac{\tau + P + D}{W^2} - P \ . \tag{5.65}$$

Isolating ϵ we obtain

$$\begin{aligned}\epsilon(P) &= \frac{\tau + P + D}{\rho_0 W^2} - \frac{P}{\rho_0} - 1 = \frac{\tau + P + D - PW^2 - \rho_0 W^2}{\rho_0 W^2} = \\ &= \frac{\tau + P(1 - W^2) + D(1 - W)}{DW} \ .\end{aligned} \tag{5.66}$$

Notice that so far we have not used the equation of state. Our pressure must be compatible with the equation of state so we demand

$$f(P) = (\Gamma - 1)\rho_0(P)\epsilon(P) - P = 0 \ . \tag{5.67}$$

The above equation must be solved numerically. A common method is the iterative Newton-Raphson method. This method requires the derivative of the function f. We can calculate the derivative directly exactly, but a faster way is to use the approximation

$$f'(P) = v^i(P)v_i(P)c_s^2(P) - 1 \ , \tag{5.68}$$

where c_s is the speed of sound

$$c_s^2 = \frac{\Gamma(\Gamma - 1)P}{(\Gamma - 1)\rho_0 + \Gamma P} \ . \tag{5.69}$$

The approximation is exact for P that solves (5.67).

5.11. The ODE solver

To integrate the ODE forward in time we must choose an appropriate integrator. The simplest choice would be to use the first order Euler's method, but this is almost never used in practice due to its low order. For second order spatial reconstruction the lowest order in time would be second order. A convenient choice is the following second order Runge-Kutta (RK) method. To solve an ODE of the form

$$\frac{d\mathbf{q}}{dt} = \mathbf{F}(\mathbf{q}) \tag{5.70}$$

we proceed in these three steps

$$\begin{aligned}\mathbf{q}^{(1)} &= \mathbf{q}^n + \Delta t \mathbf{F}(\mathbf{q}^n) \ , \\ \mathbf{q}^{(2)} &= \mathbf{q}^{(1)} + \Delta t \mathbf{F}(\mathbf{q}^{(1)}) \ , \\ \mathbf{q}^{n+1} &= \frac{1}{2}\left(\mathbf{q}^n + \mathbf{q}^{(2)}\right) \ .\end{aligned} \tag{5.71}$$

There is also a third-order method available

$$\begin{aligned}\mathbf{q}^{(1)} &= \mathbf{q}^n + \Delta t \mathbf{F}(\mathbf{q}^n) \ , \\ \mathbf{q}^{(2)} &= \mathbf{q}^{(1)} + \Delta t \mathbf{F}(\mathbf{q}^{(1)}) \ , \\ \mathbf{q}^{(3)} &= \frac{1}{4}\left(3\mathbf{q}^n + \mathbf{q}^{(2)}\right) \ , \\ \mathbf{q}^{(4)} &= \mathbf{q}^{(3)} + \Delta t \mathbf{F}(\mathbf{q}^{(3)}) \ , \\ \mathbf{q}^{n+1} &= \frac{1}{3}\left(\mathbf{q}^n + 2\mathbf{q}^{(4)}\right) \ .\end{aligned} \tag{5.72}$$

Both of the RK integrators belong to the class of total variation diminishing (TVD) methods. Note that the second (third) order RK method consists of two (three) Euler substeps.

The time step must satisfy the CFL condition, i.e., for each cell

$$\Delta t < \frac{\Delta x}{\max(\lambda^1, \ldots, \lambda^m)} \ . \tag{5.73}$$

We introduce a parameter λ such that

$$\Delta t = \lambda \Delta x \ . \tag{5.74}$$

If the characteristic speeds are of order of speed of light then a rational choice is to take λ to be a fraction of one (such as 0.5). However, this choice may be too conservative if the characteristic speeds are much smaller than the speed of light. It is always a good idea to perform a test run in order to estimate the typical characteristic speeds.

5.12. *Putting it all together*

We now have all the ingredients for a successful implementation of HRSC methods for relativistic hydrodynamic equations. Here we sketch the full algorithm

(i) Initialize the fluid variables - both primitive and conservative

(ii) Initialize ghost cells (if physically implemented)

(iii) Loop over iterations, output state variables

  (a) Each RK iteration consits of several Euler substeps and some averaging.

  (b) The Euler update is done as follows

$$\mathbf{q}^{n+1} = \mathbf{q}^n - \frac{\Delta t}{\Delta V^{(3)}_{\vec{i}}} \sum_{\vec{l}} \mathbf{f}^{\mathrm{Roe}}_{\vec{i}+\vec{l}} \Delta S_{\vec{i}+\vec{l}} + \frac{\Delta V^{(4)}_{\vec{i}}}{\Delta V^{(3)}_{\vec{i}}} \mathbf{\Sigma}_{\vec{i}}(\mathbf{q}, x^\mu) \tag{5.75}$$

  (c) To calculate Roe flux through $\Delta S_{\vec{i}+\vec{l}}$

    1. Reconstruct variables in the appropriate direction (typically we reconstruct the primitive variables).

    2. Calculate the spectral decomposition of the matrix

$$\mathbf{A}_{\vec{i}+\vec{l}} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \left(\frac{1}{2} \left[\mathbf{u}^L_{\vec{i}+\vec{l}} + \mathbf{u}^R_{\vec{i}+\vec{l}} \right] \right)$$

    3. Calculate the Roe flux according to (5.62)

$$\mathbf{f}^{\mathrm{Roe}}_{\vec{i}+\vec{l}} = \frac{1}{2} \left[\mathbf{f}(\mathbf{q}^L_{\vec{i}+\vec{l}}) + \mathbf{f}(\mathbf{q}^R_{\vec{i}+\vec{l}}) \right] - \frac{1}{2} \sum_{j=1}^{m} |\lambda^j| \Delta w^j \mathbf{r}^j \ .$$

  (d) Update ghost cells (if physically implemented)

The term

$$\frac{\Delta V^{(4)}_{\vec{i}}}{\Delta V^{(3)}_{\vec{i}}} \mathbf{\Sigma}_{\vec{i}}(\mathbf{q}, x^\mu) \tag{5.76}$$

is the approximation to the integral

$$\frac{1}{\Delta V^{(3)}_{\vec{i}}} \int_{\Delta V_{\vec{i}}} \mathbf{\Sigma}(\mathbf{q}(x^\mu), x^\mu) \sqrt{-g}\, dx^1 dx^2 dx^3 \ . \tag{5.77}$$

The source functions $\mathbf{\Sigma}$ are given by (4.27). The volume elements are defined as

$$\Delta V^{(3)}_{\vec{i}} = \int_{\Delta V_{\vec{i}}} \sqrt{\gamma}\, dx^1 dx^2 dx^3 \ , \tag{5.78}$$

$$\Delta V^{(4)}_{\vec{i}} = \int_{\Delta V_{\vec{i}}} \sqrt{-g}\, dx^1 dx^2 dx^3 \ . \tag{5.79}$$

They are typically approximated by

$$\Delta V^{(3)}_{\vec{i}} \approx \sqrt{\gamma(x^\mu_{\vec{i}})} \Delta x^1 \Delta x^2 \Delta x^3 \ , \tag{5.80}$$

$$\Delta V^{(4)}_{\vec{i}} \approx \sqrt{-g(x^\mu_{\vec{i}})} \Delta x^1 \Delta x^2 \Delta x^3 \ , \tag{5.81}$$

where $x^\mu_{\vec{i}}$ is the time and the coordinate of the center of the cell labeled by $\vec{i}$.

Further Reading

You can find additional resources on the web site

`http://laplace.physics.ubc.ca/People/msnajdr/KOREA2006/`